單元 30: 代入法積分
(課本 §6.2)

積分的代入法對應於微分的連鎖規則. 令 F 為 f 的反導函數 (不定積分), 即

$$F'(x) = f(x)$$

或

$$\int f(x)dx = F(x) + C$$

則代入

$$u = g(x),\ du = g'(x)dx$$

得

$$\begin{aligned}\int f(g(x))g'(x)dx &= \int f(u)du \\ &= F(u) + C \\ &= F(g(x)) + C\end{aligned}$$

<驗證> 根據微分的連鎖規則及不定積分的定義,

$$\begin{aligned}\frac{d}{dx}[F(g(x))] &= F'(g(x))g'(x) \\ &= f(g(x))g'(x)\end{aligned}$$

剛好就是被積函數, 故成立.

因此, 代入法爲, 取

$$u = g(x)$$

計算微分式

$$du = g'(x)dx$$

並代入, 得

$$\int f(g(x))g'(x)dx = \int f(u)du$$

註. 通常選取合成函數的內部函數作爲 u, 再求 du, 並代入整理成可用適當積分規則的式子.

例 1. 試求不定積分

$$\int 2x(x^2+3)^4dx$$

<解> 取 $u = x^2+3$, 得微分式

$$du = 2xdx$$

代入整理, 積分並將 u 反代入表成 x 的數學式, 得

$$\begin{aligned}\int 2x(x^2+3)^4dx &= \int u^4du = \frac{1}{5}u^5 + C \\ &= \frac{1}{5}(x^2+3)^5 + C\end{aligned}$$

例 2. 試求 $\int 3\sqrt{3x+1}dx$.

<解> 代

$$u = 3x+1,\ du = 3dx$$

得

$$\begin{aligned}\int 3\sqrt{3x+1}dx &= \int \sqrt{u}du = \frac{2}{3}u^{3/2}+C \\ &= \frac{2}{3}(3x+1)^{3/2}+C\end{aligned}$$

例 3. 試求 $\int x^2(x^3+1)^{3/2}dx$.

<解> 取 $u = x^3+1$, 得微分式

$$du = 3x^2dx$$

接著, 調整被積函數的係數形成微分式, 即同乘除 3, 得

$$\begin{aligned}\int x^2(x^3+1)^{3/2}dx &= \frac{1}{3}\int \underbrace{(x^3+1)^{3/2}}_{u^{3/2}}\underbrace{3x^2dx}_{du} \\ &= \frac{1}{3}\cdot\frac{2}{5}u^{5/2}+C \\ &= \frac{2}{15}(x^3+1)^{5/2}+C\end{aligned}$$

例 4. 試求 $\int e^{-3x}dx$.

<解> 代

$$u=-3x,\ du=-3dx$$

並調整係數且根據指數函數的積分公式，得

$$\begin{aligned}\int e^{-3x}dx &= -\frac{1}{3}\int \underbrace{e^{-3x}}_{e^u}\underbrace{(-3)dx}_{du}\\ &= -\frac{1}{3}e^u+C=-\frac{1}{3}e^{-3x}+C\end{aligned}$$

例 5. 試求不定積分 $\int \frac{x}{3x^2+1}dx$.

<解> 代

$$u=3x^2+1,\ du=6xdx$$

並調整係數且根據 $\frac{1}{x}$ 的積分公式，得

$$\begin{aligned}\int \frac{x}{3x^2+1}dx &= \frac{1}{6}\int \underbrace{\frac{1}{3x^2+1}}_{1/u}\underbrace{6xdx}_{du}\\ &= \frac{1}{6}\ln|u|+C\\ &= \frac{1}{6}\ln|3x^2+1|+C\end{aligned}$$

例 6. 試求 $\displaystyle\int \frac{(\ln x)^2}{2x}dx$.

