MA2008B: LINEAR ALGEBRA II

Midterm2/May 21, 2020

Please show all your work clearly for full credit! total 100 points

1. (10 pts) Let $\boldsymbol{A}$ be an $n \times n$ square matrix. Assume that λ_1 and λ_2 are two distinct eigenvalues of $\boldsymbol{A}$. Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be two eigenvectors of $\boldsymbol{A}$ for λ_1 and λ_2, respectively. Show that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are linearly independent.

   Proof:

   Let $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = 0$. We wish to show that $c_1 = c_2 = 0$.

   $\because\ c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = 0 \qquad \therefore\ \boldsymbol{A}(c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2) = \boldsymbol{A}0 = 0$

   $\therefore\ c_1\boldsymbol{A}\boldsymbol{x}_1 + c_2\boldsymbol{A}\boldsymbol{x}_2 = 0 \qquad \therefore\ c_1\lambda_1\boldsymbol{x}_1 + c_2\lambda_2\boldsymbol{x}_2 = 0 \quad \cdots\cdots\ (\star)$

   On the other hand, $\because\ c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = 0 \qquad \therefore\ c_1\lambda_2\boldsymbol{x}_1 + c_2\lambda_2\boldsymbol{x}_2 = 0 \quad \cdots\cdots\ (\star\star)$

   $(\star) - (\star\star) \Longrightarrow c_1(\lambda_1 - \lambda_2)\boldsymbol{x}_1 = 0 \qquad \because\ \lambda_1 - \lambda_2 \neq 0$ and $\boldsymbol{x}_1 \neq 0 \qquad \therefore\ c_1 = 0$

   $\therefore\ c_2\boldsymbol{x}_2 = 0 \qquad \because\ \boldsymbol{x}_2 \neq 0 \qquad \therefore\ c_2 = 0$

2. Let $\boldsymbol{A}$ be the 3×3 real matrix, $\boldsymbol{A} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

   (2a) (5 pts) Find the eigenvalues of $\boldsymbol{A}$, and for each eigenvalue find its algebraic multiplicity (AM).

   Solution:

   $$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \det\begin{bmatrix} 2-\lambda & 2 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{bmatrix} = (2-\lambda)^2(3-\lambda).$$

   Let $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$. Then we have eigenvalues: $\lambda = 2, 2, 3$.

   $\therefore\ \lambda = 2$: $AM = 2$, $\quad \lambda = 3$: $AM = 1$.

   (2b) (10 pts) For each eigenvalue of $\boldsymbol{A}$, find its geometric multiplicity (GM).

   Solution:

   (i) $\lambda = 2$: Solving

   $$\begin{bmatrix} 2-\lambda & 2 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

   we have $x = t$, $y = s$, and $z = -s$ for $t, s \in \mathbb{R}$. Therefore, the eigenvectors are given in the form:

   $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

   We have two linearly independent eigenvectors that correspond to the eigenvalue $\lambda = 2$. Therefore, $GM = 2$ for $\lambda = 2$.

   (ii) $\lambda = 3$: Solving

   $$\begin{bmatrix} 2-\lambda & 2 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\Leftrightarrow \begin{bmatrix} -1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

we have $y = 0$, $z = t$, and $x = 2t$ for $t \in \mathbb{R}$. Therefore, the eigenvectors correspond to the eigenvalue $\lambda = 3$ in the form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

We have only one linearly independent eigenvector that corresponds to the eigenvalue $\lambda = 3$. Therefore, $GM = 1$ for $\lambda = 3$.

3. Consider the second-order differential equation with two initial values:

$$\begin{cases} y''(t) = -9y(t), \\ y(0) = 3 \quad \text{and} \quad y'(0) = 0. \end{cases}$$

(3a) (5 pts) Let $\boldsymbol{u}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$. Rewrite the above IVP as a vector equation:

$$\begin{cases} \boldsymbol{u}'(t) = \boldsymbol{A}\boldsymbol{u}(t), \\ \boldsymbol{u}(0) = [3, 0]^\top. \end{cases}$$

What is the 2×2 real matrix $\boldsymbol{A}$?

Solution:

$\because \dfrac{dy}{dt} = y'(t)$ and $\dfrac{dy'}{dt} = y''(t) = -9y(t)$

$\therefore \boldsymbol{u}'(t) = \begin{bmatrix} \dfrac{dy}{dt} \\ \dfrac{dy'}{dt} \end{bmatrix} = \begin{bmatrix} y'(t) \\ -9y(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \boldsymbol{A}\boldsymbol{u}(t) \quad \therefore \boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$

(3b) (10 pts) Find the solution $\boldsymbol{u}(t)$ of problem (3a) by using the eigenvalues and eigenvectors of matrix $\boldsymbol{A}$.

