MA 2007B: Linear Algebra I – Quiz #1

Name: **Student ID number:**

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(1) (5 points) Let $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $w = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \in \mathbb{R}^2$. Find $\|v\|$, $\|w\|$, $\|v+w\|$, $v \cdot w$, and $\cos\theta$, where θ is the angle between column vectors v and w.

Solution:

$\|v\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$

$\|w\| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5.$

$\|v + w\| = \left\| \begin{bmatrix} 3+4 \\ 4+3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 7 \\ 7 \end{bmatrix} \right\| = \sqrt{7^2 + 7^2} = 7\sqrt{2}.$

$v \cdot w = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 12 + 12 = 24.$

$\cos\theta = \dfrac{v \cdot w}{\|v\|\|w\|} = \dfrac{24}{25}.$

(2) (5 points) (i) State the Cauchy-Schwarz-Buniakowsky inequality for vectors in $\mathbb{R}^n$ and then (ii) use the CSB inequality to prove that

$$\|v + w\| \leq \|v\| + \|w\|, \quad \forall\, v, w \in \mathbb{R}^n.$$

Solution:

(i). Cauchy-Schwarz-Buniakowsky inequality:

Let $v, w \in \mathbb{R}^n$. Then $|v \cdot w| \leq \|v\|\|w\|$.

(ii). *Proof:* For any $v, w \in \mathbb{R}^n$, we have

$$\begin{aligned} \|v + w\|^2 &= (v + w) \cdot (v + w) \\ &= v \cdot v + v \cdot w + w \cdot v + w \cdot w \\ &= v \cdot v + 2v \cdot w + w \cdot w \\ &= \|v\|^2 + 2v \cdot w + \|w\|^2. \end{aligned}$$

By the Cauchy-Schwarz-Buniakowsky inequality, we have

$$\begin{aligned} \|v + w\|^2 &\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 \\ &= (\|v\| + \|w\|)^2. \end{aligned}$$

Therefore, we obtain

$$\|v + w\| \leq \|v\| + \|w\|. \qquad \square$$