

MA3111: Mathematical Image Processing

Image Contrast Enhancement



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Outline of “image contrast enhancement”

In this lecture, we will briefly introduce some techniques for image contrast enhancement, including

- *Histogram equalization (HE, 直方圖均衡化)*
- *Automatic color equalization (ACE, 自動色彩均衡)*
- *Simplest color balance (SCB, 最簡色彩平衡)*
- *A variational method with split Bregman iterations*

The material of this lecture

The material of this lecture is based on the following text and papers:

- *Section 3.3: Histogram Processing* in [GW2018], pp. 133-153.
- P. Getreuer, Automatic color enhancement (ACE) and its fast implementation, *Image Processing On Line*, 2 (2012), pp. 266-277.
- P.-W. Hsieh, P.-C. Shao, and S.-Y. Yang, Adaptive variational model for contrast enhancement of low-light images, *SIAM Journal on Imaging Sciences*, 13 (2020), pp. 1-28.
- N. Limare, J.-L. Lisani, J.-M. Morel, A. B. Petro, and C. Sbert, Simplest color balance, *Image Processing On Line*, 1 (2011), pp. 297-315.

Contrast enhancement

The main purpose of contrast enhancement is to adjust the image intensity to enhance the quality and features of the image for a better human visual perception or machine vision identification.



A low-light image and its enhanced result

Histogram equalization (HE): $g(x, y) =: s = T(r) := T(f(x, y))$

- We are given a grayscale image $f : \bar{\Omega} \rightarrow [0, 1]$. The cumulative histogram (*cumulative distribution function*) T is defined by considering f as a random variable: for $\eta \in [0, 1]$, we define

$$\begin{aligned} T(\eta) &:= \text{Prob}(f \leq \eta) \\ &= \frac{1}{|\bar{\Omega}|} \left| \{(x, y) \in \bar{\Omega} : f(x, y) \leq \eta\} \right|. \end{aligned}$$

Then $T : [0, 1] \rightarrow [0, 1]$ is a monotonic increasing function.

- The histogram equalized image $g : \bar{\Omega} \rightarrow [0, 1]$ is obtained by defining

$$g(x, y) := T(f(x, y)).$$

Histogram equalized image $g \sim \mathcal{U}(0, 1)$ if T is invertible

If T is strictly increasing, then T is invertible and the *cumulative distribution function* of the histogram equalized image g is

$$\begin{aligned} \text{Prob}(g \leq \eta) &= \text{Prob}(T(f) \leq \eta) = \text{Prob}(f \leq T^{-1}(\eta)) \\ &= T(T^{-1}(\eta)) = \eta. \end{aligned}$$

Hence, the *probability density function* of g is

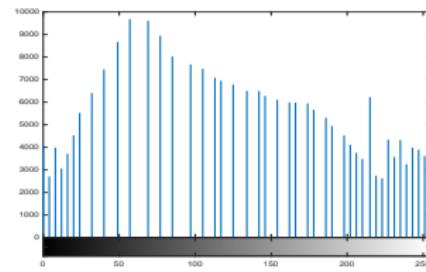
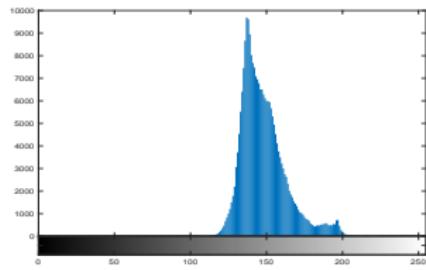
$$p(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Therefore, g has a uniform distribution, i.e., $g \sim \mathcal{U}(0, 1)$.

