影像處理專題 I

MA3113-*

Wavelet Transform and its Applications

§1 Short-time Fourier Transform and Uncertainty Principle

- Drawback of Fourier transform/series:

Classical Fourier analysis is quite inadequate for most applications.

1. The Fourier analysis assumes that signals are infinite in time or periodic, while many signals in practice are of short duration, and change substantially over their duration.

2. The formula
$$\widehat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}\,dy$$
does **not** even reflect frequencies that evolve with time.

What is really needed is for one to be able to **determine the time intervals that yield the spectral information on any desirable range of frequencies (or frequency band)**.

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§1 Short-time Fourier Transform and Uncertainty Principle

The short-time Fourier transform (STFT): for a given $f \in L^2(\mathbb{R})$, we consider the following integral

$$\mathcal{G}[f](t,\eta) = \int_{\mathbb{R}} f(\tau) e^{-i\eta\tau} \overline{w(\tau - t)}\, d\tau$$

for some $w \in L^2(\mathbb{R})$ (with compact support or decaying rapidly - the explicit condition will be given later), called the **window function**, for "extracting" local information from $\widehat{f}$.

The "optimal" window for time-localization is achieved by using any Gaussian function

$$w(t) = \frac{1}{\sqrt{4\pi\sigma^2}} \exp\Big(-\frac{t^2}{4\sigma^2}\Big),$$

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For a good choice of $\sigma > 0$, one can obtain

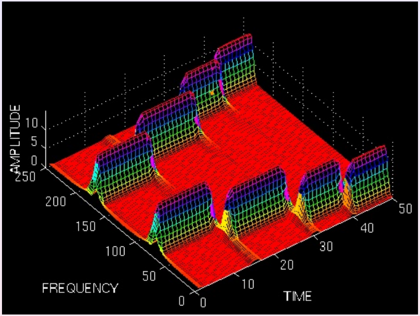

Figure 1: Figure from "the wavelet tutorial"

for the signal

$$f(t) \text{“}=\text{''} \sin(75t)\mathbf{1}_{[0,15)}(t) + \sin(50t)\mathbf{1}_{[15,30)}(t) + \sin(25t)\mathbf{1}_{[30,40)}(t) + \sin(10t)\mathbf{1}_{[40,50]}(t) .$$

§1 Short-time Fourier Transform and Uncertainty Principle

- **Uncertainty principle**

For any window function $w \in L^2(\mathbb{R})$, we define the center t^* and radius of Δ_w by

$$t^* = \frac{1}{\|w\|_2^2} \int_{\mathbb{R}} t|w(t)|^2\, dt \tag{1}$$

and

$$\Delta_w = \frac{1}{\|w\|_2} \Big(\int_{\mathbb{R}} (t - t^*)^2 |w(t)|^2\, dt \Big)^{\frac{1}{2}} . \tag{2}$$

Then using window function w, the STFT of f

$$\mathcal{G}[f](t,\eta) = \int_{\mathbb{R}} f(\tau) e^{-i\eta\tau} \overline{w(\tau - t)}\, d\tau$$

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$$\big[t^* + t - \Delta_w, t^* + t + \Delta_w \big] .$$

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§1 Short-time Fourier Transform and Uncertainty Principle

Suppose that the Fourier transform $\widehat{w}$ of w is also a window function, we can determine the center ω^* and radius $\Delta_{\widehat{w}}$, of the window function $\widehat{w}$, by using formulas analogous to (1) and (2); that is,

$$\omega^* = \frac{1}{\|\widehat{w}\|_2^2} \int_{\mathbb{R}} \omega |\widehat{w}(\omega)|^2 \, d\omega \,,$$
$$\Delta_{\widehat{w}} = \frac{1}{\|\widehat{w}\|_2^2} \Big(\int_{\mathbb{R}} (\omega - \omega^*)^2 |\widehat{w}(\omega)|^2 \, d\omega \Big)^{\frac{1}{2}} .$$

One can "show" that (something similar to the STFT of $\widehat{f}$ with window function $\widehat{w}/(2\pi)$)

$$\int_{\mathbb{R}} \widehat{f}(\omega) e^{it\omega} \frac{e^{-it\eta}}{2\pi} \overline{\widehat{w}(\omega - \eta)} \, d\omega$$

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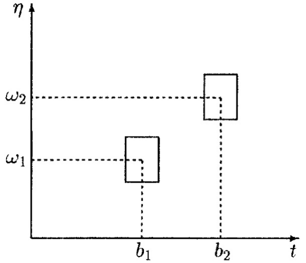
$$\big[\omega^*+\eta-\Delta_{\widehat{w}},\omega^*+\eta+\Delta_{\widehat{w}}\big].$$

§1 Short-time Fourier Transform and Uncertainty Principle

In summary, by choosing any $w \in L^2(\mathbb{R})$ such that both w and $\widehat{w}$ are both window functions to define the window Fourier transform we have a time-frequency window

$$\left[t^* + t - \Delta_w, t^* + t + \Delta_w\right] \times \left[\omega^* + \eta - \Delta_{\widehat{w}}, \omega^* + \eta + \Delta_{\widehat{w}}\right]$$

with width $2\Delta_w$ and height $2\Delta_{\widehat{w}}$ (as determined by the width of the time-window) and constant window area $4\Delta_w\Delta_{\widehat{w}}$.



§1 Short-time Fourier Transform and Uncertainty Principle

The **uncertainty principle** for STFT is given by the following

Theorem

Let $w \in L^2(\mathbb{R})$ be chosen such that both w and its Fourier transform $\widehat{w}$ are "qualified window functions". Then

$$\Delta_w \Delta_{\widehat{w}} \geqslant \frac{1}{2}.$$

Furthermore, equality is attained if and only if

$$w(t) = \frac{ce^{iat}}{4\sqrt{\pi\sigma^2}} \exp\Big(-\frac{(t-b)^2}{4\sigma^2}\Big),$$

where $c \neq 0$, and $\sigma, a, b \in \mathbb{R}$.