<解> 代

$$u = \ln x,\ du = \frac{1}{x}dx$$

並提出常數, 得

$$\begin{aligned}\int \frac{(\ln x)^2}{2x}dx &= \frac{1}{2}\int \underbrace{(\ln x)^2}_{u^2}\underbrace{\frac{1}{x}dx}_{du} \\ &= \frac{1}{6}u^3 + C = \frac{1}{6}(\ln x)^3 + C\end{aligned}$$

例 7. 太陽能面板的成本降低率爲

$$-\frac{58}{(3t+2)^2},\ 0 \le t \le 10$$

其中 $t = 0$ 對應於 1990 年年初. 另 1990 年年初的成本爲 \$10. 試求 2000 年年初的成本.

<解> 令 $C(t)$ 爲 t 年後的成本. 由題意, 得

$$C'(t) = -\frac{58}{(3t+2)^2}$$

且

$$C(0) = 10$$

故代

$$u = 3t + 2,\ du = 3dt$$

並調整係數, 得

$$\begin{aligned} C(t) &= \int \frac{-58}{(3t+2)^2} dt \\ &= -\frac{58}{3} \int \underbrace{\frac{1}{(3t+2)^2}}_{1/u^2} \underbrace{3dt}_{du} \\ &= -\frac{58}{3}\left(-\frac{1}{u}\right) + K = \frac{58}{3(3t+2)} + K \end{aligned}$$

再代 $t = 0$, $C = 10$, 得

$$10 = \frac{58}{3(3(0)+2)} + K = \frac{29}{3} + K$$

即

$$K = 10 - \frac{29}{3} = \frac{1}{3}$$

因此,

$$\begin{aligned} C(t) &= \frac{58}{3(3t+2)} + \frac{1}{3} = \frac{58 + 3t + 2}{3(3t+2)} \\ &= \frac{t+20}{3t+2} \end{aligned}$$

最後, 代 $t = 10$, 得 2000 年年出的成本為

$$C(10) = \frac{30}{32} \approx 0.94$$

例 8. 電腦上市後的月銷售量成長率為

$$2000 - 1500e^{-0.05t}, \ 0 \le t \le 60$$

試求一年後的銷售量.

<解> 令 $N(t)$ 為 t 月後的銷售量. 由題意, 得

$$N'(t) = 2000 - 1500e^{-0.05t}$$

且

$$N(0) = 0$$

故代

$$u = -0.05t, \ du = -0.05dt$$

並調整係數且逐項積分, 得

$$\begin{aligned} N(t) &= \int (2000 - 1500e^{-0.05t})dt \\ &= 2000t - \left(-\frac{1500}{0.05}\right) \int \underbrace{e^{-0.05t}}_{e^u} \underbrace{(-0.05)dt}_{du} \\ &= 2000t + 30,000e^{-0.05t} + C \end{aligned}$$

代 $t = 0$, $N = 0$, 得

$$0 = 30,000 + C$$

即

$$C = -30,000$$

因此,

$$N(t) = 2000t + 30,000(e^{-0.05t} - 1)$$

最後, 代 $t = 12$, 得 1 年後的銷售量爲

$$\begin{aligned} N(12) &= 2000(12) + 30,000(e^{-0.05(12)} - 1) \\ &\approx 10,464 \end{aligned}$$

註. 線性轉換 (即 $x \to ax + b$) 的代入法. 若

$$\int f(x)dx = F(x) + C$$

則

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$$

<證> 代

$$u = ax + b, \ du = adx$$

並調整係數，得

$$\begin{aligned}\int f(ax+b)dx &= \frac{1}{a}\int \underbrace{f(ax+b)}_{f(u)}\underbrace{adx}_{du} \\ &= \frac{1}{a}F(u)+C \\ &= \frac{1}{a}F(ax+b)+C\end{aligned}$$