Recall: Let X be a random variable and $p(t)$ the probability density function (pdf) of X . The cumulative distribution function (cdf) of X is

$$F(\eta) := \text{Prob}(X \leq \eta) = \int_{-\infty}^{\eta} p(t) dt.$$

Example of histogram equalized image



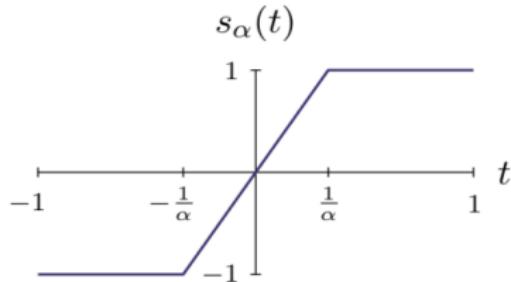
*Histogram equalization of 400×600 image
(top) before; (bottom) after; and the corresponding histograms*

Matlab commands: `imhist (A)`, `histeq (A)`

Automatic color equalization (ACE)

We are given a grayscale image $f : \bar{\Omega} \rightarrow [0, 1]$. First, the following operation is performed

$$\tilde{f}(x) = \sum_{y \in \bar{\Omega} \setminus \{x\}} \frac{s_\alpha(f(x) - f(y))}{\|x - y\|}, \quad \forall x \in \bar{\Omega}.$$

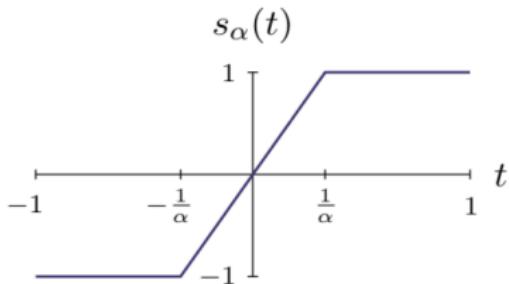


The slope function $s_\alpha(t) := \min\{\max\{\alpha t, -1\}, 1\}$ ($\alpha > 1$).

Then \tilde{f} is rescaled to $[0, 1]$ as the ACE image

$$g(x) = \frac{\tilde{f}(x) - \min \tilde{f}}{\max \tilde{f} - \min \tilde{f}}, \quad \forall x \in \bar{\Omega}.$$

ACE images for various α 's and HE image



ACE, $\alpha = 2$



ACE, $\alpha = 4$



ACE, $\alpha = 8$



HE



Simplest color balance: given a grayscale image $f(x)$

- The algorithm simply stretches, as much as it can, the values of the three channels (R, G, B), so that they occupy the maximal possible range $[0, 255]$.
- The simplest way to do so is to apply an *affine function*

$$\tilde{z} = a \underbrace{z}_{f(x)} + b$$
 to each channel such that

$$\begin{cases} az_{\min} + b = 0, \\ az_{\max} + b = 255. \end{cases}$$

We solve a and b so that the maximal value in the channel becomes 255 and the minimal value 0.

$$a = \frac{255}{z_{\max} - z_{\min}}, \quad b = -\frac{255z_{\min}}{z_{\max} - z_{\min}}.$$

That is, the intensity of the resulting image is given by

$$\tilde{f}(x) = \frac{255}{z_{\max} - z_{\min}} f(x) - \frac{255z_{\min}}{z_{\max} - z_{\min}} = 255 \left(\frac{f(x) - z_{\min}}{z_{\max} - z_{\min}} \right), x \in \overline{\Omega}.$$

Simplest color balance (cont'd)

- However, many images contain a few aberrant pixels that already occupy the 0 and 255 values. In such a case, we have

$$a = \frac{255}{z_{\max} - z_{\min}} = \frac{255}{255 - 0} = 1, \quad b = -\frac{255z_{\min}}{z_{\max} - z_{\min}} = -\frac{255 \cdot 0}{255} = 0,$$

which implies $\tilde{z} = z$ and hence, nothing changes.