Remark: A qualified window function $w \in L^2(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} t^2 |w(t)|^2 \, dt < \infty.$$

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It follows that for high-frequency spectral information, the time-interval should be relatively small to give better accuracy, and for low-frequency spectral information, the time-interval should be relatively wide to give complete information. In other words, it is important to have a flexible time-frequency window that **automatically** narrows at high "center-frequency" and widens at low center-frequency.

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§2 Continuous Wavelet Transform

Definition (Wavelet)

A wavelet is a function $\psi \in L^2(\mathbb{R})$ which satisfies the admissibility condition

$$C_\psi \equiv \int_{\mathbb{R}} \frac{\left|\widehat{\psi}(\omega)\right|^2}{|\omega|}\, d\omega < \infty\,,$$

where $\widehat{\psi}$ is the Fourier transform of ψ.

Definition (Continuous Wavelet Transform)

If $\psi \in L^2(\mathbb{R})$, and $\psi_{a,b}$ is given by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\Big(\frac{t-b}{a}\Big)\,, \tag{3}$$

then the integral transformation W_ψ defined on $L^2(\mathbb{R})$ by

$$W_\psi[f](a,b) = \langle f, \psi_{a,b}\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(t)\overline{\psi_{a,b}(t)}\, dt$$

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§2 The Wavelet Series and Multi-Resolution Analysis (MRA)

Example (The Haar Wavelet)

The Haar wavelet (Haar 1910) is one of the classic examples. It is defined by

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \leqslant t < 1/2, \\ -1 & \text{if } 1/2 \leqslant t < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

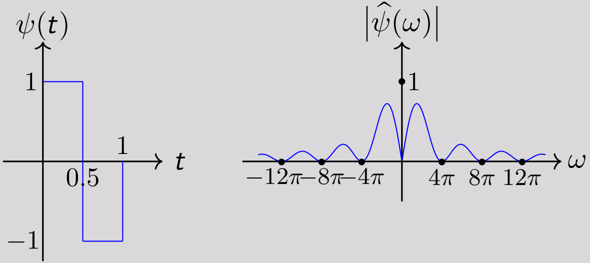

Figure 2: The Haar wavelet and its Fourier transform

§2 Continuous Wavelet Transform

Example (The Mexican Hat Wavelet)

The Mexican hat wavelet is defined by the second derivative of a Gaussian function as

$$\psi(t) = (1-t^2)\exp\left(-\frac{t^2}{2}\right) = -\frac{d^2}{dt^2}\exp\left(-\frac{t^2}{2}\right) = \psi_{1,0}(t),$$

$$\widehat{\psi}(\omega) = \widehat{\psi_{1,0}}(\omega) = \sqrt{2\pi}\,\omega^2\exp\left(-\frac{\omega^2}{2}\right).$$

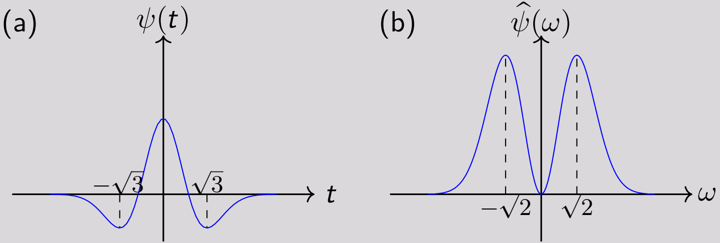

Figure 3: (a) The Mexican hat wavelet $\psi_{0,0}$. (b) its Fourier transform $\widehat{\psi_{1,0}}$

§2 Continuous Wavelet Transform

Example (The Mexican Hat Wavelet - cont'd)

Two other wavelets, $\psi_{\frac{3}{2},-2}$ and $\psi_{\frac{1}{4},\sqrt{2}}$, from the mother wavelet $\psi = \psi_{0,0}$ can be obtained. These three wavelets, $\psi_{0,0}$, $\psi_{\frac{3}{2},-2}$, and $\psi_{\frac{1}{4},\sqrt{2}}$, are shown in Figure 4(i), (ii), and (iii), respectively.

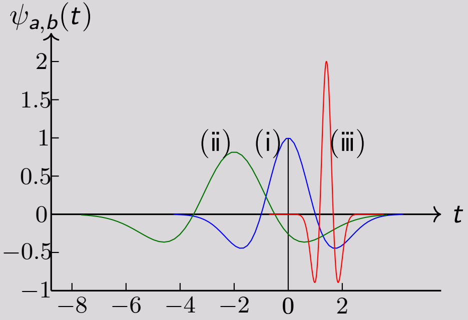

Figure 4: Three wavelets $\psi_{0,0}$, $\psi_{\frac{3}{2},-2}$, $\psi_{\frac{1}{4},\sqrt{2}}$

§2 Continuous Wavelet Transform

Theorem (Inversion Formula)

If $f \in L^2(\mathbb{R})$, then f can be reconstructed by the formula

$$f(t) = \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}} W_\psi[f](a,b)\psi_{a,b}(t)\, \frac{dbda}{a^2}\,, \tag{5}$$

where the equality holds almost everywhere.

§3 Discrete Wavelet Transform and Orthonormal Wavelets

Similar to the discrete Fourier transform (motivated from the Fourier transform/series), we would like to answer the fundamental question whether we can reconstruct f from **discrete values** of its wavelet transform $W_\psi[f]$. In particular, we would like to reconstruct f using the discrete values of $W_\psi[f]$ at $a = a_0^m$ and $b = nb_0 a_0^m$; that is,

$$(W_\psi[f])(a_0^m, nb_0 a_0^m) = a_0^{-\frac{m}{2}} \int_{\mathbb{R}} f(t)\overline{\psi}(a_0^{-m} t - nb_0)\, dt,$$

where $a_0 \neq 0$, b_0 are some given and fixed constants, and m, n are integers (later we will set $a_0 = 1/2$ and $b_0 = 1$).

