影像處理專題 I

MA3113-*

Discrete Fourier Transform (DFT)

§1 The Fourier Series

§2 Convergence of the Fourier Series

§3 The Discrete Fourier "Transform" and the Fast Fourier "Transform"

§4 Fourier Series for Functions of Two Variables

§5 Denoise using DFT

§1 The Fourier Series

Definition

For an (Riemann) integrable function $f : [-\pi, \pi] \to \mathbb{R}$, the ***Fourier series*** of f, denoted by $\mathcal{S}[f]$, is given by

$$\mathcal{S}[f](x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} (c_k \cos kx + s_k \sin kx)$$

whenever the sum makes sense, where sequences $\{c_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ given by

$$c_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx \quad \text{and} \quad s_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx$$

are called the ***Fourier coefficients*** associated with f. The n-th partial sum of the Fourier series to f, denoted by $\mathcal{S}_n[f]$, is given by

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§1 The Fourier Series

Remark: Because of the Euler identity $e^{i\theta} = \cos\theta + i\sin\theta$, we can write

$$c_k = \int_{-\pi}^{\pi} f(y)\frac{e^{iky}+e^{-iky}}{2\pi}dy \quad \text{and} \quad s_k = \int_{-\pi}^{\pi} f(y)\frac{e^{iky}-e^{-iky}}{2\pi i}dy;$$

thus

$$\mathcal{S}_n[f](x) = \frac{c_0}{2} + \sum_{k=1}^{n}\Big(c_k\frac{e^{ikx}+e^{-ikx}}{2} + s_k\frac{e^{ikx}-e^{-ikx}}{2i}\Big)$$

$$= \frac{1}{2}\Big[c_0 + \sum_{k=1}^{n}\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}dy\, e^{ikx} + \sum_{k=-n}^{-1}\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}dy\, e^{ikx}\Big].$$

Define $\widehat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}\,dy$. Then $\widehat{f}_k = \frac{c_{|k|}+is_{|k|}}{2}$, and

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The sequence $\{\widehat{f}_k\}_{k=-\infty}^{\infty}$ is also called the Fourier coefficients associated with f, and one can write the Foruier series of f as $\sum\limits_{k=-\infty}^{\infty} \widehat{f}_k e^{ikx}$ (whenever the sum makes sense).

§1 The Fourier Series

Remark: Given an integrable function g with period $2L$, let $f(x) = g\big(\frac{Lx}{\pi}\big)$ $\Big($so $f\big(\frac{\pi x}{L}\big) = g(x)\Big)$. Then f is an integrable function with period 2π, and the Fourier series of f

$$\mathcal{S}[f](x) = \frac{c_0}{2} + \sum_{k=1}^{n} (c_k \cos kx + s_k \sin kx)\,,$$

where c_k and s_k are given by

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Now, define the Fourier series of g by $\mathcal{S}[g](x) = \mathcal{S}[f]\big(\frac{\pi x}{L}\big)$. Then the Fourier series of g is given by

$$\mathcal{S}[g](x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} \Big(c_k \cos\frac{k\pi x}{L} + s_k \sin\frac{k\pi x}{L}\Big)\,,$$

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where $\{c_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ is also called the Fourier coefficients associated with g and are given by

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and similarly, $s_k = \dfrac{1}{L}\displaystyle\int_{-L}^{L} g(x)\sin\frac{k\pi x}{L}\,dx$. Similar to the previous remark, the Fourier series of g can also be written as

$$\sum_{k=-\infty}^{\infty} \widehat{g}_k e^{\frac{i\pi kx}{L}},$$

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Let $L^2(\mathbb{T})$ denote the collection of square integrable function on $[-\pi,\pi]$ modulo the relation that $f \sim g$ if $f-g=0$ except on a set of measure zero (or $f=g$ almost everywhere):

$$L^2(\mathbb{T}) = \Big\{ f:[-\pi,\pi]\to\mathbb{C} \,\Big|\, \int_{-\pi}^{\pi} |f(x)|^2\,dx < \infty \Big\} \Big/ \sim \ .$$