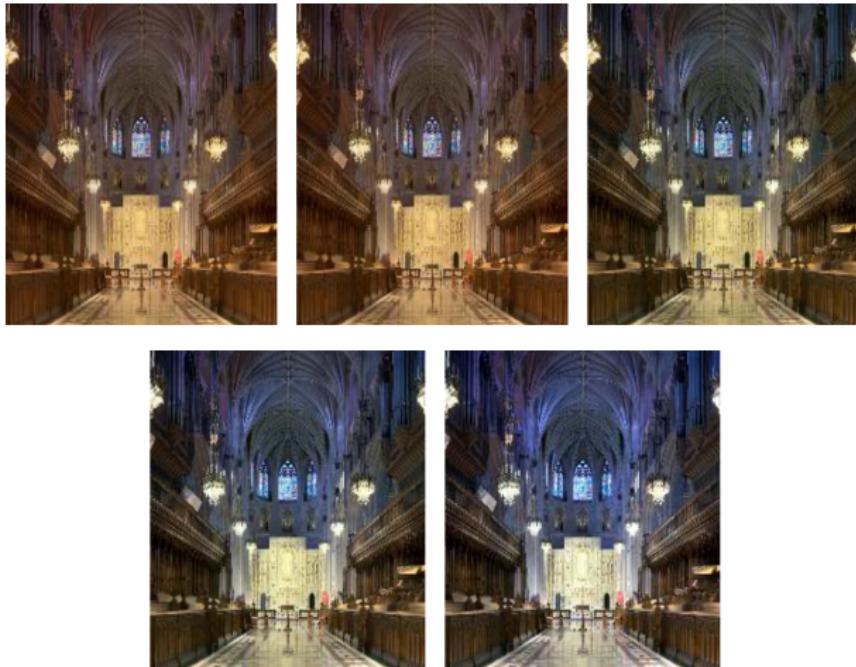
- An often spectacular image color improvement is obtained by *“clipping” a small percentage $s\%$ of the pixels with the highest values to 255 and a small percentage of the pixels with the lowest values to 0, before applying the affine transform.*
- Notice that this saturation can create flat white regions or flat black regions that may look unnatural. *Thus, the percentage of saturated pixels must be as small as possible.*

SCB images



original image, SCB images with $s\% = 0\%, 1\%, 2\%, \text{ and } 3\%$

SCB images



original image, SCB images with $s\% = 0\%, 1\%, 2\%, \text{ and } 3\%$

Image stitching (影像拼接) with SCB and $s\% = 0.1\%$



A landscape of Da-Xi (大溪)

A simple variational model

Let $f : \bar{\Omega} \rightarrow \mathbb{R}$ be a given grayscale image. The Morel-Petro-Sbert model (IPOL 2014) is given by

$$\min_u \underbrace{\frac{1}{2} \int_{\Omega} |\nabla u - \nabla f|^2 dx}_{\text{data fidelity}} + \underbrace{\frac{\lambda}{2} \int_{\Omega} (u - \bar{u})^2 dx}_{\text{regularizer}}.$$

- The constant $\bar{u} := \frac{1}{|\Omega|} \int_{\Omega} u dx$ is the mean value of u over Ω .
- The data fidelity term preserves image details presented in f and the regularizer reduces the variance of u to eliminate the effect of nonuniform illumination.
- The parameter $\lambda > 0$ balances between detail preservation and variance reduction.

Two modified variational models

- The original model is simple but difficult to solve due to the \bar{u} term. Therefore, by assuming that $\bar{u} \approx \bar{f} := \frac{1}{|\Omega|} \int_{\Omega} f \, dx$, it was simplified to

$$\min_u \frac{1}{2} \int_{\Omega} |\nabla u - \nabla f|^2 \, dx + \frac{\lambda}{2} \int_{\Omega} (u - \bar{f})^2 \, dx.$$

- Petro-Sbert-Morel (MAA 2014) further improved their model by using the L^1 norm to obtain sharper edges:

$$\min_u \int_{\Omega} |\nabla u - \nabla f| \, dx + \frac{\lambda}{2} \int_{\Omega} (u - \bar{f})^2 \, dx.$$

Note that requiring the desired image u being close to a pixel-independent constant \bar{f} highly contradicts the requirement of ∇u being close to ∇f and restrains the parameter λ to be very small.

