MA 2007B: Linear Algebra I – Quiz #5

Name: **Student ID number:**

Let $\mathcal{P}_2$ be the set of all real coefficient polynomials of degree less than or equal to 2, i.e.,

$$\mathcal{P}_2 := \{a_0 + a_1 x + a_2 x^2 |\ a_0, a_1, a_2 \in \mathbb{R}\}.$$

Let $p(x) = a_0 + a_1 x + a_2 x^2$, $q(x) = b_0 + b_1 x + b_2 x^2 \in \mathcal{P}_2$ and $c \in \mathbb{R}$. Define two closed operations $+$ and $\cdot$ by

$$\begin{aligned} p(x) + q(x) &:= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \in \mathcal{P}_2, \\ cp(x) &:= ca_0 + ca_1 x + ca_2 x^2 \in \mathcal{P}_2. \end{aligned}$$

Show that $\mathcal{P}_2$ is a vector space over $\mathbb{R}$.

Proof:

We already have two closed operations $+$ and $\cdot$ defined on $\mathcal{P}_2$. We check the 8 conditions as follows:

(1) $\forall\ p(x) = a_0 + a_1 x + a_2 x^2, q(x) = b_0 + b_1 x + b_2 x^2 \in \mathcal{P}_2$, we have

$$\begin{aligned} p(x) + q(x) &= (a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \\ &= (b_0 + a_0) + (b_1 + a_1)x + (b_2 + a_2)x^2 \\ &= (b_0 + b_1 x + b_2 x^2) + (a_0 + a_1 x + a_2 x^2) = q(x) + p(x). \end{aligned}$$

(2) $\forall\ p(x) = a_0 + a_1 x + a_2 x^2, q(x) = b_0 + b_1 x + b_2 x^2, r(x) = c_0 + c_1 x + c_2 x^2 \in \mathcal{P}_2$, we have

$$\begin{aligned} p(x) + (q(x) + r(x)) &= (a_0 + a_1 x + a_2 x^2) + \big((b_0 + c_0) + (b_1 + c_1)x + (b_2 + c_2)x^2\big) \\ &= a_0 + (b_0 + c_0) + (a_1 + (b_1 + c_1))x + (a_2 + (b_2 + c_2))x^2 \\ &= (a_0 + b_0) + c_0 + ((a_1 + b_1) + c_1)x + ((a_2 + b_2) + c_2)x^2 \\ &= \big((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2\big) + (c_0 + c_1 x + c_2 x^2) \\ &= (p(x) + q(x)) + r(x). \end{aligned}$$

(3) Let $p(x) \equiv 0 \in \mathcal{P}_2$. Then $\forall\ q(x) = b_0 + b_1 x + b_2 x^2 \in \mathcal{P}_2$, we have

$$q(x) + p(x) = (b_0 + 0) + (b_1 + 0)x + (b_2 + 0)x^2 = b_0 + b_1 x + b_2 x^2 = q(x).$$

Now, assume that $p(x) = a_0 + a_1 x + a_2 x^2 \in \mathcal{P}_2$ such that $p(x) + q(x) = q(x)$ $\forall\ q(x) = b_0 + b_1 x + b_2 x^2 \in \mathcal{P}_2$. Then

$$q(x) + p(x) = (b_0 + a_0) + (b_1 + a_1)x + (b_2 + a_2)x^2 = q(x) = b_0 + b_1 x + b_2 x^2.$$

We obtain $a_0 = a_1 = a_2 = 0$, that is, $p(x) \equiv 0$.

Therefore $p(x) \equiv 0$ is the unique zero vector!

(4) $\forall\ p(x) = a_0 + a_1 x + a_2 x^2 \in \mathcal{P}_2$, define the vector

$$-p(x) := (-a_0) + (-a_1)x + (-a_2)x^2.$$

Then $-p(x) \in \mathcal{P}_2$ and

$$\begin{aligned} p(x) + (-p(x)) &= (a_0 + (-a_0)) + (a_1 + (-a_1))x + (a_2 + (-a_2))x^2 \\ &= 0 + 0x + 0x^2 \equiv 0. \end{aligned}$$

Similar to the proof of (3), one can further show that the $-p(x)$ is unique!

(5) $\forall\ p(x) = a_0 + a_1 x + a_2 x^2 \in \mathcal{P}_2$, we have

$$1 \cdot p(x) = 1 \cdot a_0 + 1 \cdot a_1 x + 1 \cdot a_2 x^2 = a_0 + a_1 x + a_2 x^2 = p(x).$$

(6) $\forall\ p(x) = a_0 + a_1 x + a_2 x^2 \in \mathcal{P}_2$ and $c_1, c_2 \in \mathbb{R}$, we have

$$\begin{aligned} c_1(c_2 p(x)) &= c_1(c_2 a_0 + c_2 a_1 x + c_2 a_2 x^2) = c_1 c_2 a_0 + c_1 c_2 a_1 x + c_1 c_2 a_2 x^2 \\ &= (c_1 c_2) a_0 + (c_1 c_2) a_1 x + (c_1 c_2) a_2 x^2 = (c_1 c_2) p(x). \end{aligned}$$

(7) $\forall\ p(x) = a_0 + a_1 x + a_2 x^2, q(x) = b_0 + b_1 x + b_2 x^2 \in \mathcal{P}_2$, $c \in \mathbb{R}$, we have

$$\begin{aligned} c(p(x) + q(x)) &= c((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2) \\ &= (ca_0 + cb_0) + (ca_1 + cb_1)x + (ca_2 + cb_2)x^2 \\ &= (ca_0 + ca_1 x + ca_2 x^2) + (cb_0 + cb_1 x + cb_2 x^2) \\ &= cp(x) + cq(x) \end{aligned}$$

(8) $\forall\ p(x) = a_0 + a_1 x + a_2 x^2 \in \mathcal{P}_2$, $c_1, c_2 \in \mathbb{R}$, we have

$$\begin{aligned} (c_1 + c_2)p(x) &= (c_1 + c_2)a_0 + (c_1 + c_2)a_1 x + (c_1 + c_2)a_2 x^2 \\ &= (c_1 a_0 + c_1 a_1 x + c_1 a_2 x^2) + (c_2 a_0 + c_2 a_1 x + c_2 a_2 x^2) \\ &= c_1 p(x) + c_2 p(x). \end{aligned}$$