MA 2008B: Linear Algebra II – Quiz #4

Name: **Student ID number:**

Let $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Solve the following initial value problem

$$\begin{cases} \dfrac{d\boldsymbol{u}}{dt} = \boldsymbol{A}\boldsymbol{u}(t), \\ \boldsymbol{u}(0) = [4,2]^\top, \end{cases}$$

by using the eigenvalues and eigenvectors of $\boldsymbol{A}$.

Solution:

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1).$$

$\therefore$ eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = 1$ and $\lambda_2 = -1$

$$\lambda_1 = 1: \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\therefore$ eigenvectors of $\boldsymbol{A}$ are $s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for all scalar s. We take $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\lambda_2 = -1: \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\therefore$ eigenvectors of $\boldsymbol{A}$ are $s \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for all scalar s. We take $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\therefore$ The solutions of $\dfrac{d\boldsymbol{u}}{dt} = \boldsymbol{A}\boldsymbol{u}(t)$ are $Ce^{\lambda_1 t}\boldsymbol{x}_1 + De^{\lambda_2 t}\boldsymbol{x}_2 = Ce^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\because$ $\boldsymbol{u}(0) = [4,2]^\top$ $\quad \therefore \begin{cases} C + D = 4 \\ C - D = 2 \end{cases}$ $\quad \therefore$ $C = 3$ and $D = 1$

$\therefore$ The solution of the initial value problem is $\boldsymbol{u}(t) = 3e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$