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§3 Discrete Wavelet Transform and Orthonormal Wavelets

Define

$$\psi_{m,n}(x) = a_0^{-m/2}\psi(a_0^{-m}x - nb_0)\,, \tag{6}$$

where we **abuse** the use of notation here and do not confuse with

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\Big(\frac{t-b}{a}\Big) \tag{3}$$

which is used to define the continuous wavelet transform. Using (6), we have

$$(W_\psi[f])(a_0^m, nb_0a_0^m) = \langle f, \psi_{m,n}\rangle_{L^2(\mathbb{R})}\,.$$

The discrete wavelet transform represents a function by a countable set of wavelet coefficients, which correspond to points on a two dimensional grid or lattice of discrete points in the scale-time domain indexed by m and n.

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Our goal is to find a function $\psi \in L^2(\mathbb{R})$ such that the family $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ that forms a "basis" of $L^2(\mathbb{R})$. In particular, we want the family of functions $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ is obtained by

$$\psi_{m,n}(x) = 2^{m/2}\psi(2^m x - n) \qquad \forall\, m,n \in \mathbb{Z}, \tag{7}$$

for some square-integrable function $\psi\,(\equiv \psi_{0,0})$ with compact support, and an orthonormal "basis" $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ is preferable.

Suppose we have such an orthonormal "basis" $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ of $L^2(\mathbb{R})$. Then evey $f \in L^2(\mathbb{R})$ admits the expression

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, \psi_{m,n}\rangle_{L^2(\mathbb{R})}\, \psi_{m,n}$$

and the magnitude of $c_{m,n}$ gives us more information about f.

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A function $\psi \in L^2(\mathbb{R})$ is called ***an orthogonal/orthonormal wavelet*** (or o.n. wavelet), if the family $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$, as defined by (7), is an orthonormal basis of $L^2(\mathbb{R})$; that is,

$$\langle \psi_{m,n}, \psi_{k,\ell} \rangle_{L^2(\mathbb{R})} = \delta_{mk}\delta_{n\ell} \qquad \forall\, m,n,k,\ell \in \mathbb{Z}$$

and every function $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \sum_{m,n=-\infty}^{\infty} c_{m,n}\psi_{m,n}(x),$$

where $c_{m,n} = \langle f, \psi_{m,n} \rangle_{L^2(\mathbb{R})}$ and the convergence of the series above is in $L^2(\mathbb{R})$; that is,

$$\lim_{M_1,N_1,M_2,N_2\to\infty} \Big\| f - \sum_{n=-M_2}^{N_2} \sum_{m=-M_1}^{N_1} c_{m,n}\psi_{m,n} \Big\|_{L^2(\mathbb{R})} = 0.$$

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§3 Discrete Wavelet Transform and Orthonormal Wavelets

Example

The simplest example of an orthonormal wavelet is the classic Haar wavelet given by

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \leqslant t < 1/2, \\ -1 & \text{if } 1/2 \leqslant t < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

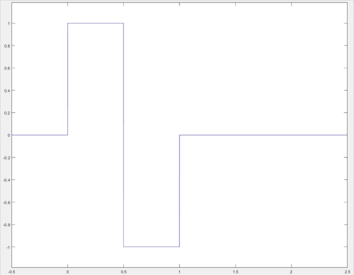
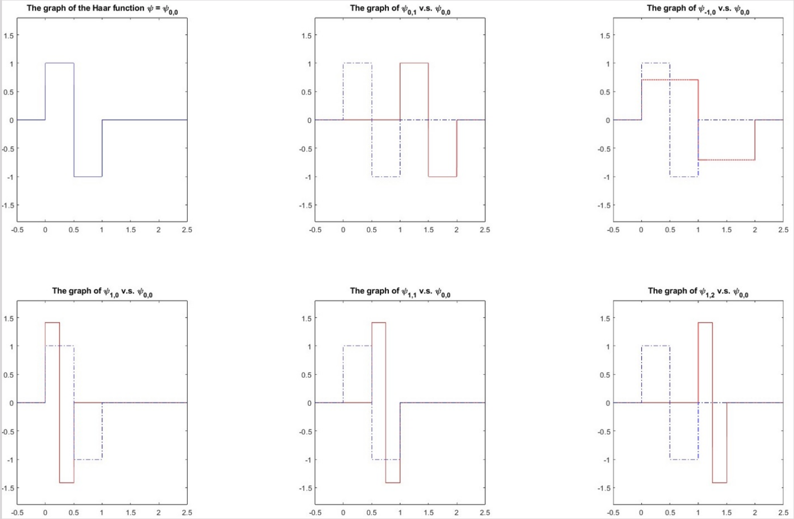


Figure 5: The graph of the Haar function

§3 Discrete Wavelet Transform and Orthonormal Wavelets

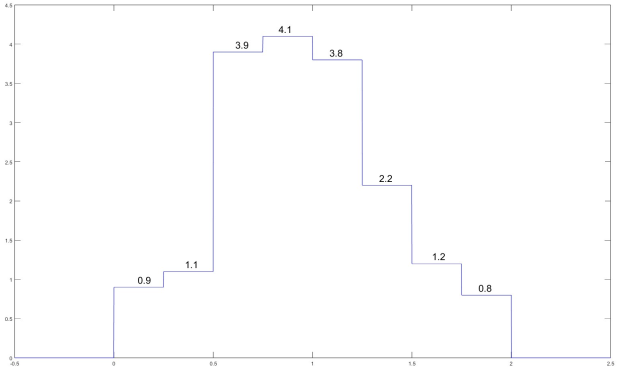
Example (cont'd)



§3 Discrete Wavelet Transform and Orthonormal Wavelets

Example (Expressing a function using wavelet series)

Consider the function f (given by its graph)



and we compute the wavelet series of f using the Haar wavelet.

