MA 2007B: Linear Algebra I – Quiz #7

Name: **Student ID number:**

---

(1) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be an invertible matrix. Show that the columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

Proof: Let $\boldsymbol{A} = [A_1, A_2, \cdots, A_n]$. Then

- $A_1, A_2, \cdots, A_n$ are linearly independent:
  Assume that $x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = \mathbf{0}$.
  Then $\boldsymbol{Ax} = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = \mathbf{0}$.
  $\therefore \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{Ax} = \boldsymbol{A}^{-1}\mathbf{0} = \mathbf{0}$
- $A_1, A_2, \cdots, A_n$ span $\mathbb{R}^n$: Let $\boldsymbol{b} \in \mathbb{R}^n$.
  $\because \boldsymbol{A}$ is invertible
  $\therefore \exists\, \boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{Ax} = \boldsymbol{b}$
  $\therefore A_1, A_2, \cdots, A_n$ span $\mathbb{R}^n$

(2) Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ be a real matrix. Assume that $\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n]$, where $\boldsymbol{a}_i \in \mathbb{R}^m$ for $i = 1, 2, \cdots, n$, and $\boldsymbol{A}^\top = [\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_m]$, where $\boldsymbol{b}_j \in \mathbb{R}^n$ for $j = 1, 2, \cdots, m$. Please find the four subspaces $C(\boldsymbol{A})$, $C(\boldsymbol{A}^\top)$, $N(\boldsymbol{A})$, $N(\boldsymbol{A}^\top)$, and state the Fundamental Theorem of Linear Algebra, Part I.

Solution:

$$
\begin{aligned}
C(\boldsymbol{A}) &= \{c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + \cdots + c_n\boldsymbol{a}_n | c_i \in \mathbb{R},\ \forall\, 1 \le i \le n\}. \\
C(\boldsymbol{A}^\top) &= \{c_1\boldsymbol{b}_1 + c_2\boldsymbol{b}_2 + \cdots + c_m\boldsymbol{b}_m | c_i \in \mathbb{R},\ \forall\, 1 \le i \le m\}. \\
N(\boldsymbol{A}) &= \{\boldsymbol{x} \in \mathbb{R}^n |\ \boldsymbol{Ax} = \mathbf{0}\}. \\
N(\boldsymbol{A}^\top) &= \{\boldsymbol{y} \in \mathbb{R}^m |\ \boldsymbol{A}^\top\boldsymbol{y} = \mathbf{0}\}.
\end{aligned}
$$

Fundamental Theorem of Linear Algebra, Part I:

- $\dim C(\boldsymbol{A}^\top) + \dim N(\boldsymbol{A}) = r + (n-r) = n = \dim \mathbb{R}^n$.
  i.e., The row space $C(\boldsymbol{A}^\top)$ has dimension r
  and the nullspce $N(\boldsymbol{A})$ has dimension $n - r$.
- $\dim C(\boldsymbol{A}) + \dim N(\boldsymbol{A}^\top) = r + (m-r) = m = \dim \mathbb{R}^m$.
  i.e., The column space $C(\boldsymbol{A})$ has dimension r
  and the left nullspce $N(\boldsymbol{A}^\top)$ has dimension $m - r$.