MA2008B: LINEAR ALGEBRA II

Final Exam/June 23, 2020

Please show all your work clearly for full credit! (total 100 points)

(1) (10 pts) State without proof the *Singular Value Decomposition* (SVD) of real matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$.

Solution: Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ be a real matrix. Then there exist real orthogonal matrices $\boldsymbol{U}$ of size $m \times m$ and $\boldsymbol{V}$ of size $n \times n$ and a diagonal rectangular matrix $\boldsymbol{\Sigma}$ of size $m \times n$,

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ & & & & \\ \end{bmatrix}_{m\times n},$$

such that

$$\boldsymbol{A} = \boldsymbol{U\Sigma V}^\top,$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are called the singular values of $\boldsymbol{A}$ and r is the rank of $\boldsymbol{A}$.

(2) (10 pts) Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ be a real matrix. Show that all the eigenvalues of $\boldsymbol{A}^\top\boldsymbol{A}$ are real and nonnegative, and explain why if $\text{rank}(\boldsymbol{A}) = r$, then there are exactly r positive eigenvalues.

Proof:

$\because\ (\boldsymbol{A}^\top\boldsymbol{A})^\top = \boldsymbol{A}^\top(\boldsymbol{A}^\top)^\top = \boldsymbol{A}^\top\boldsymbol{A}$

$\therefore\ \boldsymbol{A}^\top\boldsymbol{A}$ is symmetric and then all eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are real.

If λ is an eigenvalue of $\boldsymbol{A}^\top\boldsymbol{A}$, then $\exists\ \boldsymbol{v} \neq \boldsymbol{0}$ (eigenvector of $\boldsymbol{A}^\top\boldsymbol{A}$) such that $\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$.

$\implies \boldsymbol{v}^\top\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}^\top\boldsymbol{v} = \lambda\|\boldsymbol{v}\|^2 \implies (\boldsymbol{A}\boldsymbol{v})^\top(\boldsymbol{A}\boldsymbol{v}) = \lambda\|\boldsymbol{v}\|^2 \implies \|\boldsymbol{A}\boldsymbol{v}\|^2 = \lambda\|\boldsymbol{v}\|^2$

$\implies \lambda = \dfrac{\|\boldsymbol{A}\boldsymbol{v}\|^2}{\|\boldsymbol{v}\|^2} \geq 0 \quad \therefore$ all the eigenvalues of $\boldsymbol{A}^\top\boldsymbol{A}$ are real and nonnegative!

(i) If $\lambda = 0$ then $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}$, where $\boldsymbol{v}$ is any eigenvector of $\boldsymbol{A}^\top\boldsymbol{A}$ corresponding to $\lambda \implies \boldsymbol{v} \in N(\boldsymbol{A})$

(ii) If $\lambda > 0$ then $\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v} \implies \boldsymbol{v} = \boldsymbol{A}^\top(\frac{1}{\lambda}\boldsymbol{A}\boldsymbol{v}) \implies \boldsymbol{v} \in C(\boldsymbol{A}^\top)$

(iii) If $\text{rank}(\boldsymbol{A}) = r$ then by FTLA-Part 1, we have $\dim C(\boldsymbol{A}^\top) + \dim N(\boldsymbol{A}) = r + (n - r)$.

(iv) By the Principal Axis Theorem for the symmetric matrix $\boldsymbol{A}^\top\boldsymbol{A}$, there are n orthonormal eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ such that $\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{v}_i = \lambda_i\boldsymbol{v}_i$ for $i = 1, 2, \cdots, n$.

Combining (i)-(iv), we can conclude that there are exactly r positive eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$ of the symmetric matrix $\boldsymbol{A}^\top\boldsymbol{A}$.

(3) (10 pts) Let $\sigma_1^2, \sigma_2^2, \cdots, \sigma_r^2$, with $\sigma_i > 0$, denote the r positive eigenvalues of $\boldsymbol{A}^\top\boldsymbol{A}$ in Problem (2) and let $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r$ denote the corresponding orthonormal eigenvectors. Show that $\boldsymbol{u}_i := \dfrac{\boldsymbol{A}\boldsymbol{v}_i}{\|\boldsymbol{A}\boldsymbol{v}_i\|}$ are eigenvectors of $\boldsymbol{A}\boldsymbol{A}^\top$ and $\boldsymbol{A}\boldsymbol{v}_i = \sigma_i\boldsymbol{u}_i$ for $i = 1, 2, \cdots, r$.

