MA 2008B: Linear Algebra II – Quiz #8

Name: **Student ID number:**

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(1) (5 pts) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x-y \\ 0 \end{bmatrix}$. Show that T is a linear transformation, and find the kernel and range of T.

Solution:

(a) T **is a linear transformation from** $\mathbb{R}^2$ **to** $\mathbb{R}^2$**:**

Let $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ and $c \in \mathbb{R}$. We have

$$\begin{aligned} T(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}) &= T(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}) = \begin{bmatrix} (x_1+x_2)-(y_1+y_2) \\ 0 \end{bmatrix} = \begin{bmatrix} x_1-y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2-y_2 \\ 0 \end{bmatrix} \\ &= T(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}) + T(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}), \\ T(c\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}) &= T(\begin{bmatrix} cx_1 \\ cy_1 \end{bmatrix}) = \begin{bmatrix} cx_1-cy_1 \\ 0 \end{bmatrix} = c\begin{bmatrix} x_1-y_1 \\ 0 \end{bmatrix} = cT(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}). \end{aligned}$$

Therefore, T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(b) **kernel and range of** T**:**

$$\begin{aligned} \text{kernel}(T) &= \{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : \ T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\} = \{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : \ \begin{bmatrix} x-y \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\} \\ &= \{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : \ x = y\} = \{\begin{bmatrix} x \\ x \end{bmatrix} \in \mathbb{R}^2 : \ x \in \mathbb{R}\}, \\ \text{range}(T) &= \{\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathbb{R}^2 : \ x \in \mathbb{R}\} \quad \because \ \forall \ \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathbb{R}^2, \text{ we have } T(\begin{bmatrix} x \\ 0 \end{bmatrix}) = \begin{bmatrix} x \\ 0 \end{bmatrix}. \end{aligned}$$

(2) (5 pts) Let U, V, W be three vector spaces over $\mathbb{R}$. Let $S : U \to V$ and $T : V \to W$ be two linear transformations. Define the composition $T \circ S : U \to W$ by $(T \circ S)(u) = T(S(u))$ for all $u \in U$. Show that $T \circ S$ is a linear transformation.

Solution:

Let $u_1, u_2 \in U$ and $c \in \mathbb{R}$.

Since $S : U \to V$ and $T : V \to W$ are linear transformations and $\text{range}(S) \subseteq V$ and $\text{domain}(T) = V$, we have

$$\begin{aligned} (T \circ S)(u_1+u_2) &= T(S(u_1+u_2)) = T(S(u_1)+S(u_2)) = T(S(u_1)) + T(S(u_2)) \\ &= (T \circ S)(u_1) + (T \circ S)(u_2), \\ (T \circ S)(cu_1) &= T(S(cu_1)) = T(cS(u_1)) = cT(S(u_1)) = c(T \circ S)(u_1). \end{aligned}$$

Therefore, $T \circ S$ is a linear transformation from U to W.