Solution:

$\because \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \det \begin{bmatrix} 0 - \lambda & 1 \\ -9 & 0 - \lambda \end{bmatrix} = \lambda^2 + 9.$

$\therefore$ eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = 3i,\ \lambda_2 = -3i$

$\lambda_1 = 3i : \begin{bmatrix} -3i & 1 \\ -9 & -3i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} -3i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ eigenvectors of λ_1 are $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s \begin{bmatrix} 1 \\ 3i \end{bmatrix}, \forall s.$ We take $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 3i \end{bmatrix}$.

$\lambda_2 = -3i : \begin{bmatrix} 3i & 1 \\ -9 & 3i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 3i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ eigenvectors of λ_2 are $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s \begin{bmatrix} 1 \\ -3i \end{bmatrix}, \forall s.$ We take $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -3i \end{bmatrix}$.

$\therefore$ the complete solution is $\boldsymbol{u}(t) = Ce^{3it} \begin{bmatrix} 1 \\ 3i \end{bmatrix} + De^{-3it} \begin{bmatrix} 1 \\ -3i \end{bmatrix}$

$\because \boldsymbol{u}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \therefore \begin{cases} C + D &= 3 \\ 3iC - 3iD &= 0 \end{cases} \quad \therefore C = \dfrac{3}{2}, D = \dfrac{3}{2}$

$\therefore \boldsymbol{u}(t) = \dfrac{3}{2} e^{3it} \begin{bmatrix} 1 \\ 3i \end{bmatrix} + \dfrac{3}{2} e^{-3it} \begin{bmatrix} 1 \\ -3i \end{bmatrix} = \begin{bmatrix} 3\cos(3t) \\ -9\sin(3t) \end{bmatrix}$

4. (15 pts) State without proof the *Principal Axis Theorem.* Find the symmetric diagonalization of

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Solution:

(5 pts) **Principal Axis Theorem:** Let A be an $n \times n$ real symmetric matrix. Then A can be factorized as $A = Q\Lambda Q^{-1}$, where Λ is a diagonal matrix with real eigenvalues of A and Q is an orthogonal matrix, $Q^\top Q = I$, with eigenvectors in its columns.

(10 pts) **Symmetric diagonalization:**

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3).$$

Then the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$.

$\lambda_1 = 1$:

$$(A - I)x = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{gives unit eigenvector } x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(or $x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$).

$\lambda_2 = 3$:

$$(A - 3I)x = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{gives unit eigenvector } x_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(or $x_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$).

Therefore, we have

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(other choices of Q are possible!)

5. (15 pts) Let A be an $n \times n$ real symmetric matrix. Prove that if A is positive definite (i.e., all eigenvalues of A are positive), then $x^\top Ax > 0$ for all $x \in \mathbb{R}^n$ and $x \neq \mathbf{0}$. (Hint: use the *Principal Axis Theorem*)

Proof:

$\because$ A is a real symmetric matrix

$\therefore$ By the Principal Axis Theorem stated in Problem (4), we have $A = Q\Lambda Q^\top$

Let $Q = [q_1, q_2, \cdots, q_n]$.

$\because$ $q_1, q_2, \cdots, q_n$ are orthonormal vectors

$\therefore$ They are a basis of R^n

Let $x \in \mathbb{R}^n$ and $x \neq \mathbf{0}$.

$$\text{Then } x = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n = Q \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} := Qc \text{ and } c \neq 0$$

$\therefore$ $x^\top Ax = c^\top Q^\top Q\Lambda Q^\top Qc = c^\top \Lambda c = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \cdots + \lambda_n c_n^2 > 0$

($\because$ $Q^\top Q = I$ and $\lambda_i > 0$ for all i and $c_i \neq 0$ for some i)

6. Let $\boldsymbol{M}$ be an $n \times n$ invertible matrix and $\boldsymbol{B} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$.

   (6a) (5 pts) Show that $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same characteristic polynomials.

   (6b) (10 pts) Show that if $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$, then $\boldsymbol{M}^{-1}\boldsymbol{x}$ is an eigenvector of $\boldsymbol{B}$.

   Proof:

   (6a) $\because\ \det(\boldsymbol{M}^{-1})\det(\boldsymbol{M}) = 1$
   $\therefore\ \det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \det(\boldsymbol{M}^{-1})\det(\boldsymbol{A} - \lambda\boldsymbol{I})\det(\boldsymbol{M}) = \det(\boldsymbol{M}^{-1}(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{M})$
   $= \det(\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M} - \lambda\boldsymbol{M}^{-1}\boldsymbol{I}\boldsymbol{M}) = \det(\boldsymbol{B} - \lambda\boldsymbol{I})$
   $\therefore$ $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same characteristic polynomials.