Proof:

$\because\ \boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{v}_i = \sigma_i^2\boldsymbol{v}_i$ for $i = 1, 2, \cdots, r$

$\therefore\ \boldsymbol{v}_i^\top\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{v}_i = \sigma_i^2\boldsymbol{v}_i^\top\boldsymbol{v}_i \implies \|\boldsymbol{A}\boldsymbol{v}_i\|^2 = \sigma_i^2\|\boldsymbol{v}_i\|^2 = \sigma_i^2 \implies 0 < \sigma_i = \|\boldsymbol{A}\boldsymbol{v}_i\|$

$\because\ \boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{v}_i = \sigma_i^2\boldsymbol{v}_i$ for $i = 1, 2, \cdots, r$

$\therefore\ \boldsymbol{A}\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{v}_i = \sigma_i^2\boldsymbol{A}\boldsymbol{v}_i$ and $\boldsymbol{A}\boldsymbol{v}_i \neq \boldsymbol{0}$ for $i = 1, 2, \cdots, r \Longrightarrow\ \boldsymbol{A}\boldsymbol{v}_i$ is an eigenvector of $\boldsymbol{A}\boldsymbol{A}^\top$ w.r.t. σ_i^2

$\Longrightarrow\ \boldsymbol{u}_i := \dfrac{\boldsymbol{A}\boldsymbol{v}_i}{\|\boldsymbol{A}\boldsymbol{v}_i\|}$ are eigenvectors of $\boldsymbol{A}\boldsymbol{A}^\top$ and $\boldsymbol{A}\boldsymbol{v}_i = \|\boldsymbol{A}\boldsymbol{v}_i\|\boldsymbol{u}_i = \sigma_i\boldsymbol{u}_i$ for $i = 1, 2, \cdots, r$

(4) (15 pts) Let $\boldsymbol{A} \in \mathbb{R}^{2\times 3}$ be the rectangular matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Find the singular value decomposition (SVD) of matrix $\boldsymbol{A}$: $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$.

Solution: Note that $r = \operatorname{rank}(\boldsymbol{A}) = 2$ and $\boldsymbol{A}^\top\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Find the eigenvalues of $\boldsymbol{A}^\top\boldsymbol{A}$:

$$\det(\boldsymbol{A}^\top\boldsymbol{A} - \lambda\boldsymbol{I}) = \det\begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} = \cdots\cdots = (1-\lambda)(\lambda^2 - 3\lambda).$$

Eigenvalues of $\boldsymbol{A}^\top\boldsymbol{A}$ are $\lambda_1 = 3$, $\lambda_2 = 1$, $\lambda_3 = 0 \Longrightarrow \sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$.

Eigenvectors for $\lambda_1 = 3$: $(\boldsymbol{A}^\top\boldsymbol{A} - \lambda_1\boldsymbol{I})\boldsymbol{x} = \begin{bmatrix} 1-3 & 1 & 0 \\ 1 & 2-3 & 1 \\ 0 & 1 & 1-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Longleftrightarrow \begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \therefore\ \boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Longrightarrow \boldsymbol{v}_1 := \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\Longrightarrow \boldsymbol{u}_1 = \frac{\boldsymbol{A}\boldsymbol{v}_1}{\sigma_1} = \frac{1}{\sqrt{3}}\begin{bmatrix} \frac{3}{\sqrt{6}} \\ \frac{3}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Eigenvectors for $\lambda_2 = 1$: $(\boldsymbol{A}^\top\boldsymbol{A} - \lambda_2\boldsymbol{I})\boldsymbol{x} = \begin{bmatrix} 1-1 & 1 & 0 \\ 1 & 2-1 & 1 \\ 0 & 1 & 1-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Longleftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \therefore\ \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Longrightarrow \boldsymbol{v}_2 := \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$\Longrightarrow \boldsymbol{u}_2 = \frac{\boldsymbol{A}\boldsymbol{v}_2}{\sigma_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Find a basis vector $\boldsymbol{v}_3$ of the nullspace of $\boldsymbol{A}$: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \Longleftrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Longrightarrow \boldsymbol{v}_3 := \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\therefore\ \boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top \Longleftrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

(5) (15 pts) Let $\boldsymbol{A} := [a_{ij}]$ be a 3×3 invertible real matrix. Define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$, $\forall\ \boldsymbol{v} \in \mathbb{R}^3$. Show that the matrix representation of T with respect to the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ for both $\mathbb{R}^3$ is $\boldsymbol{A}$, and show that if $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent then $T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), T(\boldsymbol{v}_3)$ are linearly independent.

