MA 2007B: Linear Algebra I – Quiz #4

Name: **Student ID number:**

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(1) (5 pts) Using elimination, we can factor the matrix $\boldsymbol{A}$ into $\boldsymbol{LU}$, i.e., $\boldsymbol{A} = \boldsymbol{LU}$,

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}.$$

Find the solution of the following linear system $\boldsymbol{Ax} = \boldsymbol{b}$ by solving two triangular systems associated with $\boldsymbol{L}$ and $\boldsymbol{U}$:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Solution: First, we solve the lower triangular system $\boldsymbol{Lc} = \boldsymbol{b}$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

By forward substitution, we have

$$c_1 = 1 \Longrightarrow c_2 = \frac{1}{2}c_1 = \frac{1}{2} \Longrightarrow c_3 = \frac{2}{3}c_2 = \frac{1}{3} \Longrightarrow c_4 = 1 + \frac{3}{4}c_3 = \frac{5}{4}.$$

Next, we solve the upper triangular system $\boldsymbol{Ux} = \boldsymbol{c}$,

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 5/4 \end{bmatrix}.$$

By backward substitution, we have $x_4 = 1 \Longrightarrow x_3 = 1 \Longrightarrow x_2 = 1 \Longrightarrow x_1 = 1$.

(2) (5 pts) Assume that $\boldsymbol{A}^{-1}$ exists. Show that $\boldsymbol{A}^\top$ is invertible and $(\boldsymbol{A}^\top)^{-1} = (\boldsymbol{A}^{-1})^\top$.

Proof:

(i) $\because\ \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I} \quad \therefore\ (\boldsymbol{A}^{-1}\boldsymbol{A})^\top = \boldsymbol{I}^\top = \boldsymbol{I} \quad \therefore\ \boldsymbol{A}^\top(\boldsymbol{A}^{-1})^\top = \boldsymbol{I}$

(ii) $\because\ \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I} \quad \therefore\ (\boldsymbol{A}\boldsymbol{A}^{-1})^\top = \boldsymbol{I}^\top = \boldsymbol{I} \quad \therefore\ (\boldsymbol{A}^{-1})^\top\boldsymbol{A}^\top = \boldsymbol{I}$

By (i) (ii), $\boldsymbol{A}^\top$ is invertible and $(\boldsymbol{A}^\top)^{-1} = (\boldsymbol{A}^{-1})^\top$.