§3 Discrete Wavelet Transform and Orthonormal Wavelets

Example (Expressing a function using wavelet series - cont'd)

1. $m \leqslant -2$: $\langle f, \psi_{m,0}\rangle_{L^2(\mathbb{R})} = 9 \cdot 2^{m/2-1}$, and $\langle f, \psi_{m,n}\rangle_{L^2(\mathbb{R})} = 0$ for all $n \neq 0$.
2. $m = -1$: $\langle f, \psi_{m,0}\rangle_{L^2(\mathbb{R})} = 2^{m/2-1}$, and $\langle f, \psi_{m,n}\rangle_{L^2(\mathbb{R})} = 0$ for all $n \neq 0$.
3. $m = 0$: $\langle f, \psi_{m,n}\rangle_{L^2(\mathbb{R})} = 0$ if $n \geqslant 2$ or $n < 0$, and
$$\langle f, \psi_{m,0}\rangle_{L^2(\mathbb{R})} = -\frac{3}{2}, \qquad \langle f, \psi_{m,1}\rangle_{L^2(\mathbb{R})} = 1.$$
4. $m = 1$: $\langle f, \psi_{m,n}\rangle_{L^2(\mathbb{R})} = 0$ if $n \geqslant 4$ or $n < 0$, and
$$\langle f, \psi_{m,0}\rangle_{L^2(\mathbb{R})} = -\frac{\sqrt{2}}{20}, \qquad \langle f, \psi_{m,1}\rangle_{L^2(\mathbb{R})} = -\frac{\sqrt{2}}{20},$$
$$\langle f, \psi_{m,2}\rangle_{L^2(\mathbb{R})} = \frac{2\sqrt{2}}{5}, \qquad \langle f, \psi_{m,3}\rangle_{L^2(\mathbb{R})} = \frac{\sqrt{2}}{10}.$$
5. $m \geqslant 2$: $\langle f, \psi_{m,n}\rangle_{L^2(\mathbb{R})} = 0$ for all $n \in \mathbb{Z}$.

§3 Discrete Wavelet Transform and Orthonormal Wavelets

Example (Expressing a function using wavelet series - cont'd)

Therefore, the sum $d_m \equiv \sum_{n=-\infty}^{\infty} \langle f, \psi_{m,n} \rangle_{L^2(\mathbb{R})} \psi_{m,n}$ (not summing over m yet) is

1. $m \leqslant -2$: $d_m = 9 \cdot 2^{m-1} \Big[\mathbf{1}_{[0, 2^{-m-1})} - \mathbf{1}_{[2^{-m-1}, 2^{-m})} \Big]$.
2. $m = -1$: $d_m = \frac{1}{4} \Big[\mathbf{1}_{[0,1)} - \mathbf{1}_{[1,2)} \Big]$.
3. $m = 0$: $d_m = -\frac{3}{2} \Big[\mathbf{1}_{[0,\frac{1}{2})} - \mathbf{1}_{[\frac{1}{2},1)} \Big] + \Big[\mathbf{1}_{[1,\frac{3}{2})} - \mathbf{1}_{[\frac{3}{2},2)} \Big]$.
4. $m = 1$:
$$d_m = -\frac{1}{10} \Big[\mathbf{1}_{[0,\frac{1}{4})} - \mathbf{1}_{[\frac{1}{4},\frac{1}{2})} \Big] - \frac{1}{10} \Big[\mathbf{1}_{[\frac{1}{2},\frac{3}{4})} - \mathbf{1}_{[\frac{3}{4},1)} \Big] + \frac{4}{5} \Big[\mathbf{1}_{[1,\frac{5}{4})} - \mathbf{1}_{[\frac{5}{4},\frac{3}{2})} \Big] + \frac{1}{5} \Big[\mathbf{1}_{[\frac{3}{2},\frac{7}{4})} - \mathbf{1}_{[\frac{7}{4},2)} \Big].$$
5. For $m \geqslant 2$, $d_m = 0$.

§3 Discrete Wavelet Transform and Orthonormal Wavelets

Example (Expressing a function using wavelet series - cont'd)

Define $f_k \equiv \sum_{m=-\infty}^{k} d_m = \sum_{m=-\infty}^{k} \sum_{n=-\infty}^{\infty} \langle f, \psi_{m,n} \rangle_{L^2(\mathbb{R})} \psi_{m,n}$. Then

1. $k \leqslant -2$:
$$f_k = \sum_{m=-\infty}^{k} 9 \cdot 2^{m-1} \Big[\mathbf{1}_{[0,2^{-m-1})} - \mathbf{1}_{[2^{-m-1},2^{-m})} \Big] = 9 \cdot 2^k \mathbf{1}_{[0,2)} .$$

2. $k = -1$:
$$f_k = f_{k-1} + d_k = \frac{9}{4}\mathbf{1}_{[0,2)} + \frac{1}{4}\Big[\mathbf{1}_{[0,1)} - \mathbf{1}_{[1,2)} \Big] = \frac{5}{2}\mathbf{1}_{[0,1)} + 2\mathbf{1}_{[1,2)} .$$

3. $k = 0$:
$$\begin{aligned} f_k &= f_{k-1} + d_k \\ &= \frac{5}{2}\mathbf{1}_{[0,1)} + 2\mathbf{1}_{[1,2)} - \frac{3}{2}\Big[\mathbf{1}_{[0,\frac{1}{2})} - \mathbf{1}_{[\frac{1}{2},1)} \Big] + \Big[\mathbf{1}_{[1,\frac{3}{2})} - \mathbf{1}_{[\frac{3}{2},2)} \Big] \\ &= \mathbf{1}_{[0,\frac{1}{2})} + 4\mathbf{1}_{[\frac{1}{2},1)} + 3\mathbf{1}_{[1,\frac{3}{2})} + \mathbf{1}_{[\frac{3}{2},2)} . \end{aligned}$$

§3 Discrete Wavelet Transform and Orthonormal Wavelets

Example (Expressing a function using wavelet series - cont'd)