例如，由

$$\int \frac{1}{x}dx = \ln|x|+C$$

得

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|+C$$

且

$$\int \frac{1}{3x+5}dx = \frac{1}{3}\ln|3x+5|+C$$

已知，當 $n \neq -1$,

$$\int x^n dx = \frac{1}{n+1}x^{n+1}+C$$

得

$$\int (ax+b)^n dx = \frac{1}{a}\frac{1}{n+1}(ax+b)^{n+1}+C$$

且

$$\begin{aligned}\int \frac{1}{(1-2x)^7}dx &= -\frac{1}{2}\left(-\frac{1}{6}\right)(1-2x)^{-6}+C \\ &= \frac{1}{12(1-2x)^6}+C\end{aligned}$$

又根據

$$\int e^x dx = e^x + C$$

得

$$\int e^{ax+b}dx = \frac{1}{a}e^{ax+b}+C$$

以及

$$\begin{aligned}\int e^{3-0.05t}dt &= -\frac{1}{0.05}e^{3-0.05t}+C \\ &= -20e^{3-0.05t}+C\end{aligned}$$

Self-Check Exercises

3. 試求 $\int xe^{2x^2-1}dx$

<解> 代

$$u = 2x^2-1,\ du = 4xdx$$

並調整係數, 得

$$\int xe^{2x^2-1}dx = \frac{1}{4}\int \underbrace{e^{2x^2-1}}_{e^u}\underbrace{(4x)dx}_{du} = \frac{1}{4}e^{2x^2-1} + C$$

Exercises

試求下列各項不定積分.

29. $\int \frac{2e^{\sqrt{x}}}{\sqrt{x}}dx$

<解> 代

$$u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}dx$$

並調整係數, 得

$$\int \frac{2e^{\sqrt{x}}}{\sqrt{x}}dx = 2(2)\int \underbrace{e^{\sqrt{x}}}_{e^u}\underbrace{\frac{1}{2\sqrt{x}}dx}_{du} = 4e^{\sqrt{x}} + C$$

33. $\int e^{2x}(e^{2x}+1)^3dx$

<解> 代

$$u = e^{2x} + 1,\ du = 2e^{2x}dx$$

並調整係數，得

$$\begin{aligned}&\int e^{2x}(e^{2x}+1)^3 dx \\ =\ & \frac{1}{2}\int \underbrace{(e^{2x}+1)^3}_{u^3}\underbrace{2e^{2x}dx}_{du} \\ =\ & \frac{1}{2}\left(\frac{1}{4}\right)u^4 + C = \frac{1}{8}(e^{2x}+1)^4 + C\end{aligned}$$

37. $\displaystyle\int \frac{3}{x\ln x}dx$

<解> 代

$$u = \ln x,\ du = \frac{1}{x}dx$$

並根據積分規則，得

$$\begin{aligned}\int \frac{3}{x\ln x}dx &= 3\int \underbrace{\frac{1}{\ln x}}_{1/u}\underbrace{\frac{1}{x}dx}_{du} \\ &= 3\ln|u| + C = 3\ln|\ln x| + C\end{aligned}$$

39. $\displaystyle\int \frac{\sqrt{\ln x}}{x}dx$

<解> 代

$$u = \ln x,\ du = \frac{1}{x}dx$$

並根據積分規則, 得

$$\begin{aligned}\int \frac{\sqrt{\ln x}}{x}dx &= \int \underbrace{\sqrt{\ln x}}_{\sqrt{u}}\underbrace{\frac{1}{x}dx}_{du} \\ &= \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\ln x)^{3/2} + C\end{aligned}$$

42. $\displaystyle\int \left(xe^{-x^2} + \frac{2e^x}{e^x+3}\right)dx$

<解> 逐項積分, 並將

$$u = -x^2,\ du = -2xdx$$

代入第一項且調整係數, 得

$$\begin{aligned}\int xe^{-x^2}dx &= -\frac{1}{2}\int \underbrace{e^{-x^2}}_{e^u}\underbrace{(-2x)dx}_{du} \\ &= -\frac{1}{2}e^{-x^2} + C\end{aligned}$$