An adaptive variational model

Hsieh-Shao-Yang (*SIIMS 2020*) proposed two adaptive functions g and h to replace \bar{f} and the original input image f ,

$$\min_u \int_{\Omega} |\nabla u - \nabla h| dx + \frac{\lambda}{2} \int_{\Omega} (u - g)^2 dx + \chi_{[0,255]}(u),$$

where g and h are devised respectively as

$$g(x) = \begin{cases} \alpha \bar{f}, & x \in \Omega_d, \\ f(x), & x \in \Omega_b, \end{cases} \quad h(x) = \begin{cases} \beta f(x), & x \in \Omega_d, \\ f(x), & x \in \Omega_b, \end{cases}$$

$$\Omega_d := \{x \in \bar{\Omega} : f(x) \leq \bar{f}\}, \quad \Omega_b := \{x \in \bar{\Omega} : f(x) > \bar{f}\},$$

with a brightness parameter $\alpha > 0$ and a contrast-level parameter $\beta > 1$, and the characteristic function is defined as

$$\chi_{[0,255]}(u) = \begin{cases} 0, & \text{range}(u) \subseteq [0, 255], \\ \infty, & \text{otherwise.} \end{cases}$$

Generally speaking, Ω_d contains relatively dim elements, while Ω_b contains relatively bright elements.

Differentiability of h

To ensure the differentiability of h , in practice we smooth the coefficients and redefine the adaptive function h as

$$h(\mathbf{x}) = G * (\beta 1_{\Omega_d}(\mathbf{x}) + 1_{\Omega_b}(\mathbf{x}))f(\mathbf{x}), \quad \mathbf{x} \in \overline{\Omega},$$

where the indicator function $1_A(\mathbf{x}) = 1$, if $\mathbf{x} \in A$, otherwise $1_A(\mathbf{x}) = 0$, and $G*$ represents suitable Gaussian convolution such that ∇h is well-defined.

Color RGB images

- The domain division for color RGB images (f_R, f_G, f_B) is conducted as follows. First, we define the maximum image as

$$f_{\max}(x) := \max\{f_R(x), f_G(x), f_B(x)\}, \quad \forall x \in \overline{\Omega}.$$

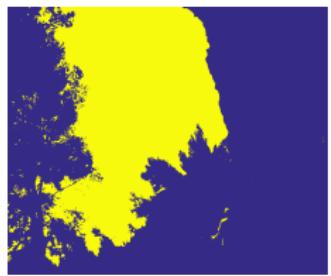
- Let $\bar{f}_{\max} := \frac{1}{|\Omega|} \int_{\Omega} f_{\max} dx$. Then we divide the image domain Ω into two parts

$$\Omega_d := \{x \in \overline{\Omega} : f_{\max}(x) \leq \bar{f}_{\max}\},$$

$$\Omega_b := \{x \in \overline{\Omega} : f_{\max}(x) > \bar{f}_{\max}\}.$$

- **Example:** Consider an element $x^* \in \overline{\Omega}$ with color intensities $(f_R(x^*), f_G(x^*), f_B(x^*)) = (25, 25, 200)$, then $f_{\max}(x^*) = 200$, a large value which should be classified into Ω_b .

Domain division for color images



(top row): low-light images (bottom row): domain-division results

Adaptive variational model for color images

- With the help of the maximum image f_{\max} , we can now process color images channelwise. For every $f \in \{f_R, f_G, f_B\}$, we solve

$$\min_u \int_{\Omega} |\nabla u - \nabla h_c| \, d\mathbf{x} + \frac{\lambda}{2} \int_{\Omega} (u - g_c)^2 \, d\mathbf{x} + \chi_{[0,255]}(u),$$

where the adaptive functions g_c and h_c are defined as

$$g_c(\mathbf{x}) := \begin{cases} \alpha \bar{f}, & \mathbf{x} \in \Omega_d, \\ f(\mathbf{x}), & \mathbf{x} \in \Omega_b, \end{cases}$$

and

$$h_c(\mathbf{x}) := \begin{cases} \beta f(\mathbf{x}), & \mathbf{x} \in \Omega_d, \\ f(\mathbf{x}), & \mathbf{x} \in \Omega_b. \end{cases}$$

- There is no evidence shown that chooses different λ , α , and β for each channel separately can have specific benefit. Therefore, for simplicity, we fix λ , α , and β across channel.