   (6b) $\because\ \boldsymbol{B} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$
   $\therefore\ \boldsymbol{A} = \boldsymbol{M}\boldsymbol{B}\boldsymbol{M}^{-1}$
   Suppose that $\boldsymbol{x} \neq 0$ is an eigenvector of $\boldsymbol{A}$ for eigenvalue λ.
   Then $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$.
   $\therefore\ \boldsymbol{M}\boldsymbol{B}\boldsymbol{M}^{-1}\boldsymbol{x} = \lambda\boldsymbol{x}$
   $\therefore\ \boldsymbol{B}\boldsymbol{M}^{-1}\boldsymbol{x} = \boldsymbol{M}^{-1}\lambda\boldsymbol{x}$
   $\therefore\ \boldsymbol{B}(\boldsymbol{M}^{-1}\boldsymbol{x}) = \lambda(\boldsymbol{M}^{-1}\boldsymbol{x})$
   $\because\ \boldsymbol{x} \neq 0 \qquad \therefore\ \boldsymbol{M}^{-1}\boldsymbol{x} \neq 0$ (otherwise $\boldsymbol{x} = 0$)
   $\therefore\ \boldsymbol{M}^{-1}\boldsymbol{x}$ is an eigenvector of $\boldsymbol{B}$

7. Let $\boldsymbol{A}$ be an $n \times n$ real matrix with rank $r < n$ and let $\boldsymbol{B}$ be a real matrix similar to $\boldsymbol{A}$.

   (7a) (5 pts) Explain why the dimension of the nullspace $N(\boldsymbol{A})$ is $n - r$.

   (7b) (10 pts) Show that if $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_{n-r}\}$ is a basis of $N(\boldsymbol{A})$, then $\{\boldsymbol{M}^{-1}\boldsymbol{x}_1, \boldsymbol{M}^{-1}\boldsymbol{x}_2, \cdots, \boldsymbol{M}^{-1}\boldsymbol{x}_{n-r}\}$ is a basis of $N(\boldsymbol{B})$.

   Proof:

   (7a) $\because$ By the FTLA-part 1, we have $n = \dim C(\boldsymbol{A}^\top) + \dim N(\boldsymbol{A}) = r + \dim N(\boldsymbol{A})$
   $\therefore\ \dim N(\boldsymbol{A}) = n - r$

   (7b) Claim: $\{\boldsymbol{M}^{-1}\boldsymbol{x}_1, \boldsymbol{M}^{-1}\boldsymbol{x}_2, \cdots, \boldsymbol{M}^{-1}\boldsymbol{x}_{n-r}\}$ is linearly independent
   Let $c_1\boldsymbol{M}^{-1}\boldsymbol{x}_1 + c_2\boldsymbol{M}^{-1}\boldsymbol{x}_2 + \cdots + c_{n-r}\boldsymbol{M}^{-1}\boldsymbol{x}_{n-r} = 0$.
   Then $\boldsymbol{M}^{-1}(c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_{n-r}\boldsymbol{x}_{n-r}) = 0$
   $\therefore\ c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_{n-r}\boldsymbol{x}_{n-r} = 0$
   $\because\ \{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_{n-r}\}$ is a basis of $N(\boldsymbol{A})$
   $\therefore\ c_1 = c_2 = \cdots = c_{n-r} = 0$

   Claim: $\{\boldsymbol{M}^{-1}\boldsymbol{x}_1, \boldsymbol{M}^{-1}\boldsymbol{x}_2, \cdots, \boldsymbol{M}^{-1}\boldsymbol{x}_{n-r}\}$ spans $N(\boldsymbol{B})$
   Let $\boldsymbol{x} \in N(\boldsymbol{B})$. Then $\boldsymbol{B}\boldsymbol{x} = 0$
   $\therefore\ \boldsymbol{B}\boldsymbol{x} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}\boldsymbol{x} = \boldsymbol{0}$
   $\therefore\ \boldsymbol{A}\boldsymbol{M}\boldsymbol{x} = \boldsymbol{0}$
   $\therefore\ \boldsymbol{M}\boldsymbol{x} \in N(\boldsymbol{A})$
   $\therefore\ \boldsymbol{M}\boldsymbol{x} = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_{n-r}\boldsymbol{x}_{n-r}$ for some $c_1, c_2, \cdots, c_{n-r} \in \mathbb{R}$
   $\therefore\ \boldsymbol{x} = c_1\boldsymbol{M}^{-1}\boldsymbol{x}_1 + c_2\boldsymbol{M}^{-1}\boldsymbol{x}_2 + \cdots + c_{n-r}\boldsymbol{M}^{-1}\boldsymbol{x}_{n-r}$