Define a bilinear function $\langle\cdot,\cdot\rangle$ on $L^2(\mathbb{T})\times L^2(\mathbb{T})$ by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx\,.$$

Then $\langle\cdot,\cdot\rangle$ is an inner product on $L^2(\mathbb{T})$. Therefore, $(L^2(\mathbb{T}),\langle\cdot,\cdot\rangle)$ is an inner product space, and the norm induced by the inner product is denoted by $\|\cdot\|_{L^2(\mathbb{T})}$; that is,

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For $k \in \mathbb{Z}$, define $\mathbf{e}_k : [-\pi, \pi] \to \mathbb{C}$ by $\mathbf{e}_k(x) = e^{ikx}$. Then $\{\mathbf{e}_k\}_{k=-\infty}^{\infty}$ is an orthonormal set in $L^2(\mathbb{T})$ since

$$\langle \mathbf{e}_k, \mathbf{e}_\ell \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-i\ell x}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-\ell)x}\, dx = \begin{cases} 1 & \text{if } k = \ell\,, \\ 0 & \text{if } k \neq \ell\,. \end{cases}$$

Let

$$\begin{aligned} \mathcal{V}_n &= \operatorname{span}(\mathbf{e}_{-n}, \mathbf{e}_{-n+1}, \cdots, \mathbf{e}_0, \mathbf{e}_1, \cdots, \mathbf{e}_n) \\ &= \Big\{ \sum_{k=-n}^{n} a_k \mathbf{e}_k \,\Big|\, \{a_k\}_{k=-n}^{n} \subseteq \mathbb{C} \Big\}. \end{aligned}$$

For each vector $f \in L^2(\mathbb{T})$, the orthogonal projection of f onto $\mathcal{V}_n$ is, conceptually, given by

$$\sum_{k=-n}^{n} \langle f, \mathbf{e}_k \rangle \mathbf{e}_k = \sum_{k=-n}^{n} \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx \Big) \mathbf{e}_k = \sum_{k=-n}^{n} \widehat{f}_k \mathbf{e}_k\,.$$

By the definition of $\mathbf{e}_k$, we obtain that the orthogonal projection of f on $\mathcal{V}_n$ is exactly $\mathcal{S}_n[f]$.

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§2 Convergence of the Fourier Series

Let $f\colon [-\pi,\pi]$ be Riemann integrable. Using the formula

$$\mathcal{S}_n[f](x) = \sum_{k=-n}^{n} \widehat{f}_k e^{ikx}, \qquad \widehat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}\,dy,$$

we find that

$$\mathcal{S}_n[f](x) = \int_{-\pi}^{\pi} f(y)\Big(\frac{1}{2\pi}\sum_{k=-n}^{n} e^{ik(x-y)}\Big)\,dy.$$

Define $\mathrm{D}_n(x) = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{ikx}$. Then D_n is 2π-periodic, and

$$\mathcal{S}_n[f](x) = \int_{-\pi}^{\pi} f(y)\mathrm{D}_n(x-y)\,dy.$$

For 2π-periodic Riemann integrable functions f and g, we define the convolution of f and g on the circle by

$$(f\star g)(x) = \int_{-\pi}^{\pi} f(y)g(x-y)\,dy.$$

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The function $\mathrm{D}_n : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathrm{D}_n(x) = \begin{cases} \dfrac{\sin(n+\frac{1}{2})x}{2\pi \sin\frac{x}{2}} & \text{if } x \notin \{2k\pi \mid k \in \mathbb{Z}\}, \\ \dfrac{2n+1}{2\pi} & \text{if } x \in \{2k\pi \mid k \in \mathbb{Z}\}, \end{cases}$$

is called the ***Dirichlet kernel***.

The convergence of the Fourier series of f is usually expressed in the following terms:

For a given f, does $\mathrm{D}_n \star f$ converge to f?