5. $k = 1$:

$$\begin{aligned} f_k &= f_{k-1} + d_k \\ &= \mathbf{1}_{[0,\frac{1}{2})} + 4 \cdot \mathbf{1}_{[\frac{1}{2},1)} + 3 \cdot \mathbf{1}_{[1,\frac{3}{2})} + \mathbf{1}_{[\frac{3}{2},2)} \\ &\quad - \frac{1}{10}\Big[\mathbf{1}_{[0,\frac{1}{4})} - \mathbf{1}_{[\frac{1}{4},\frac{1}{2})}\Big] - \frac{1}{10}\Big[\mathbf{1}_{[\frac{1}{2},\frac{3}{4})} - \mathbf{1}_{[\frac{3}{4},1)}\Big] \\ &\quad + \frac{4}{5}\Big[\mathbf{1}_{[1,\frac{5}{4})} - \mathbf{1}_{[\frac{5}{4},\frac{3}{2})}\Big] + \frac{1}{5}\Big[\mathbf{1}_{[\frac{3}{2},\frac{7}{4})} - \mathbf{1}_{[\frac{7}{4},2)}\Big] \\ &= 0.9 \cdot \mathbf{1}_{[0,\frac{1}{4})} + 1.1 \cdot \mathbf{1}_{[\frac{1}{4},\frac{1}{2})} + 3.9 \cdot \mathbf{1}_{[\frac{1}{2},\frac{3}{4})} + 4.1 \cdot \mathbf{1}_{[\frac{3}{4},1)} \\ &\quad + 3.8 \cdot \mathbf{1}_{[1,\frac{5}{4})} + 2.2 \cdot \mathbf{1}_{[\frac{5}{4},\frac{3}{2})} + 1.2 \cdot \mathbf{1}_{[\frac{3}{2},\frac{7}{4})} + 0.8 \cdot \mathbf{1}_{[\frac{7}{4},2)} \\ &= f. \end{aligned}$$

6. $k \geqslant 2$: since $d_k = 0$ for all $k \geqslant 2$, $f_k = f$ for all $k \geqslant 2$.

The process of making the sum $f_{k-1} + d_k$ is to add detail information d_k to the coarse information f_{k-1} to obtain a finer f_k.

§3 Discrete Wavelet Transform and Orthonormal Wavelets

Example (Expressing a function using wavelet series - cont'd)

5. $k = 1$:

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§3 Discrete Wavelet Transform and Orthonormal Wavelets

- **Matlab commands dwt**:

The level 1 discrete wavelet transform of $\boldsymbol{x} \equiv \left[x_0, x_1, \cdots, x_{2^n-1}\right]$, using Haar wavelet, can be obtained by the matlab command

$$[\mathrm{c},\mathrm{d}] = \mathrm{dwt}(\boldsymbol{x}, \text{'haar'});$$

where the output c is the coarse vector

$$\left[\frac{x_0 + x_1}{\sqrt{2}}, \frac{x_2 + x_3}{\sqrt{2}}, \cdots, \frac{x_{2^n-2} + x_{2^n-1}}{\sqrt{2}}\right]$$

and the output d is the detail vector

$$\left[\frac{x_0 - x_1}{\sqrt{2}}, \frac{x_2 - x_3}{\sqrt{2}}, \cdots, \frac{x_{2^n-2} - x_{2^n-1}}{\sqrt{2}}\right].$$

In general, the matlab command

$$[\mathrm{c},\mathrm{d}] = \mathrm{dwt}(\text{input\_vector}, \text{'name\_of\_wavelet'});$$

gives the coarse vector c and detail vector d after 1 level of “average” using the specific wavelet named name_of_wavelet.

§3 Discrete Wavelet Transform and Orthonormal Wavelets

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§3 Discrete Wavelet Transform and Orthonormal Wavelets

- **Matlab commands wavedec**:

The level 1 discrete wavelet transform of $\boldsymbol{x} \equiv \big[x_0, x_1, \cdots, x_{2^n-1}\big]$, using Haar wavelet, can also be obtained by the matlab command

$$[\mathrm{c,l}] = \mathrm{wavedec}(\boldsymbol{x}, 1, \text{‘haar’});$$

where c is the vector

$$\left[\frac{x_0+x_1}{\sqrt{2}}, \frac{x_2+x_3}{\sqrt{2}}, \cdots, \frac{x_{2^n-2}+x_{2^n-1}}{\sqrt{2}}, \frac{x_0-x_1}{\sqrt{2}}, \frac{x_2-x_3}{\sqrt{2}}, \cdots, \frac{x_{2^n-2}-x_{2^n-1}}{\sqrt{2}}\right]$$

In general, the matlab command

$$[\mathrm{c,l}] = \mathrm{wavedec}(\text{input\_vector}, \text{level}, \text{‘name\_of\_wavelet’});$$

gives

c: this output consists of the final coarse vector after level-times “average” and the detail information for each level. The structure of c is similar to $[f_{-2}, d_{-1}, d_0, d_1]$ if level $= 3$;

l: the length of (a) the final coarse vector, (b) the detail information at each level, and (c) the input.

§3 Discrete Wavelet Transform and Orthonormal Wavelets

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The level 1 discrete wavelet transform of $\boldsymbol{x} \equiv \left[x_0, x_1, \cdots, x_{2^n-1}\right]$, using Haar wavelet, can also be obtained by the matlab command

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§3 Discrete Wavelet Transform and Orthonormal Wavelets

- **Matlab commands for reconstruction**:

The inverse operation of dwt and wavedec are idwt and waverec, respectively, given below:

$$\begin{aligned}[c,d] &= \text{dwt}(x,\text{`name\_of\_wavelet'});\\ &\Leftrightarrow \quad x = \text{idwt}(c,d,\text{'name\_of\_wavelet'});\end{aligned}$$

and

$$\begin{aligned}[c,l] &= \text{wavedec}(x,\text{level},\text{`name\_of\_wavelet'});\\ &\Leftrightarrow \quad x = \text{waverec}(c,l,\text{'name\_of\_wavelet'});\end{aligned}$$

§4 Multi-Resolution Analysis (MRA)

Before proceeding, let us introduce the translation T and the dilation operator d, D that are often used (to simplify the notation) in the study of the wavelet theory: for $f:\mathbb{R}^n \to \mathbb{R}$, $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}\backslash\{0\}$,

$$(T_y f)(x) = f(x-y), \quad (d_c f)(x) = f\Big(\frac{x}{c}\Big), \quad (D_c f)(x) = \frac{1}{\sqrt{|c|}} f\Big(\frac{x}{c}\Big).$$

Therefore,

$$\psi_{m,n}(x) \equiv 2^{m/2}\psi(2^m x - n) = (D_{2^{-m}} T_n \psi)(x).$$

Remark: For the dilation operator d_c (or D_c), the smaller the number c, the finer the "resolution".