再將

$$v = e^x + 3,\ dv = e^x dx$$

代入第二項, 得

$$\begin{aligned}\int \frac{2e^x}{e^x+3}dx &= 2\int \underbrace{\frac{1}{e^x+3}}_{1/v}\underbrace{e^x dx}_{dv} \\ &= 2\ln|e^x+3|+C\end{aligned}$$

最後, 合併, 得

$$\begin{aligned}&\int \left(xe^{-x^2}+\frac{2e^x}{e^x+3}\right)dx \\ = &-\frac{1}{2}e^{-x^2}+2\ln|e^x+3|+C\end{aligned}$$

44. $2\int \dfrac{e^{-u}-1}{e^{-u}+u}du$

<解> 代

$$v=e^{-u}+u,\ dv=(-e^{-u}+1)du$$

並調整係數, 得

$$\begin{aligned}&2\int \frac{e^{-u}-1}{e^{-u}+u}du \\ = &\ 2(-1)\int \underbrace{\frac{1}{e^{-u}+u}}_{1/v}\underbrace{(1-e^{-u})du}_{dv} \\ = &-2\ln|e^{-u}+u|+C\end{aligned}$$

45. $\int x(x-2)^5 dx$

<解> 代

$$u = x - 2,\ du = dx,\ x = u + 2$$

並展開後, 逐項積分, 得

$$\begin{aligned}\int x(x-2)^5 dx &= \int (u+2)u^5 du \\ &= \int (u^6 + 2u^5) du = \frac{1}{7}u^7 + \frac{1}{3}u^6 \\ &= \frac{1}{7}(x-2)^7 + \frac{1}{3}(x-2)^6 + C\end{aligned}$$

46. $\int \frac{2t}{t+1} dt$

<解> 整理分子, 或用長除法, 再逐項積分並根據註解的線性轉換代入法, 得

$$\begin{aligned}\int \frac{2t}{t+1} dt &= \int \frac{2(t+1)-2}{t+1} dt \\ &= \int \left(2 - \frac{2}{t+1}\right) dt \\ &= 2t - 2\ln|t+1| + C\end{aligned}$$

47. $\int \frac{1-\sqrt{x}}{1+\sqrt{x}} dx$

<解> 代

$$u = 1+\sqrt{x},\ du = \frac{1}{2\sqrt{x}}dx$$

以及

$$\sqrt{x} = u-1,\ dx = 2\sqrt{x}du = 2(u-1)du$$

並整理後, 逐項積分, 得

$$\begin{aligned}
\int \frac{1-\sqrt{x}}{1+\sqrt{x}}dx &= \int \frac{1-(u-1)}{u}2(u-1)du \\
&= 2\int \frac{(2-u)(u-1)}{u}du \\
&= 2\int \frac{-u^2+3u-2}{u}du \\
&= 2\int \left(-u+3-\frac{2}{u}\right)du \\
&= 2\left(-\frac{1}{2}u^2+3u-2\ln|u|\right)+C \\
&= -(1+\sqrt{x})^2+6(1+\sqrt{x}) \\
&\qquad -4\ln(1+\sqrt{x})+C
\end{aligned}$$

50. $\int x^3(x^2+1)^{3/2}dx$

<解> 代

$$u = x^2+1,\ du = 2xdx,\ x^2 = u-1$$

並調整係數, 展開後逐項積分, 得

$$
\begin{aligned}
&\int x^3(x^2+1)^{3/2}dx \\
&= \frac{1}{2}\int x^2(x^2+1)^{3/2}(2x)dx \\
&= \frac{1}{2}\int(u-1)u^{3/2}du = \frac{1}{2}\int(u^{5/2}-u^{3/2})du \\
&= \frac{1}{2}\left(\frac{2}{7}u^{7/2}-\frac{2}{5}u^{5/2}\right)+C \\
&= \frac{1}{7}(x^2+1)^{7/2}-\frac{1}{5}(x^2+1)^{5/2}+C
\end{aligned}
$$