MA3113: Topics in Mathematical Image Processing I
ROF Model and Split Bregman Iterations

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February 14, 2025

The bounded variation space $BV(\Omega)$

Let Ω be an open subset of $\mathbb{R}^2$. The space of functions of bounded variation $BV(\Omega)$ is defined as the space of real-valued function $u \in L^1(\Omega)$ such that the total variation is finite, i.e.,

$$BV(\Omega) = \{u \in L^1(\Omega) : \|u\|_{TV(\Omega)} < \infty\},$$

where

$$\|u\|_{TV(\Omega)} := \sup \left\{ \int_\Omega u(\nabla \cdot \varphi)\, d\boldsymbol{x} : \varphi \in C_c^1(\Omega, \mathbb{R}^2), \|\varphi\|_{(L^\infty(\Omega))^2} \le 1 \right\},$$

$C_c^1(\Omega, \mathbb{R}^2)$ is the space of continuously differentiable vector functions with compact support in Ω, $L^1(\Omega)$ and $L^\infty(\Omega)$ are the usual $L^p(\Omega)$ space for $p = 1$ and $p = \infty$, respectively.

Then $BV(\Omega)$ is a Banach space with the norm,

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \|u\|_{TV(\Omega)}.$$

The ROF total-variation regularization model

Let $f : \overline{\Omega} \subset \mathbb{R}^2 \to \mathbb{R}$ be a given noisy image. Rudin, Osher, and Fatemi (*Physica D, 1992*) proposed the model for image denoising:

$$\min_{u \in BV(\Omega) \cap L^2(\Omega)} \Big(\underbrace{\|u\|_{TV(\Omega)}}_{regularizer} + \underbrace{\frac{\lambda}{2} \int_\Omega \left(u(\boldsymbol{x}) - f(\boldsymbol{x})\right)^2 d\boldsymbol{x}}_{data\ fidelity} \Big),$$

where $\lambda > 0$ is a tuning parameter which controls the regularization strength. Notice that

- A smaller value of λ will lead to a more regular solution.
- The space of functions with bounded variation help remove spurious oscillations (noise) and preserve sharp signals (edges).
- The *TV* term allows the solution to have discontinuities.

The existence, uniqueness and stability of solution

Theorem: *If u is smooth, then $\|u\|_{TV(\Omega)} = \int_\Omega |\nabla u| \, d\boldsymbol{x}$.*

Theorem: *If $f \in L^2(\Omega)$, the minimizer exists and is unique and is stable in L^2 with respect to perturbations in f.*

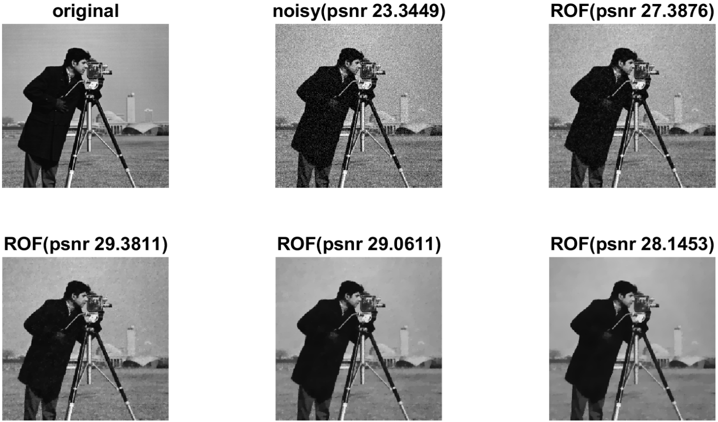
ROF model for image denoising: Below we assume that u is smooth, and we consider the model

$$\min_{u \in \mathcal{V}} \left(\int_\Omega |\nabla u| \, d\boldsymbol{x} + \frac{\lambda}{2} \int_\Omega \left(u(\boldsymbol{x}) - f(\boldsymbol{x})\right)^2 d\boldsymbol{x} \right).$$

Let $E[\cdot]$ be the functional over the vector space $\mathcal{V}$,

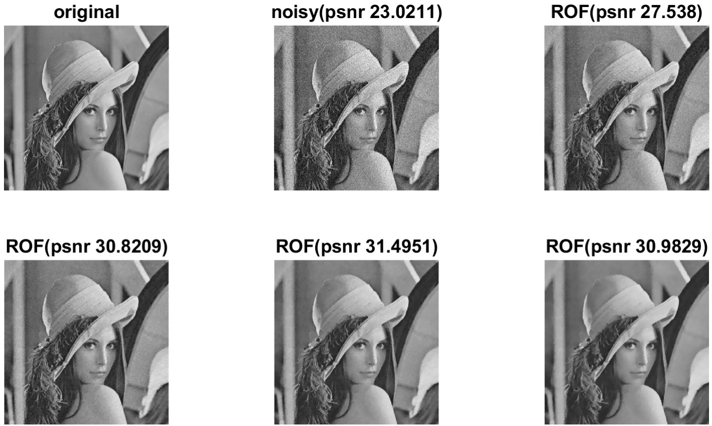
$$E[u] := \int_\Omega |\nabla u| \, d\boldsymbol{x} + \frac{\lambda}{2} \int_\Omega \left(u(\boldsymbol{x}) - f(\boldsymbol{x})\right)^2 d\boldsymbol{x}.$$

Numerical experiments (Cameraman)



$$\lambda h = 1/5, 1/10, 1/15, 1/20$$

Numerical experiments (Lena)



$$\lambda h = 1/5, 1/10, 1/15, 1/20$$

References

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