MA 2008B: Linear Algebra II – Quiz #5

Name: **Student ID number:**

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(1) (4 pts) Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric matrix. State the principal axis theorem.

Solution:

A can be factorized as $A = Q\Lambda Q^{-1}$, where Λ is a diagonal matrix with real eigenvalues of A and Q is an orthogonal matrix, $Q^\top Q = I$, with eigenvectors in its columns.

(2) (6 pts) Let A be the real symmetric 2×2 matrix,

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Find the eigenvalue matrix Λ and the eigenvector matrix Q of A.

Solution:

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3).$$

Then the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$.

$\lambda_1 = 1$:

$(A - I)x = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives unit eigenvector $x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(or $x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$).

$\lambda_2 = 3$:

$(A - 3I)x = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives unit eigenvector $x_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(or $x_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$).

Therefore, we have

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(other choices of Q are possible ....)