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§4 Multi-Resolution Analysis (MRA)

Definition

An MRA consists of a sequence $\{V_m\}_{m\in\mathbb{Z}}$ of embedded closed subspaces of $L^2(\mathbb{R})$ that satisfy the following conditions:

1. $\cdots \subseteq V_{-2} \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$.
2. $\bigcup\limits_{m=-\infty}^{\infty} V_m$ is dense in $L^2(\mathbb{R})$ and $\bigcap\limits_{m=-\infty}^{\infty} V_m = \{0\}$.
3. $f \in V_m$ if and only if $d_{1/2} f \in V_{m+1}$ for all $m \in \mathbb{Z}$;
4. there exists a function $\phi \in V_0$ such that $\{\phi_{0,n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for V_0; that is,

$$\|f\|^2_{L^2(\mathbb{R})} = \sum_{n=-\infty}^{\infty} \left|\langle f, \phi_{0,n}\rangle_{L^2(\mathbb{R})}\right|^2 \qquad \forall\, f \in V_0\,.$$

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§4 Multi-Resolution Analysis (MRA)

- **Consequences of the Definition of MRA**

(i) A repeated application of condition ③ implies that $f \in V_m$ if and only if $d_{2^m} f \in V_0$ for all $m \in \mathbb{Z}$. Moreover, the fact that $\{\phi_{0,n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of V_0 shows that $\{\phi_{m,n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for V_m.

(ii) Condition ② can be expressed in terms of the orthogonal projections P_m onto V_m; that is, for all $f \in L^2(\mathbb{R})$,

$$\lim_{m\to-\infty} \mathrm{P}_m f = 0 \qquad \text{and} \qquad \lim_{m\to\infty} \mathrm{P}_m f = f.$$

The projection $\mathrm{P}_m f$ can be considered as an approximation of f at the scale 2^{-m}. Indeed, we have

$$\mathrm{P}_m f = \sum_{n=-\infty}^{\infty} \langle f, \phi_{m,n} \rangle_{L^2(\mathbb{R})} \phi_{m,n},$$

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§4 Multi-Resolution Analysis (MRA)

- (iii) Since $V_0 \subseteq V_1$, the scaling function ϕ that leads to a basis for V_0 also belongs to V_1. Since $\phi \in V_1$ and $\{\phi_{1,n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for V_1, ϕ can be expressed in the form

$$\phi(x) = \sum_{n=-\infty}^{\infty} \langle \phi, \phi_{1,n} \rangle_{L^2(\mathbb{R})} \phi_{1,n}(x) .$$

The equation above is called the **dilation equation**, **two-scale equation** or **refinement equation**.

§4 Multi-Resolution Analysis (MRA)

The real importance of an MRA lies in the simple fact that it enables us to **construct an orthonormal wavelet** for $L^2(\mathbb{R})$. Suppose that $\{V_m\}_{m=-\infty}^{\infty}$ is an MRA. Define W_m as the orthogonal complement of V_m in V_{m+1} for every $m \in \mathbb{Z}$, so that we have for $m \in \mathbb{N}$,

$$\begin{aligned} V_{m+1} &= V_m \oplus W_m = \Big(V_{m-1} \oplus W_{m-1}\Big) \oplus W_m = \cdots \\ &= V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_m = V_0 \oplus \Big(\bigoplus_{k=0}^{m} W_k\Big) \end{aligned}$$

and $V_n \perp W_m$ for $n < m$.

Since $\bigcup_{m=-\infty}^{\infty} V_m$ is dense in $L^2(\mathbb{R})$, we may take the limit as $m \to \infty$ to obtain

$$V_0 \oplus \Big(\bigoplus_{m=0}^{\infty} W_m\Big) = L^2(\mathbb{R}).$$

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§4 Multi-Resolution Analysis (MRA)

Similarly, we may go in the other direction to write

$$\begin{aligned} V_0 &= V_{-1} \oplus W_{-1} = (V_{-2} \oplus W_{-2}) \oplus W_{-1} = \cdots \\ &= V_{-m} \oplus W_{-m} \oplus \cdots \oplus W_{-1} . \end{aligned}$$

Since $\bigcap_{m=-\infty}^{\infty} V_m = \{0\}$, it follows that $\lim_{m\to\infty} V_{-m} = \{0\}$ which further implies that

$$V_0 = \bigoplus_{m=-\infty}^{-1} W_m .$$

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§4 Multi-Resolution Analysis (MRA)

Finally, the difference between the two successive approximations $\mathrm{P}_m f$ and $\mathrm{P}_{m+1} f$ is given by the orthogonal projection $\mathrm{Q}_m f$ of f onto the orthogonal complement W_m of V_m in V_{m+1} so that

$$\mathrm{Q}_m f = \mathrm{P}_{m+1} f - \mathrm{P}_m f.$$

It follows from conditions ①-④ in the definition of MRA that the spaces W_m are also scaled versions of W_0 and, for $f \in L^2(\mathbb{R})$,

$$f \in W_m \quad \text{if and only if} \quad d_{2^m} f \in W_0 \qquad \forall\, m \in \mathbb{Z}$$

since

$$\begin{aligned} f \in W_m &\Leftrightarrow f \in V_{m+1} \text{ and } f \perp V_m \\ &\Leftrightarrow d_{2^m} f \in V_1 \text{ and } d_{2^m} f \perp V_0 \\ &\Leftrightarrow d_{2^m} f \in W_0 . \end{aligned}$$

