MA 2008B: Linear Algebra II – Quiz #6

Name: **Student ID number:**

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(1) (5 pts) Let $\boldsymbol{A} = \boldsymbol{R}^\top \boldsymbol{R}$, where

$$\boldsymbol{R} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Is $\boldsymbol{A}$ positive definite? Give your reason without using the eigenvalues of $\boldsymbol{A}$.

Solution: $\boldsymbol{A}$ is positive definite!

$$\boldsymbol{A} = \boldsymbol{R}^\top \boldsymbol{R} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} \text{ is symmetric.}$$

Method 1: Claim that the columns of $\boldsymbol{R}$ are linearly independent ($\Rightarrow$ $\boldsymbol{A}$ is PD)

Let $\alpha \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then $\alpha + \beta = 0, \alpha + 2\beta = 0, 2\alpha + \beta = 0$.

$\Rightarrow$ $\alpha = 0$ and $\beta = 0 \Rightarrow$ the columns of $\boldsymbol{R}$ are linearly independent $\Rightarrow$ $\boldsymbol{A}$ is PD

Method 2: Claim $\boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x} > 0, \forall\, \boldsymbol{x} = [x, y]^\top \in \mathbb{R}^2$ and $\boldsymbol{x} \neq \boldsymbol{0}$ ($\Rightarrow$ $\boldsymbol{A}$ is PD)

$$\boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 6x^2 + 10xy + 6y^2 = \cdots\cdots > 0.$$

(2) (5 pts) Consider the tilted ellipse

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1.$$

Find the directions of the axes of the tilted ellipse, and find their half-lengths.

Solution:

$$\because \det \begin{bmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{bmatrix} = \lambda^2 - 10\lambda + 9 = (\lambda - 9)(\lambda - 1) \quad \therefore \lambda_1 = 9, \lambda_2 = 1$$

$$\lambda_1 = 9: \quad \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \forall s \in \mathbb{R}, s \neq 0$$

$$\lambda_2 = 1: \quad \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \forall s \in \mathbb{R}, s \neq 0$$

$\therefore$ The axes point along $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$,

and the half-lengths are $\dfrac{1}{\sqrt{\lambda_1}} = \dfrac{1}{3}$ and $\dfrac{1}{\sqrt{\lambda_2}} = 1$