The bounded variation space $BV(\Omega)$

Let Ω be an open subset of \mathbb{R}^2 . The space of functions of bounded variation $BV(\Omega)$ is defined as the space of real-valued function $u \in L^1(\Omega)$ such that the total variation is finite, i.e.,

$$BV(\Omega) = \{u \in L^1(\Omega) : \|u\|_{TV(\Omega)} < \infty\},$$

where

$$\begin{aligned}\|u\|_{TV(\Omega)} &:= \int_{\Omega} |Du| \\ &:= \sup \left\{ \int_{\Omega} u(\nabla \cdot \varphi) dx : \varphi \in C_c^1(\Omega, \mathbb{R}^2), \|\varphi\|_{(L^\infty(\Omega))^2} \leq 1 \right\},\end{aligned}$$

$C_c^1(\Omega, \mathbb{R}^2)$ is the space of continuously differentiable vector functions with compact support in Ω , $L^1(\Omega)$ and $L^\infty(\Omega)$ are the $L^p(\Omega)$ space for $p = 1$ and $p = \infty$, respectively, equipped with the $\|\cdot\|_{L^p(\Omega)}$ norm.

Then $BV(\Omega)$ is a Banach space with the norm,

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \|u\|_{TV(\Omega)}.$$

For a sufficiently smooth function u , we have $\int_{\Omega} |Du| = \int_{\Omega} |\nabla u| dx$.

Existence and uniqueness of minimizer

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with Lipschitz boundary and let $h \in BV(\Omega)$ be the input image. Then the variational problem

$$\min_u \int_{\Omega} |\nabla u - \nabla h| dx + \frac{\lambda}{2} \int_{\Omega} (u - g)^2 dx + \chi_{[0,255]}(u)$$

admits a unique minimizer in $BV(\Omega) \cap L^2(\Omega)$.

Remarks:

- $\int_{\Omega} |\nabla u| dx$ should be realized as the total variation $\int_{\Omega} |Du|$.
- Let $w = u - h$, then the energy can be rewritten as the TV denoising one proposed by Goldstein-Osher (SIIMS 2009).
- direct method (Lebesgue dominated convergence) \rightarrow existence.
- strict convexity \rightarrow uniqueness.

The alternating minimization algorithm

- The discrete gradient of u is defined as $\nabla u_{i,j} = (\nabla_x^+ u_{i,j}, \nabla_y^+ u_{i,j})$,

$$\begin{aligned}\nabla_x^+ u_{i,j} &:= \begin{cases} (u_{i,j+1} - u_{i,j})/h, & 1 \leq j \leq N-1, \\ 0, & j = N, \end{cases} \\ \nabla_y^+ u_{i,j} &:= \begin{cases} (u_{i+1,j} - u_{i,j})/h, & 1 \leq i \leq N-1, \\ 0, & i = N, \end{cases}\end{aligned}$$

- The continuous model can be discretized as

$$\min_u \sum_{i,j} |\nabla u_{i,j} - \nabla h_{i,j}| + \frac{\lambda}{2} (u_{i,j} - g_{i,j})^2 + \chi_{[0,255]}(u).$$

- Applying the operator splitting, it is then equivalent to

$$\min_{u,d,v} \sum_{i,j} \left(|d_{i,j}| + \frac{\lambda}{2} (u_{i,j} - g_{i,j})^2 \right) + \chi_{[0,255]}(v),$$

subject to $d = \nabla u - \nabla h$ and $v = u$.