Solution:

Matrix representation of T:

$$
\begin{aligned}
T(\boldsymbol{e}_1) &= \boldsymbol{A}\boldsymbol{e}_1 = \boldsymbol{A}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = a_{11}\boldsymbol{e}_1 + a_{21}\boldsymbol{e}_2 + a_{31}\boldsymbol{e}_3, \\
T(\boldsymbol{e}_2) &= \boldsymbol{A}\boldsymbol{e}_2 = \boldsymbol{A}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = a_{12}\boldsymbol{e}_1 + a_{22}\boldsymbol{e}_2 + a_{32}\boldsymbol{e}_3, \\
T(\boldsymbol{e}_3) &= \boldsymbol{A}\boldsymbol{e}_3 = \boldsymbol{A}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = a_{13}\boldsymbol{e}_1 + a_{23}\boldsymbol{e}_2 + a_{33}\boldsymbol{e}_3.
\end{aligned}
$$

Therefore, the matrix representation of T with respect to the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ for both $\mathbb{R}^3$ is

$$
\boldsymbol{M} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \boldsymbol{A}.
$$

linearly independent: Let

$$
c_1 T(\boldsymbol{v}_1) + c_2 T(\boldsymbol{v}_2) + c_3 T(\boldsymbol{v}_3) = \boldsymbol{0}.
$$

Then

$$
\begin{aligned}
T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3) = \boldsymbol{0} &\iff \boldsymbol{A}(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3) = \boldsymbol{0} \\
&\iff \boldsymbol{A}^{-1}\boldsymbol{A}(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3) = \boldsymbol{A}^{-1}\boldsymbol{0} = \boldsymbol{0}
\end{aligned}
$$

Therefore,

$$
c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \boldsymbol{0}.
$$

Since $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent, we have $c_1 = c_2 = c_3 = 0$. Therefore, $T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), T(\boldsymbol{v}_3)$ are linearly independent.

(6) (15 pts) Let $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ in Problem (5). Find the matrix representation $\boldsymbol{B}$ of the linear transformation T with respect to the basis $\{\boldsymbol{e}_2, \boldsymbol{e}_1, \boldsymbol{e}_3\}$ for both $\mathbb{R}^3$ and show that the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar.

Solution:

Matrix representation of T:

$$
\begin{aligned}
T(\boldsymbol{e}_2) &= \boldsymbol{A}\boldsymbol{e}_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = 2\boldsymbol{e}_2 + (-1)\boldsymbol{e}_1 + (-1)\boldsymbol{e}_3, \\
T(\boldsymbol{e}_1) &= \boldsymbol{A}\boldsymbol{e}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = (-1)\boldsymbol{e}_2 + 2\boldsymbol{e}_1 + 0\boldsymbol{e}_3, \\
T(\boldsymbol{e}_3) &= \boldsymbol{A}\boldsymbol{e}_3 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = (-1)\boldsymbol{e}_2 + 0\boldsymbol{e}_1 + 2\boldsymbol{e}_3.
\end{aligned}
$$

Therefore, the matrix representation $\boldsymbol{B}$ of the linear transformation T with respect to the basis $\{\boldsymbol{e}_2, \boldsymbol{e}_1, \boldsymbol{e}_3\}$ for both $\mathbb{R}^3$ is

$$\boldsymbol{B} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

***A* and *B* are similar:** Define two identity transformations:

$$I_1 : \mathbb{R}^3 \to \mathbb{R}^3, \qquad I_1(\boldsymbol{v}) = \boldsymbol{v}, \forall\, \boldsymbol{v} \in \mathbb{R}^3,$$
$$I_2 : \mathbb{R}^3 \to \mathbb{R}^3, \qquad I_2(\boldsymbol{v}) = \boldsymbol{v}, \forall\, \boldsymbol{v} \in \mathbb{R}^3.$$

Then the change of basis matrix of I_2 with respect to basis $\{\boldsymbol{e}_2, \boldsymbol{e}_1, \boldsymbol{e}_3\}$ for the domain $\mathbb{R}^3$ and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ for the range $\mathbb{R}^3$ is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} := \boldsymbol{M}.$$

The change of basis matrix of I_1 with respect to basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ for the domain $\mathbb{R}^3$ and $\{\boldsymbol{e}_2, \boldsymbol{e}_1, \boldsymbol{e}_3\}$ for the range $\mathbb{R}^3$ is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{M} = \boldsymbol{M}^{-1}.$$

Note that

$$(I_1 \circ T \circ I_2)(\boldsymbol{v}) = (I_1 \circ T)(I_2(\boldsymbol{v})) = (I_1 \circ T)(\boldsymbol{v}) = I_1(T(\boldsymbol{v})) = T(\boldsymbol{v}),\ \forall\, \boldsymbol{v} \in \mathbb{R}^3.$$

Therefore, we have $I_1 \circ T \circ I_2 = T$ and then $\boldsymbol{B} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$. That is, $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar.

(7) (10 pts) Let $\mathcal{P}_n$ denote the vector space of all real-coefficient polynomials with degree less than or equal to n. Define the "derivative transformation" $T : \mathcal{P}_4 \to \mathcal{P}_3$ by $T(\boldsymbol{v}) = \dfrac{d\boldsymbol{v}}{dx}$. Show that T is a linear transformation and find the matrix representation of T with respect to the basis $\{1, x, x^2, x^3, x^4\}$ for $\mathcal{P}_4$ and the basis $\{1, x, x^2, x^3\}$ for $\mathcal{P}_3$.

