MA 2008B: Linear Algebra II – Quiz #3

Name: **Student ID number:**

Let $\boldsymbol{A}$ be the 3×3 real matrix, $\boldsymbol{A} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

(1) (2 pts) Find the eigenvalues of $\boldsymbol{A}$.

Solution:

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \det \begin{bmatrix} 2-\lambda & 2 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{bmatrix} = (2-\lambda)^2(3-\lambda).$$

Let $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$. Then we have eigenvalues: $\lambda = 2, 2, 3$.

(2) (8 pts) For each eigenvalue of $\boldsymbol{A}$, find its algebraic multiplicity (AM) and geometric multiplicity (GM). Please show all your work clearly.

Solution:

(i) $\lambda = 2$: $AM = 2$. Solving

$$\begin{bmatrix} 2-\lambda & 2 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

we have $x = t$, $y = s$, and $z = -s$ for $t, s \in \mathbb{R}$. Therefore, the eigenvectors are given in the form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

We have two linearly independent eigenvectors that correspond to the eigenvalue $\lambda = 2$. Therefore, the geometric multiplicity of $\lambda = 2$ is $GM = 2$.

(ii) $\lambda = 3$: $AM = 1$. Solving

$$\begin{bmatrix} 2-\lambda & 2 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\Leftrightarrow \begin{bmatrix} -1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

we have $y = 0$, $z = t$, and $x = 2t$ for $t \in \mathbb{R}$. Therefore, the eigenvectors correspond to the eigenvalue $\lambda = 3$ in the form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

We have only one linearly independent eigenvector that corresponds to the eigenvalue $\lambda = 3$. Therefore, the geometric multiplicity of $\lambda = 3$ is $GM = 1$.