The Bregman iterations

- Introducing the penalty parameter $\gamma > 0$ and $\delta > 0$, we arrive at the following unconstrained minimization problem:

$$\begin{aligned} \min_{u,d,v} \sum_{i,j} \left(|d_{i,j}| + \frac{\lambda}{2} (u_{i,j} - g_{i,j})^2 + \frac{\gamma}{2} |d_{i,j} - \nabla u_{i,j} + \nabla h_{i,j} - b_{i,j}|^2 \right. \\ \left. + \frac{\delta}{2} (v_{i,j} - u_{i,j} - c_{i,j})^2 \right) + \chi_{[0,255]}(v), \end{aligned}$$

where b and c are the variables related to the Bregman iterations.

- The problem can be solved by alternating the search directions of u , d , and v , using the Bregman iteration.

The split Bregman iterations: 3 subproblems + 2 identities

- **u -subproblem:**

$$\begin{aligned} u^{n+1} = \arg \min_u \sum_{i,j} & \left(\frac{\lambda}{2} (u_{i,j} - g_{i,j})^2 + \frac{\gamma}{2} |d_{i,j}^n - (\nabla u)_{i,j} + (\nabla h)_{i,j} - b_{i,j}^n|^2 \right. \\ & \left. + \frac{\delta}{2} (v_{i,j}^n - u_{i,j} - c_{i,j}^n)^2 \right). \end{aligned}$$

- **d -subproblem:**

$$d^{n+1} = \arg \min_d \sum_{i,j} \left(|d_{i,j}| + \frac{\gamma}{2} |d_{i,j} - (\nabla u^{n+1})_{i,j} + (\nabla h)_{i,j} - b_{i,j}^n|^2 \right).$$

- **v -subproblem:**

$$v^{n+1} = \arg \min_v \sum_{i,j} \left(\frac{\delta}{2} (v_{i,j} - u_{i,j}^{n+1} - c_{i,j}^n)^2 \right) + \chi_S(v).$$

- **Bregman variables b and c :**

$$b^{n+1} = b^n + \nabla u^n - \nabla h - d^{n+1}, \quad c^{n+1} = c^n + u^{n+1} - v^{n+1}.$$

u-subproblem

u-subproblem:

$$\begin{aligned} u^{n+1} = \arg \min_u \sum_{i,j} \left(\frac{\lambda}{2} (u_{i,j} - g_{i,j})^2 + \frac{\gamma}{2} |d_{i,j}^n - (\nabla u)_{i,j} + (\nabla h)_{i,j} - b_{i,j}^n|^2 \right. \\ \left. + \frac{\delta}{2} (v_{i,j}^n - u_{i,j} - c_{i,j}^n)^2 \right). \end{aligned}$$

It can be viewed as the discretization of the minimization problem:

$$\begin{aligned} \min_u \frac{\lambda}{2} \int_{\Omega} (u - g)^2 dx + \frac{\gamma}{2} \int_{\Omega} |d - \nabla u + \nabla h - b|^2 dx \\ + \frac{\delta}{2} \int_{\Omega} (v - u - c)^2 dx. \end{aligned}$$

The EL equation of the above minimization problem is given by

$$(\lambda + \delta)u - \gamma \Delta u = \lambda g - \gamma (\operatorname{div}(d + \nabla h - b)) + \delta(v - c).$$

Note: $\frac{\partial L}{\partial u} - \nabla \cdot \left(\frac{\partial L}{\partial u_x}, \frac{\partial L}{\partial u_y} \right)^\top = 0$ in Ω , $\frac{\partial L}{\partial u_x} n_1 + \frac{\partial L}{\partial u_y} n_2 = 0$ on $\partial\Omega$.

u-subproblem (cont'd)

We obtain the discrete equations:

$$(\lambda + \delta)u_{i,j}^{n+1} - \gamma(\Delta u^{n+1})_{i,j} = \lambda g_{i,j} - \gamma(\operatorname{div}(d^n + \nabla h - b^n))_{i,j} + \delta(v_{i,j}^n - c_{i,j}^n).$$