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$$\mathrm{Q}_m f = \mathrm{P}_{m+1} f - \mathrm{P}_m f.$$

It follows from conditions ①-④ in the definition of MRA that the spaces W_m are also scaled versions of W_0 and, for $f \in L^2(\mathbb{R})$,

$$f \in W_m \quad \text{if and only if} \quad d_{2^m} f \in W_0 \qquad \forall\, m \in \mathbb{Z}$$

since

$$\begin{aligned} f \in W_m &\Leftrightarrow f \in V_{m+1} \text{ and } f \perp V_m \\ &\Leftrightarrow d_{2^m} f \in V_1 \text{ and } d_{2^m} f \perp V_0 \\ &\Leftrightarrow d_{2^m} f \in W_0 . \end{aligned}$$

§4 Multi-Resolution Analysis (MRA)

Moreover, W_m's are mutually orthogonal spaces generating all $L^2(\mathbb{R})$:

$$W_m \perp W_k \text{ if } m \neq k \quad \text{and} \quad \bigoplus_{m\in\mathbb{Z}} W_m = L^2(\mathbb{R}),$$

and are translation-invariant for the discrete translations $n \in \mathbb{Z}$:

$$f \in W_0 \quad \text{if and only if} \quad T_n f \in W_0,$$

where the translation-invariant is due to the following equivalence:

$$\begin{aligned} f \in W_0 &\Leftrightarrow f \in V_1 \text{ and } f \perp V_0 \\ &\Leftrightarrow f \overset{L^2}{=} \sum_{k=-\infty}^{\infty} c_k \phi_{1,k} \text{ for some } \{c_k\}_{k=-\infty}^{\infty} \in \ell^2 \text{ and } f \perp V_0 \\ &\Leftrightarrow T_n f \overset{L^2}{=} \sum_{k=-\infty}^{\infty} c_{k-2n} \phi_{1,k} \text{ for some } \{c_k\}_{k=-\infty}^{\infty} \in \ell^2 \\ &\qquad \text{and } T_n f \perp V_0 \\ &\Leftrightarrow T_n f \in V_1 \text{ and } T_n f \perp V_0 \Leftrightarrow T_n f \in W_0 . \end{aligned}$$

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&\Leftrightarrow T_n f \in V_1 \text{ and } T_n f \perp V_0 \Leftrightarrow T_n f \in W_0 .
\end{aligned}$$

§4 Multi-Resolution Analysis (MRA)

Therefore, if one can show that there exists a function $\psi \in W_0$ such that $\{\psi_{0,n}\}_{n\in\mathbb{Z}}$ constitutes an orthonormal basis for W_0, then it follows from the fact that

$$f \in W_m \quad \text{if and only if} \quad d_{2^m} f \in W_0 \qquad \forall\, m \in \mathbb{Z}$$

$\{\psi_{m,n}\}_{n\in\mathbb{Z}}$ constitute an orthonormal basis for W_m. Since

$$\bigoplus_{m=-\infty}^{\infty} W_m = L^2(\mathbb{R}),$$

we then also conclude that the family $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ represents an orthonormal basis of wavelets for $L^2(\mathbb{R})$.

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§5 Construction of Father and Mother Wavelets

Theorem

Let ϕ be a bounded function with compact support, $\widehat{\phi}(0) = 1$, and $\{\phi_{0,n}\}_{n\in\mathbb{Z}}$ is an orthonormal system in $L^2(\mathbb{R})$. If it holds the two-scale equation

$$\phi(x) = \sum_{n=-\infty}^{\infty} \langle \phi, \phi_{1,n} \rangle_{L^2(\mathbb{R})} \phi_{1,n}(x)\,,$$

then V_m defined $V_m = \mathrm{closure}_{\|\cdot\|_{L^2(\mathbb{R})}}\big(\mathrm{span}\big(\{\phi_{m,n}\}_{n\in\mathbb{Z}}\big)\big)$ forms an MRA $\{V_m\}_{m\in\mathbb{Z}}$.

§5 Construction of Father and Mother Wavelets

Theorem

If $\{V_n\}_{n\in\mathbb{Z}}$ is an MRA with the scaling function ϕ, then there is a mother wavelet ψ given by

$$\begin{aligned}\psi(x) &= \sqrt{2}\sum_{n=-\infty}^{\infty}(-1)^{-n-1}\overline{c_{-n-1}}\phi(2x-n)\\ &= \sum_{n=-\infty}^{\infty}(-1)^{-n-1}\overline{c_{-n-1}}\phi_{1,n}(x),\end{aligned}$$

where the coefficients c_n are given by

$$c_n = \langle \phi, \phi_{1,n}\rangle_{L^2(\mathbb{R})} = \sqrt{2}\int_{\mathbb{R}}\phi(x)\overline{\phi(2x-n)}\,dx.$$

§5 Construction of Father and Mother Wavelets

Remark: The study of the father wavelet ϕ is usually transformed to the study of its **generating function** $\widehat{m}$ defined by

$$\widehat{m}(\omega) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \langle \phi, \phi_{1,n} \rangle_{L^2(\mathbb{R})} e^{-in\omega}$$

which satisfies

$$\widehat{\phi}(\omega) = \widehat{m}\Big(\frac{\omega}{2}\Big)\widehat{\phi}\Big(\frac{\omega}{2}\Big).$$

The reason for the name "generating function" is that if $\phi \in L^1(\mathbb{R})$ and $\widehat{\phi}(0) = 1$, then

$$\begin{aligned}\widehat{\phi}(\omega) &= \widehat{m}\Big(\frac{\omega}{2}\Big)\widehat{\phi}\Big(\frac{\omega}{2}\Big) = \widehat{m}\Big(\frac{\omega}{2}\Big)\Big[\widehat{m}\Big(\frac{\omega}{4}\Big)\widehat{\phi}\Big(\frac{\omega}{4}\Big)\Big] = \cdots \\ &= \widehat{m}\Big(\frac{\omega}{2}\Big)\widehat{m}\Big(\frac{\omega}{4}\Big)\cdots\widehat{m}\Big(\frac{\omega}{2^k}\Big)\widehat{\phi}\Big(\frac{\omega}{2^k}\Big) \\ &= \Big[\prod_{\ell=1}^{k}\widehat{m}\Big(\frac{\omega}{2^\ell}\Big)\Big]\widehat{\phi}\Big(\frac{\omega}{2^k}\Big) \to \prod_{\ell=1}^{\infty}\widehat{m}\Big(\frac{\omega}{2^\ell}\Big) \quad \text{as } k \to \infty,\end{aligned}$$

so one can "generate" the father wavelet ϕ using $\widehat{m}$.