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A function $f \in \mathcal{C}(\mathbb{T})$ is said to be ***Hölder continuous with exponent*** $\alpha \in (0,1]$, denoted by $f \in \mathcal{C}^{0,\alpha}(\mathbb{T})$, if $\displaystyle\sup_{x,y\in\mathbb{R}, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty$. Let $\|\cdot\|_{\mathcal{C}^{0,\alpha}(\mathbb{T})}$ be defined by

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Then $\|\cdot\|_{\mathcal{C}^{0,\alpha}(\mathbb{T})}$ is a norm on $\mathcal{C}^{0,\alpha}(\mathbb{T})$, and

$$\mathcal{C}^{0,\alpha}(\mathbb{T}) = \Big\{ f \in \mathcal{C}(\mathbb{T}) \,\Big|\, \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{T})} < \infty \Big\}.$$

In particular, when $\alpha = 1$, a function in $\mathcal{C}^{0,1}(\mathbb{T})$ is said to be Lipschitz continuous on $\mathbb{T}$; thus $\mathcal{C}^{0,1}(\mathbb{T})$ consists of Lipschitz continuous functions on $\mathbb{T}$.

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Theorem

*For any $f \in \mathcal{C}^{0,\alpha}(\mathbb{T})$ with $\alpha \in (0,1]$, the Fourier series of f converges **uniformly** to f on $\mathbb{R}$.*

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*Let $f : (-\pi, \pi) \to \mathbb{R}$ be piecewise Hölder continuous with exponent $\alpha \in (0,1]$. If f is continuous on (a, b), then the Fourier series of f converges uniformly to f **on any compact subsets** of (a, b). In particular, $\lim_{n\to\infty} \mathcal{S}_n[f](x_0) = f(x_0)$ if f is continuous at x_0. In other words, the Fourier series of f converges **pointwise** to f except the discontinuities.*

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§2 Convergence of the Fourier Series

Concerning the convergence of square integrable functions, we have the following

Theorem

Let $f \in L^2(\mathbb{T})$. *Then*

$$\lim_{n\to\infty} \left\| f - \mathcal{S}_n[f] \right\|_{L^2(\mathbb{T})} = 0$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx = \sum_{k=-\infty}^{\infty} |\widehat{f}_k|^2 . \qquad \text{(Parseval's identity)}$$

§2 Convergence of the Fourier Series

- **Gibbs phenomenon**

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be $2L$-periodic piecewise Hölder continuous with exponent $\alpha \in (0,1]$. Then

$$\lim_{n\to\infty} \mathcal{S}_n[f](x_0) = \frac{f(x_0^+) + f(x_0^-)}{2} \qquad \forall\, x_0 \in \mathbb{R}\,.$$

Moreover, if x_0 is a jump discontinuity of f so that

$$f(x_0^+) - f(x_0^-) = a \neq 0\,,$$

then there exists a constant $c > 0$, independent of f, x_0 and L (in fact, $c = \frac{1}{\pi}\int_0^\pi \frac{\sin x}{x}\,dx - \frac{1}{2} \approx 0.089490$), such that

$$\lim_{n\to\infty} \mathcal{S}_n[f]\Big(x_0 \pm \frac{L}{n}\Big) = f(x_0^\pm) \pm ca\,.$$

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§3 The Discrete Fourier "Transform" and the Fast Fourier "Transform"

Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period L and f is Riemann integrable on $[0, L)$. The Fourier series of f (defined in second remark of this slide) can be written as

$$\mathcal{S}_n[f](x) = \sum_{k=-\infty}^{\infty} \widehat{f}_k e^{\frac{2\pi ikx}{L}},$$

where $\widehat{f}_k = \frac{1}{L}\int_0^L f(y)e^{\frac{-2\pi iky}{L}}\,dy$, and $\widehat{f}_k$ can be approximated by the Riemann sum

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In other words, the values of f at N evenly distributed points can be used to determine an approximation of the Fourier coefficients of f.

