MA2008B: LINEAR ALGEBRA II

Midterm1/April 09, 2020

Please show all your work clearly for full credit! total 100 points

(1) Let $\boldsymbol{u}$ be any unit vector in $\mathbb{R}^n$. Then $\boldsymbol{Q} = \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^\top$ is a reflection matrix.

(a) (5 pts) Show that $\boldsymbol{Q}\boldsymbol{u} = -\boldsymbol{u}$, that is, the mirror is perpendicular to $\boldsymbol{u}$.

(b) (5 pts) Show that $\boldsymbol{Q}$ is symmetric.

(c) (5 pts) Show that $\boldsymbol{Q}$ is an orthogonal matrix.

Proof:

(a) $\boldsymbol{Q}\boldsymbol{u} = (\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^\top)\boldsymbol{u} = \boldsymbol{u} - (2\boldsymbol{u}\boldsymbol{u}^\top)\boldsymbol{u} = \boldsymbol{u} - 2\boldsymbol{u}(\boldsymbol{u}^\top\boldsymbol{u}) = \boldsymbol{u} - 2\boldsymbol{u} = -\boldsymbol{u}$.

(b) $\because\ \boldsymbol{Q}^\top = (\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^\top)^\top = \boldsymbol{I}^\top - 2(\boldsymbol{u}^\top)^\top\boldsymbol{u}^\top = \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^\top = \boldsymbol{Q}$

$\therefore\ \boldsymbol{Q}$ is symmetric.

(c) $\because\ \boldsymbol{Q}^\top\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q} = (\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^\top)(\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^\top)$

$= \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^\top - 2\boldsymbol{u}\boldsymbol{u}^\top + 4\boldsymbol{u}\boldsymbol{u}^\top\boldsymbol{u}\boldsymbol{u}^\top$

$= \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^\top - 2\boldsymbol{u}\boldsymbol{u}^\top + 4\boldsymbol{u}(\boldsymbol{u}^\top\boldsymbol{u})\boldsymbol{u}^\top$

$= \boldsymbol{I} - 4\boldsymbol{u}\boldsymbol{u}^\top + 4\boldsymbol{u}\boldsymbol{u}^\top = \boldsymbol{I}$

$\therefore\ \boldsymbol{Q}$ is an orthogonal matrix

(2) Let $\boldsymbol{A}$ be the 3×3 real matrix,

$$\boldsymbol{A} = [\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}] = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}.$$

(a) (5 pts) Use the Gram-Schmidt process to find the orthonormal vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$ from the independent vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

(b) (5 pts) Find the $\boldsymbol{QR}$ factorization of matrix $\boldsymbol{A}$, that is, $\boldsymbol{A} = \boldsymbol{QR}$.

(c) (5 pts) What is the projection of $[1, 1, 1]^\top$ onto the column space $C(\boldsymbol{A})$? Please give your reasons.

Solution:

(a) By the Gram-Schmidt process, we have

$$A := \boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \boldsymbol{q}_1 = \frac{A}{\|A\|} = \frac{1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$B = \boldsymbol{b} - \frac{A^\top\boldsymbol{b}}{A^\top A}A = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \frac{2}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \Longrightarrow \boldsymbol{q}_2 = \frac{B}{\|B\|} = \frac{1}{3}\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$C = \boldsymbol{c} - \frac{A^\top\boldsymbol{c}}{A^\top A}A - \frac{B^\top\boldsymbol{c}}{B^\top B}B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{4}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{18}{9}\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \boldsymbol{q}_3 = \frac{C}{\|C\|} = \frac{1}{5}\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

(b) From part (a), we obtain

$$A = QR = [q_1 \quad q_2 \quad q_3]\begin{bmatrix} q_1^\top a & q_1^\top b & q_1^\top c \\ 0 & q_2^\top b & q_2^\top c \\ 0 & 0 & q_3^\top c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}.$$

(c) <u>Method I:</u>

$\because\ C(A) = C(Q) = \mathbb{R}^3$

$\therefore$ the projection of the vector $[1,1,1]^\top$ onto $\mathbb{R}^3$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

<u>Method II:</u>

$\because\ C(A) = C(Q)$

$\therefore$ we want to find the projection of the vector $[1,1,1]^\top$ onto $C(Q)$

$\therefore$ the projection $p = Q\widehat{x}$ and $Q^\top Q\widehat{x} = Q^\top[1,1,1]^\top$

$\because\ Q^\top Q = I \implies \widehat{x} = Q^\top[1,1,1]^\top \implies p = QQ^\top[1,1,1]^\top$

$$p = q_1(q_1^\top \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) + q_2(q_2^\top \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) + q_3(q_3^\top \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = q_1 + q_2 + q_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(3) (10 pts) Let A be an $n \times n$ real matrix. Show that $\det(A^\top) = \det(A)$.

