MA 2008B: Linear Algebra II – Quiz #1

Name: **Student ID number:**

Let $\boldsymbol{A}$ be the 3×3 matrix,

$$\boldsymbol{A} = [\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}] = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}.$$

(1) Find the orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ from the independent vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ by using the Gram-Schmidt process.

Solution: By the Gram-Schmidt process, we have

$$\begin{aligned} A &:= \boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\ \Longrightarrow \boldsymbol{q}_1 &= \frac{A}{\|A\|} = \frac{1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \\ B &= \boldsymbol{b} - \frac{A^\top \boldsymbol{b}}{A^\top A} A = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \frac{2}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \\ \Longrightarrow \boldsymbol{q}_2 &= \frac{B}{\|B\|} = \frac{1}{3}\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \\ C &= \boldsymbol{c} - \frac{A^\top \boldsymbol{c}}{A^\top A} A - \frac{B^\top \boldsymbol{c}}{B^\top B} B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{4}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{18}{9}\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}, \\ \Longrightarrow \boldsymbol{q}_3 &= \frac{C}{\|C\|} = \frac{1}{5}\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \end{aligned}$$

(2) Find the $\boldsymbol{QR}$ factorization of matrix $\boldsymbol{A}$, that is, $\boldsymbol{A} = \boldsymbol{QR}$, where $\boldsymbol{Q} = [\boldsymbol{q}_1 \quad \boldsymbol{q}_2 \quad \boldsymbol{q}_3]$.

Solution: From problem (1), we obtain

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{QR} = [\boldsymbol{q}_1 \quad \boldsymbol{q}_2 \quad \boldsymbol{q}_3] \begin{bmatrix} \boldsymbol{q}_1^\top \boldsymbol{a} & \boldsymbol{q}_1^\top \boldsymbol{b} & \boldsymbol{q}_1^\top \boldsymbol{c} \\ 0 & \boldsymbol{q}_2^\top \boldsymbol{b} & \boldsymbol{q}_2^\top \boldsymbol{c} \\ 0 & 0 & \boldsymbol{q}_3^\top \boldsymbol{c} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}. \end{aligned}$$