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Let $f:\mathbb{R}\to\mathbb{R}$ be a periodic function with period L. Suppose that we know the values of f at N evenly distributed points $\left\{\frac{Lj}{N}\right\}_{j=0}^{N-1}$ in $[0,L)$. The discrete Fourier transform of $\left\{f\left(\frac{Lj}{N}\right)\right\}_{j=0}^{N-1}$ are coefficients $\{X_\ell\}_{\ell=0}^{N-1}$ such that the series

$$\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{\frac{2\pi ikx}{L}} .$$

agrees with f at these N points. Such $\{X_\ell\}_{\ell=0}^{N-1}$ must satisfy

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \cdots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \cdots & \omega_N^{2(N-1)} \\ \vdots & & & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \cdots & \omega_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} f(0) \\ f(L/N) \\ \vdots \\ f((N-1)L/N) \end{bmatrix},$$

where $\omega_N = e^{-\frac{2\pi i}{N}}$. This motivates the following

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Definition

The ***discrete Fourier transform***, symbolized by DFT, of a sequence of N complex numbers $\{x_0, x_1, \cdots, x_{N-1}\}$ is a sequence $\{X_k\}_{k\in\mathbb{Z}}$ defined by

$$X_k = \sum_{\ell=0}^{N-1} x_\ell e^{\frac{-2\pi i k\ell}{N}} \qquad \forall\, k \in \mathbb{Z}\,.$$

We note that the sequence $\{X_k\}_{k\in\mathbb{Z}}$ is N-periodic; that is, $X_{k+N} = X_k$ for all $k \in \mathbb{Z}$. Therefore, often time we only focus on one of the following N consecutive terms $\{X_0, X_1, \cdots, X_{N-1}\}$ of the DFT.

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§3 The Discrete Fourier "Transform" and the Fast Fourier "Transform"

- **The inversion formula**

Let $\{X_k\}_{k=0}^{N-1}$ be the discrete Fourier transform of $\{x_\ell\}_{\ell=0}^{N-1}$. Then $\{x_\ell\}_{\ell=0}^{N-1}$ can be recovered given $\{X_k\}_{k=0}^{N-1}$ by the inversion formula

$$x_\ell = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{\frac{2\pi ik\ell}{N}}\,.$$

The map from $\{X_k\}_{k=0}^{N-1}$ to $\{x_\ell\}_{\ell=0}^{N-1}$ is called the ***discrete inverse Fourier transform***.

§3 The Discrete Fourier "Transform" and the Fast Fourier "Transform"

Remark: Given a sample data $[x_0, x_1, \cdots, x_{N-1}]$ which is the values of a function f on N evenly distributed points on $[0, L)$ (for some unknown $L > 0$), the DFT $[X_0, X_1, \cdots, X_{N-1}]$ also satisfies that

$$x_\ell = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{\frac{2\pi i k\ell}{N}} = \frac{1}{N}\sum_{k=0}^{[\frac{N-1}{2}]} X_k e^{\frac{2\pi i k\ell}{N}} + \frac{1}{N}\sum_{k=[\frac{N-1}{2}]+1}^{N-1} X_{k-N} e^{\frac{2\pi i(k-N)\ell}{N}}$$

$$= \frac{1}{N}\sum_{k=0}^{[\frac{N-1}{2}]} X_k e^{\frac{2\pi i k\ell}{N}} + \frac{1}{N}\sum_{k=-[\frac{N}{2}]}^{-1} X_k e^{\frac{2\pi i k\ell}{N}} = \frac{1}{N}\sum_{k=-[\frac{N}{2}]}^{[\frac{N-1}{2}]} X_k e^{\frac{2\pi i k\ell}{N}};$$

thus we may also consider the following approximation:

$$f(x) \approx \frac{1}{N}\sum_{k=-[\frac{N}{2}]}^{[\frac{N-1}{2}]} X_k e^{\frac{2\pi i k x}{L}},$$

where $\approx$ becomes $=$ if $x = \frac{L\ell}{N}$, $0 \leqslant \ell \leqslant N-1$.

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we find that for $0 \leqslant k \leqslant [\frac{N-1}{2}]$ each X_k is the coefficient associated with the wave with frequency k/L. To determine L, we introduce the ***sampling frequency*** F_s which is the number of samples per unit time/length. Then $F_s = N/L$ so that X_k is the coefficient associated with the wave with frequency $\frac{F_s}{N}k$.