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&= \widehat{m}\Big(\frac{\omega}{2}\Big)\widehat{m}\Big(\frac{\omega}{4}\Big)\cdots\widehat{m}\Big(\frac{\omega}{2^k}\Big)\widehat{\phi}\Big(\frac{\omega}{2^k}\Big) \\
&= \Big[\prod_{\ell=1}^{k} \widehat{m}\Big(\frac{\omega}{2^\ell}\Big)\Big]\widehat{\phi}\Big(\frac{\omega}{2^k}\Big) \to \prod_{\ell=1}^{\infty} \widehat{m}\Big(\frac{\omega}{2^\ell}\Big) \quad \text{as } k \to \infty,
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so one can "generate" the father wavelet ϕ using $\widehat{m}$.

§5 Construction of Father and Mother Wavelets

It can be shown that if $\{c_n\}_{n\in\mathbb{Z}} \in \ell^2$ satisfying

$$\sum_{n=-\infty}^{\infty} c_n = \sqrt{2},$$

and certain decay conditions, and $\widehat{m}$ defined by

$$\widehat{m}(\omega) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} c_n e^{-in\omega}$$

satisfies

$$\left|\widehat{m}(\omega)\right|^2 + \left|\widehat{m}(\omega+\pi)\right|^2 = 1 \quad \forall\, \omega \in \mathbb{R},$$

then $\widehat{m}$ generates a father wavelet ϕ whose Fourier transform is given by

$$\widehat{\phi}(\omega) = \prod_{\ell=1}^{\infty} \widehat{m}\left(\frac{\omega}{2^\ell}\right).$$

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$$\widehat{\phi}(\omega) = \prod_{\ell=1}^{\infty} \widehat{m}\left(\frac{\omega}{2^\ell}\right).$$

§5 Construction of Father and Mother Wavelets

Theorem

Let ϑ be a 2π-periodic continuous function satisfying

$$\left|\vartheta(\omega)\right|^2 + \left|\vartheta(\omega+\pi)\right|^2 = 1 \qquad \forall\, \omega \in \mathbb{R},$$

and $\{d_n\}_{n\in\mathbb{Z}}$ be the "Fourier coefficients" of ϑ satisfying

$$\vartheta(\omega) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} d_n e^{-in\omega}.$$

If $\vartheta(\omega_0) = 0$ and $\{d_n\}_{n\in\mathbb{Z}}$ "decays fast enough", then $\widehat{m}$ defined by

$$\widehat{m}(\omega) = \frac{e^{i\alpha}}{\sqrt{2}} \sum_{n=-\infty}^{\infty} (-1)^n d_n e^{-in(\omega+\omega_0)},$$

where α is chosen so that $\widehat{m}(0) = 1$, is a generating function of a father wavelet ϕ; that is, the Fourier transform of ϕ is given by

$$\widehat{\phi}(\omega) = \prod_{\ell=1}^{\infty} \widehat{m}\left(\frac{\omega}{2^\ell}\right).$$

§5 Construction of Father and Mother Wavelets

Example (Daubechies' Wavelets)

The Daubechies' (father) wavelet ϕ is a "smooth" wavelet with compact support which is obtained by assuming that the function ϑ in the previous theorem takes the form

$$\vartheta(\omega) = \Big(\frac{1+e^{-i\omega}}{2}\Big)^N \widehat{L}(\omega)$$

for some $N \in \mathbb{N}$, where $\widehat{L}(\omega)$ is a polynomial of $\cos\omega$ with $\widehat{L}(0) = 1$ and $\widehat{L}(\pi) \neq 0$. We note that in this case only finitely many d_n's are non-zero; thus the decay condition is fulfilled automatically. Moreover, that only finitely many d_n's are non-zero implies that $\widehat{m}$ is smooth.

The number N is highly associated with the differentiability of ϕ: larger N implies higher order of differentiability. However, the case $N = 1$ and $L = 1$ reduces to the case of the Haar wavelet.

§5 Construction of Father and Mother Wavelets

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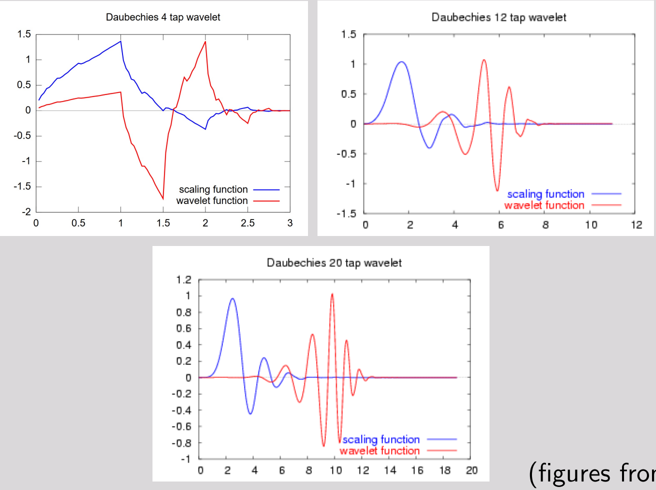
§5 Construction of Father and Mother Wavelets

Example (Daubechies' Wavelets - cont'd)



(figures from wiki)

Figure 6: Some Daubechies' wavelets and corresponding scaling functions