The discrete operators div and Δ are defined as follows:

- Given $p = (p^1, p^2)$ with $p^1, p^2 \in \mathbb{R}^{N \times N}$, we define

$$(\operatorname{div} p)_{i,j} := (\nabla_x^- p^1)_{i,j} + (\nabla_y^- p^2)_{i,j} := (p_{i,j}^1 - p_{i,j-1}^1) + (p_{i,j}^2 - p_{i-1,j}^2).$$

- The discrete Laplacian is then defined as the composite of ∇ and div as $\Delta u := \operatorname{div}(\nabla u)$.
- Since the discretized problem produces a symmetric and diagonally dominant linear system, some iterative solvers such as Jacobi method or Gauss-Seidel method can be employed for efficiently solving u .

d-subproblem

d-subproblem:

$$d^{n+1} = \arg \min_d \sum_{i,j} \left(|d_{i,j}| + \frac{\gamma}{2} |d_{i,j} - (\nabla u^{n+1})_{i,j} + (\nabla h)_{i,j} - b_{i,j}^n|^2 \right).$$

The objective function is strictly convex and it has the following closed-form solution:

$$\begin{aligned} d_{i,j}^{n+1} &= \frac{(\nabla u^{n+1})_{i,j} - (\nabla h)_{i,j} + b_{i,j}^n}{|(\nabla u^{n+1})_{i,j} - (\nabla h)_{i,j} + b_{i,j}^n|} \\ &\quad \times \max \left\{ |(\nabla u^{n+1})_{i,j} - (\nabla h)_{i,j} + b_{i,j}^n| - \frac{1}{\gamma}, 0 \right\}. \end{aligned}$$

v-subproblem, Bregman variables, and initialization

***v*-subproblem:**

$$v^{n+1} = \arg \min_v \sum_{i,j} \left(\frac{\delta}{2} (v_{i,j} - u_{i,j}^{n+1} - c_{i,j}^n)^2 \right) + \chi_S(v).$$

For the *v*-subproblem, it can be solved by pixel-wise orthogonal projection of $u + c$ onto the predefined interval $S := [s_1, s_2]$

$$v_{i,j} = \min \left\{ \max \{u_{i,j} + c_{i,j}, s_1\}, s_2 \right\}.$$

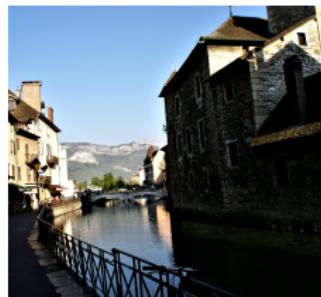
Note that we take $S = [s_1, s_2] := [0, 255]$.

Bregman variables *b* and *c*:

$$b^{n+1} = b^n + \nabla u^n - \nabla h - d^{n+1}, \quad c^{n+1} = c^n + u^{n+1} - v^{n+1}.$$

Initialization: $u = h, v = h, d = 0, b = 0, c = 0$.

Numerical experiments and comparisons



(T): f, u_{MPS}, u_{HE} (B): $u_{VCE}, u_{CLAHE}, u_{MLHE-HE}$

Numerical experiments and comparisons



(T): $u_{ACE}(\alpha = 2, 4, 6)$ (B): $u_{Adaptive}(\alpha = 0.8, 1.0, 1.2), \beta = 3\alpha$

Surprisingly, under the same parameter setting, the iteration number of our model is far less than that of the MPS model.