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§3 The Discrete Fourier "Transform" and the Fast Fourier "Transform"

- **The fast Fourier transform**

Let

$$F_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \cdots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \cdots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \cdots & \omega_N^{(N-1)(N-1)} \end{bmatrix}, \quad \omega_N = \exp\Big(-\frac{2\pi i}{N}\Big).$$

For $v = [x_0, x_1, \cdots, x_{N-1}]^{\mathrm{T}}$, a naive way of computing the DFT $\hat{v} = F_N v$ of v just does the matrix-vector multiplication to compute all the entries of $\hat{v}$. This would take $\mathcal{O}(N^2)$ steps to compute the vector $\hat{v}$. However, there is a more efficient way of computing $\hat{v}$. This algorithm is called the **Fast Fourier Transform** (**FFT**, due to Cooley and Tukey in 1965), and takes only $\mathcal{O}(N \log_2 N)$ steps.

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§3 The Discrete Fourier "Transform" and the Fast Fourier "Transform"

We will assume $N = 2^n$, which is usually fine because we can add zeroes to our vector to make its dimension a power of 2 (but similar FFTs can be given also directly for most N that are not a power of 2). The key to the FFT is to rewrite the entries of $\widehat{v}$ as follows:

$$\begin{aligned}\widehat{v}_j = \sum_{k=0}^{N-1} \omega_N^{jk} v_k &= \sum_{k \text{ even}} \omega_N^{jk} v_k + \sum_{k \text{ odd}} \omega_N^{jk} v_k \\ &= \sum_{k \text{ even}} \omega_{N/2}^{jk/2} v_k + \omega_N^j \sum_{k \text{ odd}} \omega_{N/2}^{j(k-1)/2} v_k \,.\end{aligned}$$

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Note that we have rewritten the entries of the N-dimensional discrete Fourier transform $\widehat{v}$ in terms of two $\frac{N}{2}$-dimensional discrete Fourier transforms, one of the even-numbered entries of v, and one of the odd-numbered entries of v. This suggests a recursive procedure for computing $\widehat{v}$: first separately compute the discrete Fourier transform $\widehat{v_{\text{even}}}$ of the $\frac{N}{2}$-dimensional vector of even-numbered entries of v and the discrete Fourier transform $\widehat{v_{\text{odd}}}$ of the $\frac{N}{2}$-dimensional vector of odd-numbered entries of v, and then compute the N entries using

$$\widehat{v}_j = (\widehat{v_{\text{even}}})_j + \omega_N^j (\widehat{v_{\text{odd}}})_j \qquad \forall\, 0 \leqslant j \leqslant \frac{N}{2} - 1\,,$$

$$\widehat{v}_{j+\frac{N}{2}} = (\widehat{v_{\text{even}}})_j - \omega_N^j (\widehat{v_{\text{odd}}})_j \qquad \forall\, 0 \leqslant j \leqslant \frac{N}{2} - 1\,.$$

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§3 The Discrete Fourier "Transform" and the Fast Fourier "Transform"

The computation time $T(N)$ it takes to implement F_N this way can be written recursively as $T(N) = 2T(\frac{N}{2}) + 2N$, because we need to compute two $\frac{N}{2}$-dimensional discrete Fourier transforms and do $2N$ additional operations (additions and multiplications) to compute $\widehat{v}$.

This works out to time $T(N) = \mathcal{O}(N \log_2 N)$, as promised. Similarly, we have an equally efficient algorithm for the inverse discrete Fourier transform $F_N^{-1} = \frac{1}{N} F_N^*$, whose (j,k)-entries are $\frac{1}{N}\omega_N^{jk}$.