(Hint: consider GE and probably with row exchanges!)

Proof:

If A is singular, then $A^\top$ is also singular $\implies \det(A) = 0 = \det(A^\top)$.

Suppose A is nonsingular. By GE, we have $PA = LU$, where L is a lower triangular matrix with all diagonal entries 1, U is a upper triangular matrix, and P is a permutation matrix.

$\therefore\ (PA)^\top = (LU)^\top \implies A^\top P^\top = U^\top L^\top$

$\implies \det(P)\det(A) = \det(L)\det(U)$ and $\det(A^\top)\det(P^\top) = \det(U^\top)\det(L^\top)$

$\because\ \det(U) = \det(U^\top)$ ($\because$ have the same diagonal),

$\det(L) = \det(L^\top) = 1$ ($\because$ both have 1's on the diagonal),

$\therefore\ \det(P)\det(A) = \det(A^\top)\det(P^\top)$

$\because\ \det(P) = \det(P^\top)$ ($\because\ P^\top P = I \Rightarrow \det(P^\top P) = \det(P^\top)\det(P) = 1$

$\Rightarrow \det(P) = 1 = \det(P^\top)$ or $\det(P) = -1 = \det(P^\top)$)

$\therefore\ \det(A) = \det(A^\top)$

(4) (10 pts) Compute the determinants of A and B,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Are their columns linearly independent? Please give your reasons.

Solution: By cofactor formula, we have

$$\det A = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 1 \times \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - 1 \times \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = -1 - 1 = -2.$$

$\because \det \boldsymbol{A} = -2 \neq 0$

$\therefore$ $\boldsymbol{A}$ is nonsingular

$\therefore$ the columns of $\boldsymbol{A}$ are linearly independent

$$
\begin{aligned}
\det \boldsymbol{B} &= \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1 \times \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 4 \times \det \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix} + 7 \times \det \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \\
&= -3 + 24 - 21 = 0.
\end{aligned}
$$

$\because \det \boldsymbol{B} = 0$

$\therefore$ $\boldsymbol{B}$ is singular

$\therefore$ the columns of $\boldsymbol{B}$ are linearly dependent

(5) Let $\boldsymbol{A}$ be the tridiagonal matrix

$$
\boldsymbol{A} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{n \times n} .
$$

(a) (5 pts) Find an upper triangular matrix $\boldsymbol{U}$ by applying row operations to $\boldsymbol{A}$.

(b) (5 pts) Use part (a) to compute $\det(\boldsymbol{A})$.

(c) (5 pts) Let $D_n = \det(\boldsymbol{A})$. Use the cofactor formula to show that $D_n = 2D_{n-1} - D_{n-2}$.

Solution:

(a) See textbook, page 256, we have

$$
\boldsymbol{A} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & \ddots & \ddots & & \\ & & -\frac{n-2}{n-1} & 1 & \\ & & & -\frac{n-1}{n} & 1 \end{bmatrix}}_{\boldsymbol{L}} \underbrace{\begin{bmatrix} 2 & -1 & & & \\ & \frac{3}{2} & -1 & & \\ & & \ddots & \ddots & \\ & & & \frac{n}{n-1} & -1 \\ & & & & \frac{n+1}{n} \end{bmatrix}}_{\boldsymbol{U}} .
$$

(b) $\det(\boldsymbol{A}) = \det(\boldsymbol{L}) \times \det(\boldsymbol{U}) = (1) \times (2 \times \dfrac{3}{2} \times \cdots \times \dfrac{n+1}{n}) = n + 1.$

(c) By the cofactor formula, we have

$$
\begin{aligned}
D_n &= 2 \det \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{(n-1)\times(n-1)} - (-1) \det \begin{bmatrix} -1 & -1 & & & \\ & 2 & -1 & & \\ & & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{(n-1)\times(n-1)} \\
&= 2D_{n-1} + (-1) \det \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{(n-2)\times(n-2)} = 2D_{n-1} - D_{n-2}.
\end{aligned}
$$

(6) Consider the following linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$:

$$\begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) (5 pts) Use Cramer's rule to solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

(b) (5 pts) Find the inverse matrix $\boldsymbol{A}^{-1} = \dfrac{\boldsymbol{C}^\top}{\det \boldsymbol{A}}$, where $\boldsymbol{C}$ is the cofactor matrix.