Solution:

***T* is a linear transformation:** Let $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{P}_4$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} T(\boldsymbol{v} + \boldsymbol{w}) &= \frac{d(\boldsymbol{v} + \boldsymbol{w})}{dx} = \frac{d\boldsymbol{v}}{dx} + \frac{d\boldsymbol{w}}{dx} = T(\boldsymbol{v}) + T(\boldsymbol{w}), \\ T(\alpha\boldsymbol{v}) &= \frac{d(\alpha\boldsymbol{v})}{dx} = \alpha\frac{d\boldsymbol{v}}{dx} = \alpha T(\boldsymbol{v}). \end{aligned}$$

Therefore, T is a linear transformation.

Matrix representation of *T*:

$$\begin{aligned} T(1) &= 0 = 0 \times 1 + 0 \times x + 0 \times x^2 + 0 \times x^3, \\ T(x) &= 1 = 1 \times 1 + 0 \times x + 0 \times x^2 + 0 \times x^3, \\ T(x^2) &= 2x = 0 \times 1 + 2 \times x + 0 \times x^2 + 0 \times x^3, \\ T(x^3) &= 3x^2 = 0 \times 1 + 0 \times x + 3 \times x^2 + 0 \times x^3, \\ T(x^4) &= 4x^3 = 0 \times 1 + 0 \times x + 0 \times x^2 + 4 \times x^3. \end{aligned}$$

Therefore, the matrix representation $\boldsymbol{A}$ of T with respect to the basis $\{1, x, x^2, x^3, x^4\}$ for $\mathcal{P}_4$ and the basis $\{1, x, x^2, x^3\}$ for $\mathcal{P}_3$ is

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

(8) (15 pts) Let $w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $w_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$, $w_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, and $w_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$. Then $\{w_1, w_2, w_3, w_4\}$ is the Harr wavelet basis for $\mathbb{R}^4$ and for any $v \in \mathbb{R}^4$ there exists a unique $c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \in \mathbb{R}^4$ such that $v = c_1 w_1 + c_2 w_2 + c_3 w_3 + c_4 w_4$. Define a function $T : \mathbb{R}^4 \to \mathbb{R}^4$ by $T(v) = c$. Show that T is a linear transformation and find the matrix representation of T with respect to the standard basis for both $\mathbb{R}^4$.

Solution:

T **is a linear transformation:** Let $v, w \in \mathbb{R}^4$, $\alpha \in \mathbb{R}$. Assume that

$$\begin{aligned} v &= c_1 w_1 + c_2 w_2 + c_3 w_3 + c_4 w_4, \\ w &= d_1 w_1 + d_2 w_2 + d_3 w_3 + d_4 w_4, \end{aligned}$$

Then

$$T(v) = c := \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \quad \text{and} \quad T(w) = d := \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix},$$

and we have

$$\begin{aligned} T(v + w) &= T\big((c_1 + d_1)w_1 + (c_2 + d_2)w_2 + (c_3 + d_3)w_3 + (c_4 + d_4)w_4\big) \\ &= \begin{bmatrix} c_1 + d_1 \\ c_2 + d_2 \\ c_3 + d_3 \\ c_4 + d_4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = c + d = T(v) + T(w), \\ T(\alpha v) &= T\big(\alpha(c_1 w_1 + c_2 w_2 + c_3 w_3 + c_4 w_4)\big) = T\big(\alpha c_1 w_1 + \alpha c_2 w_2 + \alpha c_3 w_3 + \alpha c_4 w_4\big) \\ &= \begin{bmatrix} \alpha c_1 \\ \alpha c_2 \\ \alpha c_3 \\ \alpha c_4 \end{bmatrix} = \alpha \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \alpha c = \alpha T(v). \end{aligned}$$

Therefore, T is a linear transformation.

Matrix representation of T**:** Let $v \in \mathbb{R}^4$ and $v = c_1 w_1 + c_2 w_2 + c_3 w_3 + c_4 w_4 := Wc$, where

$$W = [w_1, w_2, w_3, w_4].$$

Then

$$T(v) = c = W^{-1} v$$

Similar to Problem (5), the matrix representation of T with respect to the standard basis for both $\mathbb{R}^4$ is W^{-1}. Since w_1, w_2, w_3, w_4 are orthogonal, we have

$$W^{-1} = \begin{bmatrix} 1/4 & & & \\ & 1/4 & & \\ & & 1/2 & \\ & & & 1/2 \end{bmatrix} W^\top = \begin{bmatrix} 1/4 & & & \\ & 1/4 & & \\ & & 1/2 & \\ & & & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$