§3 The Discrete Fourier "Transform" and the Fast Fourier "Transform"

The computation time $T(N)$ it takes to implement F_N this way can be written recursively as $T(N) = 2T(\frac{N}{2}) + 2N$, because we need to compute two $\frac{N}{2}$-dimensional discrete Fourier transforms and do $2N$ additional operations (additions and multiplications) to compute $\widehat{v}$. This works out to time $T(N) = \mathcal{O}(N \log_2 N)$, as promised. Similarly, we have an equally efficient algorithm for the inverse discrete Fourier transform $F_N^{-1} = \frac{1}{N} F_N^*$, whose (j, k)-entries are $\frac{1}{N} \omega_N^{jk}$.

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§4 Fourier Series for Functions of Two Variables

Let $\Omega \equiv [-L_1, L_1] \times [-L_2, L_2]$ and define

$$L^2(\Omega) = \left\{ f : \Omega \to \mathbb{C} \,\middle|\, \int_\Omega \left|f(x_1, x_2)\right|^2 d(x_2, x_2) < \infty \right\} \Big/ \sim$$

equipped with the inner product

$$\langle f, g \rangle \equiv \frac{1}{\nu(\Omega)} \int_\Omega f(x_1, x_2)\overline{g(x_1, x_2)}\, d(x_1, x_2)\,,$$

where $\nu(\Omega)$ denotes the area of Ω and $\sim$ again denotes the equivalence relation defined by $f \sim g$ if and only if $f - g = 0$ except on a set of measure zero. Denote the norm induced by the inner product $\langle \cdot, \cdot \rangle$ by $\| \cdot \|_{L^2(\Omega)}$.

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Let $\mathbf{e}_{k\ell}(\boldsymbol{x}) = e^{i\pi(\frac{k}{L_1}, \frac{\ell}{L_2})\cdot \boldsymbol{x}}$, here $\boldsymbol{x} = (x_1, x_2)$. Then for each $f \in L^2(\Omega)$, by defining the partial sum

$$\mathcal{S}_{n,m}[f](\boldsymbol{x}) = \sum_{k=-n}^{n} \sum_{\ell=-m}^{m} \langle f, \mathbf{e}_{k\ell} \rangle \mathbf{e}_{k\ell}(\boldsymbol{x})$$

we have $\lim\limits_{n,m\to\infty} \big\| f - \mathcal{S}_{n,m}[f] \big\|_{L^2(\Omega)} = 0$. The limit of $\mathcal{S}_{n,m}[f]$, as $n, m \to \infty$, in the inner product space $\big(L^2(\Omega), \langle \cdot, \cdot \rangle\big)$ is denoted by

$$\mathcal{S}[f] = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \langle f, \mathbf{e}_{k\ell} \rangle \mathbf{e}_{k\ell}$$

and is called the Fourier series of f.

§4 Fourier Series for Functions of Two Variables

The discrete Fourier transform (or DFT) of a collection of data $\{x_{mn}\}_{0\leqslant n\leqslant M-1, 0\leqslant n\leqslant N-1}$ is a double sequence $\{X_{k\ell}\}_{k,\ell\in\mathbb{Z}}$ given by

$$X_{k\ell} = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x_{mn}\,\omega_M^{mk}\omega_N^{n\ell},$$

where $\omega_M = e^{-\frac{2\pi i}{M}}$ and $\omega_N = e^{-\frac{2\pi i}{N}}$. The double sequence $\{X_{k\ell}\}_{k,\ell\in\mathbb{Z}}$ is doubly periodic satisfying $X_{k+M,\ell+N}$ for all $k,\ell\in\mathbb{Z}$; thus we usually only focus on the terms $\{X_{k\ell}\}_{0\leqslant k\leqslant M-1,0\leqslant \ell\leqslant N-1}$. The discrete inverse Fourier transform of a double sequence $\{X_{k\ell}\}_{0\leqslant k\leqslant M-1,0\leqslant \ell\leqslant N-1}$ is a double sequence $\{x_{mn}\}_{m,n\in\mathbb{Z}}$ defined by

$$x_{mn} = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{\ell=0}^{N-1} X_{k\ell}\,\overline{\omega_M}^{mk}\overline{\omega_N}^{n\ell},$$

where $\overline{\omega_M}$ and $\overline{\omega_N}$ are complex conjugate of ω_M and ω_N defined above.

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