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A Hybrid Neural-Network and MAC Scheme for Stokes Interface Problems

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Abstract. In this paper, we present a hybrid neural-network and MAC (Marker-And-Cell) scheme for solving Stokes equations with singular forces on an embedded interface in regular domains. As known, the solution variables (the pressure and velocity) exhibit non-smooth behaviors across the interface so extra discretization efforts must be paid near the interface in order to have small order of local truncation errors in finite difference schemes. The present hybrid approach avoids such additional difficulty. It combines the expressive power of neural networks with the convergence of finite difference schemes to ease the code implementation and to achieve good accuracy at the same time. The key idea is to decompose the solution into singular and regular parts. The neural network learning machinery incorporating the given jump conditions finds the singular part solution, while the standard MAC scheme is used to obtain the regular part solution with associated boundary conditions. The two- and three-dimensional numerical results show that the present hybrid method converges with second-order accuracy for the velocity and first-order accuracy for the pressure, and it is comparable with the traditional immersed interface method in literature.

AMS subject classifications: 35J25, 65N06, 68T07

Key words: Stokes interface problems, neural networks, MAC scheme, hybrid method.

1. Introduction

In this paper, we consider d-dimensional ($d = 2$ or 3) Stokes equations in a regular domain $\Omega \subseteq \mathbb{R}^d$, in which an embedded interface Γ with codimension $d-1$ (assumed to be smooth and closed) separates the domain into Ω^- and Ω^+. Denoting the interface

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position by $\mathbf{X}$, an interfacial force $\mathbf{F}(\mathbf{X})$ defined only along the interface Γ is exerted to the surrounding fluid and affects the flow behavior accordingly. Assuming that the viscosity of both fluid subdomains are identical, one can write a single Stokes fluid system with an external body force $\mathbf{g}$, viz.

$$\begin{aligned} &-\nabla p(\mathbf{x})+\mu\Delta\mathbf{u}(\mathbf{x})+\int_\Gamma \mathbf{F}(\mathbf{X})\delta^d(\mathbf{x}-\mathbf{X})\,\mathrm{d}\mathbf{X}+\mathbf{g}(\mathbf{x})=\mathbf{0}, && \mathbf{x}\in\Omega,\\ &\nabla\cdot\mathbf{u}(\mathbf{x})=0, && \mathbf{x}\in\Omega,\\ &\mathbf{u}(\mathbf{x})=\mathbf{u}_b(\mathbf{x}), && \mathbf{x}\in\partial\Omega, \end{aligned} \tag{1.1}$$

where $\mathbf{u}(\mathbf{x})$ and $p(\mathbf{x})$ are the velocity and the pressure, respectively, μ is the constant viscosity, and $\mathbf{u}_b(\mathbf{x})$ is the velocity boundary condition. Notice that, the force term appeared in the first equation is singular and expressed in the Immersed Boundary (IB) formulation [17] in which the integral involves a d-dimensional Dirac delta function δ^d over a $(d-1)$-dimensional surface resulting in one-dimensional delta function singularity. The above system (1.1) can be solved efficiently by the IB method [17, 21]. That is, the integral involving the delta function δ^d (line integral for $d=2$ and surface integral for $d=3$) can be regularized via a discrete delta function (a regularized form of the Dirac delta function) so the interfacial force $\mathbf{F}$ can be spread into the fluid grid points near the interface. However, this singular force spreading process results in first-order accuracy for the velocity [16] and has $\mathcal{O}(1)$ error for the pressure [2].

Due to the delta function singularity in Eq. (1.1), the pressure and velocity are no longer smooth across the interface so that the problem can be reformulated as the immersed interface formulation [13]

$$-\nabla p(\mathbf{x})+\mu\Delta\mathbf{u}(\mathbf{x})+\mathbf{g}(\mathbf{x})=\mathbf{0},\quad \mathbf{x}\in\Omega^-\cup\Omega^+, \tag{1.2}$$

$$\nabla\cdot\mathbf{u}(\mathbf{x})=0,\quad \mathbf{x}\in\Omega^-\cup\Omega^+, \tag{1.3}$$

$$\mathbf{u}(\mathbf{x})=\mathbf{u}_b(\mathbf{x}),\quad \mathbf{x}\in\partial\Omega, \tag{1.4}$$

where the pressure and velocity fields are subjected to the following jump conditions (see the derivation in [11]):

$$[\![p(\mathbf{X})]\!]=F_n(\mathbf{X}),\quad \mathbf{X}\in\Gamma, \tag{1.5}$$

$$[\![\mathbf{u}(\mathbf{X})]\!]=\mathbf{0},\quad \mu\left[\!\!\left[\frac{\partial\mathbf{u}}{\partial\mathbf{n}}(\mathbf{X})\right]\!\!\right]=-\big(\mathbf{F}(\mathbf{X})-F_n(\mathbf{X})\mathbf{n}(\mathbf{X})\big),\quad \mathbf{X}\in\Gamma. \tag{1.6}$$

Here, $F_n(\mathbf{X})=\mathbf{F}(\mathbf{X})\cdot\mathbf{n}(\mathbf{X})$ denotes the normal component of the interfacial force with $\mathbf{n}(\mathbf{X})$ being the unit outward normal vector at $\mathbf{X}\in\Gamma$. We use the double bracket $[\![\cdot]\!]$ to denote the jump of a quantity evaluated by the quantity from the Ω^+ side minus the one from the Ω^- side. From the jump condition (1.5), one can see that the pressure is discontinuous across Γ when $F_n\neq 0$. Also, from the jump condition (1.6), the velocity is continuous across the interface while its normal derivative is discontinuous and determined by the tangential part of $\mathbf{F}$. As a result, the velocity has a cusp behavior across the interface Γ.

As mentioned above, the pressure and velocity exhibit non-smooth behaviors across the interface, so extra discretization efforts must be paid near the interface in order to