Numerical experiments and comparisons



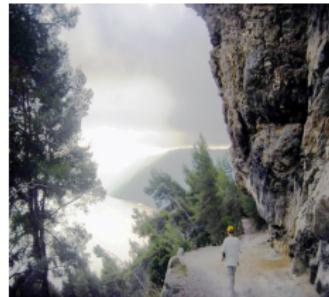
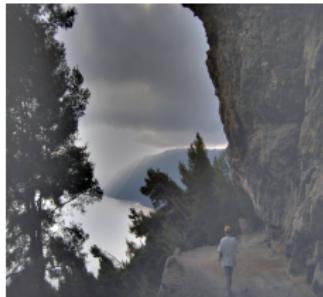
(T): f, u_{MPS}, u_{HE} (B): $u_{VCE}, u_{CLAHE}, u_{MLHE-HE}$

Numerical experiments and comparisons



(T): $u_{ACE}(\alpha = 2, 4, 6)$ (B): $u_{Adaptive}(\alpha = 0.8, 1.0, 1.2), \beta = 3\alpha$)

Numerical experiments and comparisons



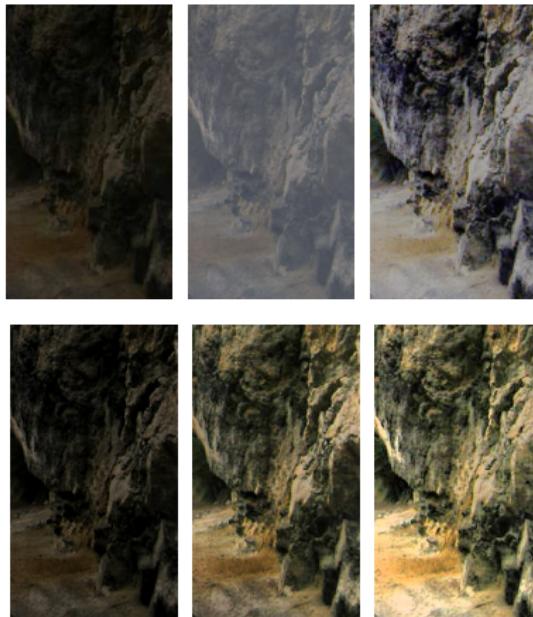
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Numerical experiments and comparisons



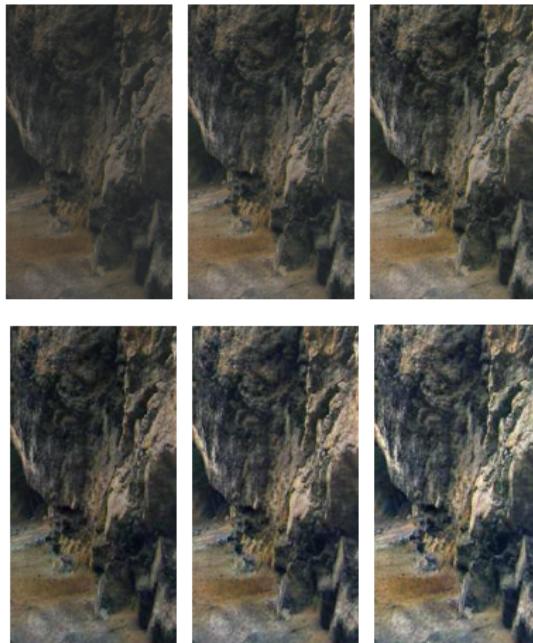
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Numerical experiments and comparisons



(T): f, u_{MPS}, u_{HE} (B): $u_{VCE}, u_{CLAHE}, u_{MLHE-HE}$

Numerical experiments and comparisons



(T): $u_{ACE}(\alpha = 2, 4, 6)$ (B): $u_{Adaptive}(\alpha = 0.8, 1.0, 1.2, \beta = 3\alpha)$

Numerical results of the proposed method



(T): low-light images (B): enhanced images

Numerical results of the proposed method



(T): low-light images (B): enhanced images

Summary of the proposed adaptive variational model

- ① We have proposed a simple and efficient adaptive variational model for image contrast enhancement.
- ② This model is designed for enhancing low-light images by dividing the image domain into bright and dim parts.
- ③ The existence and uniqueness of minimizer for the minimization problem is established, and a convergent algorithm is provided.
- ④ The most distinguished feature of our model is that colors are preserved as close as possible to the original ones.