MA 2008B: Linear Algebra II – Quiz #2

Name: **Student ID number:**

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(1) Let $\boldsymbol{A}$ be the 3×3 Vandermonde matrix,

$$\boldsymbol{A} = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}.$$

Show that $\det \boldsymbol{A} = (b-a)(c-a)(c-b)$.

Solution:

$$\begin{aligned}
\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} &= \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} \quad \text{(by rule 5)} \\
&= \det \begin{bmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{bmatrix} \quad \text{(by cofactor formula)} \\
&= (b-a)(c-a)(c+a) - (c-a)(b-a)(b+a) \\
&= (b-a)(c-a)(c+a-b-a) \\
&= (b-a)(c-a)(c-b).
\end{aligned}$$

(2) Compute the determinants of $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Are their columns linearly independent? Please give your reasons?

Solution: By cofactor formula, we have

$$\det \boldsymbol{A} = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 1 \times \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - 1 \times \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = -1 - 1 = -2.$$

$\because$ $\det \boldsymbol{A} = -2 \neq 0$

$\therefore$ $\boldsymbol{A}$ is nonsingular

$\therefore$ the columns of $\boldsymbol{A}$ are linearly independent

$$\begin{aligned}
\det \boldsymbol{B} &= \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1 \times \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 4 \times \det \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix} + 7 \times \det \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \\
&= -3 + 24 - 21 = 0.
\end{aligned}$$

$\because$ $\det \boldsymbol{B} = 0$

$\therefore$ $\boldsymbol{B}$ is singular

$\therefore$ the columns of $\boldsymbol{B}$ are linearly dependent