Solution: (Please see textbook, page 278, 5.3 B, for details)

(a) By direct computations, we have

$$\det(\boldsymbol{A}) = \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} = 2,$$

$$\det B_1 = \det \begin{bmatrix} 0 & 6 & 2 \\ 0 & 4 & 2 \\ 1 & 9 & 0 \end{bmatrix} = 4, \quad \det B_2 = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 2 \\ 5 & 1 & 0 \end{bmatrix} = -2, \quad \det B_3 = \begin{bmatrix} 2 & 6 & 0 \\ 1 & 4 & 0 \\ 5 & 9 & 1 \end{bmatrix} = 2.$$

Therefore, $\boldsymbol{x} = [2, -1, 1]^\top$.

(b) By direct computations, we have

$$\boldsymbol{A}^{-1} = \frac{\boldsymbol{C}^\top}{\det \boldsymbol{A}} = \frac{1}{2} \begin{bmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{bmatrix}.$$

(7) Let

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

If there exists a nonzero vector $\boldsymbol{x}$ such that $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ for some scalar λ, then we say that $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ corresponding to the eigenvalue λ.

(a) (5 pts) Derive the characteristic polynomial $p(\lambda)$ of $\boldsymbol{A}$.

(b) (5 pts) Explain why $p(\lambda)$ is a polynomial in λ of degree exactly n with leading term $(-1)^n \lambda^n$.

(c) (5 pts) Assume that $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $\boldsymbol{A}$. Show that $\det(\boldsymbol{A}) = \lambda_1 \lambda_2 \cdots \lambda_n$. (Hint: use part (b).)

Solution:

(a) λ is an eigenvalue of $\boldsymbol{A}$ $\Leftrightarrow$ $\exists\, \boldsymbol{x} \neq \boldsymbol{0}$ such that $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ $\Leftrightarrow$ $\exists\, \boldsymbol{x} \neq \boldsymbol{0}$ such that $\boldsymbol{A}\boldsymbol{x} - \lambda \boldsymbol{I}\boldsymbol{x} = \boldsymbol{0}$ $\Leftrightarrow$ $\exists\, \boldsymbol{x} \neq \boldsymbol{0}$ such that $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$ $\Leftrightarrow$ $\boldsymbol{A} - \lambda \boldsymbol{I}$ is singular $\Leftrightarrow$ $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$

$p(\lambda) := \det(\boldsymbol{A} - \lambda \boldsymbol{I})$ is called the characteristic polynomial $p(\lambda)$ of $\boldsymbol{A}$.

(b) From part (a), we have

$$p(\lambda) := \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

By the cofactor formula, we have

$$\begin{aligned} p(\lambda) &= (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) + (\text{terms of degree} \leq n-2) \\ &= (-1)^n \lambda^n + \text{lower degree terms} \end{aligned}$$

(c) Since $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $\boldsymbol{A}$, i.e., zeros of $p(\lambda)$, and

$$p(\lambda) = (-1)^n \lambda^n + \text{lower degree terms},$$

we have

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda).$$

Then

$$\det(\boldsymbol{A}) = p(0) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

(8) (10 pts) Find the eigenvalues of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Is $\boldsymbol{A}$ nonsingular? Please give your reasons. (Hint: use Problem 7(c))

Solution:

$$\begin{aligned} \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \det \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} &= (1-\lambda)(2-\lambda)(1-\lambda) - 2(1-\lambda) \\ &= (1-\lambda)\big((2-\lambda)(1-\lambda) - 2\big) \\ &= (1-\lambda)(-\lambda)(3-\lambda). \end{aligned}$$

Therefore, the eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3$.

Since $\det(\boldsymbol{A}) = \lambda_1 \lambda_2 \lambda_3 = 0$, $\boldsymbol{A}$ is not nonsingular ($\boldsymbol{A}$ is singular).