MA2007B: LINEAR ALGEBRA I

Midterm2/December 05, 2019

Please show all your work clearly for full credit! each problem 10 points, total 100 points)

(1) Find the symmetric factorization $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^\top$ of the 3×3 real symmetric matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}.$$

Solution: By the elimination, we have

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix} \xrightarrow{\ell_{21}=4,\ \ell_{31}=5} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -14 & -14 \\ 0 & -14 & -22 \end{bmatrix} \xrightarrow{\ell_{32}=1} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -14 & -14 \\ 0 & 0 & -8 \end{bmatrix} = \boldsymbol{U}.$$

Therefore, we obtain the symmetric factorization

$$\begin{aligned} \boldsymbol{A} = \boldsymbol{L}\boldsymbol{U} &= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -14 & -14 \\ 0 & 0 & -8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^\top. \end{aligned}$$

(2) Show that if an $n \times n$ real matrix $\boldsymbol{A}$ is invertible, then $\boldsymbol{A}^\top$ is also invertible and $(\boldsymbol{A}^\top)^{-1} = (\boldsymbol{A}^{-1})^\top$.

Proof:

$\because$ $\boldsymbol{A}$ is invertible

$\therefore$ $\exists\ \boldsymbol{A}^{-1}$ such that $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$ and $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$

$\therefore$ $(\boldsymbol{A}^{-1}\boldsymbol{A})^\top = \boldsymbol{I}^\top = \boldsymbol{I}$ and $(\boldsymbol{A}\boldsymbol{A}^{-1})^\top = \boldsymbol{I}^\top = \boldsymbol{I}$

$\therefore$ $\boldsymbol{A}^\top(\boldsymbol{A}^{-1})^\top = \boldsymbol{I}$ and $(\boldsymbol{A}^{-1})^\top\boldsymbol{A}^\top = \boldsymbol{I}$

$\therefore$ $\boldsymbol{A}^\top$ is invertible and $(\boldsymbol{A}^\top)^{-1} = (\boldsymbol{A}^{-1})^\top$

(3) Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times p}$. Show that $C(\boldsymbol{A})$ is a subspace of $\mathbb{R}^m$ and $C(\boldsymbol{AB}) \subseteq C(\boldsymbol{A})$, where $C(\boldsymbol{M})$ denotes the column space of matrix $\boldsymbol{M}$.

Proof:

(i) Let $\boldsymbol{A} = [A_1, A_2, \cdots, A_n]$, where $A_i \in \mathbb{R}^m$, $1 \le i \le n$, are the column vectors of $\boldsymbol{A}$. Then the column space of $\boldsymbol{A}$ is given by

$$C(\boldsymbol{A}) = \{c_1A_1 + c_2A_2 + \cdots + c_nA_n |\ c_1, c_2, \cdots, c_n \in \mathbb{R}\} \subseteq \mathbb{R}^m.$$

Let $\boldsymbol{v} := c_1A_1 + c_2A_2 + \cdots + c_nA_n$, $\boldsymbol{w} := d_1A_1 + d_2A_2 + \cdots + d_nA_n \in C(\boldsymbol{A})$ and $c \in \mathbb{R}$. Then

$$\begin{aligned} \boldsymbol{v} + \boldsymbol{w} &= (c_1A_1 + c_2A_2 + \cdots + c_nA_n) + (d_1A_1 + d_2A_2 + \cdots + d_nA_n) \\ &= (c_1 + d_1)A_1 + (c_2 + d_2)A_2 + \cdots + (c_n + d_n)A_n \in C(\boldsymbol{A}), \\ c\boldsymbol{v} &= c(c_1A_1 + c_2A_2 + \cdots + c_nA_n) \\ &= cc_1A_1 + cc_2A_2 + \cdots + cc_nA_n \in C(\boldsymbol{A}). \end{aligned}$$

Therefore, $C(\boldsymbol{A})$ is a subspace of $\mathbb{R}^m$.

(ii) Let $\boldsymbol{B} = [B_1, B_2, \cdots, B_p]$, where $B_i \in \mathbb{R}^n$, $i = 1, 2, \cdots, p$, are the column vectors of $\boldsymbol{B}$. Then $\boldsymbol{AB} = \boldsymbol{A}[B_1, B_2, \cdots, B_p] = [\boldsymbol{A}B_1, \boldsymbol{A}B_2, \cdots, \boldsymbol{A}B_p]$, and

$$C(\boldsymbol{AB}) = \text{the set of all linear combinations of } \boldsymbol{A}B_1, \boldsymbol{A}B_2, \cdots, \boldsymbol{A}B_p.$$

Let $\boldsymbol{v} \in C(\boldsymbol{AB})$. Then $\boldsymbol{v}$ is a linear combination of $\boldsymbol{A}B_1, \boldsymbol{A}B_2, \cdots, \boldsymbol{A}B_p$.
$\because$ For each i, $\boldsymbol{A}B_i$ is a linear combination of columns of $\boldsymbol{A}$
$\therefore$ $\boldsymbol{v}$ is a linear combination of columns of $\boldsymbol{A}$
$\therefore$ $\boldsymbol{v} \in C(\boldsymbol{A})$
$\therefore$ $C(\boldsymbol{AB}) \subseteq C(\boldsymbol{A})$

(4) Let $\boldsymbol{V}$ be a vector space over the field $\mathbb{F}$ and let $\boldsymbol{K}$ be a nonempty subset of $\boldsymbol{V}$. Define

$$\boldsymbol{S} := \{c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_N\boldsymbol{v}_N \mid \forall\, \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_N \in \boldsymbol{K}, c_1, c_2, \cdots, c_N \in \mathbb{F}, \text{and } N \in \mathbb{N}\}.$$

Show that $\boldsymbol{S}$ is a subspace of $\boldsymbol{V}$ and $\boldsymbol{S}$ is the smallest subspace of $\boldsymbol{V}$ that contains $\boldsymbol{K}$.

Proof:

(i) Obviously, $\boldsymbol{S} \subseteq \boldsymbol{V}$. Let $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{S}$ and $c \in \mathbb{F}$. Then

$$\begin{aligned} \boldsymbol{v} &= c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_N\boldsymbol{v}_N \text{ for some } \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_N \in \boldsymbol{K} \text{ and } c_1, c_2, \cdots, c_N \in \mathbb{F}, \\ \boldsymbol{w} &= d_1\boldsymbol{w}_1 + d_2\boldsymbol{w}_2 + \cdots + d_M\boldsymbol{w}_M \text{ for some } \boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_M \in \boldsymbol{K} \text{ and } d_1, d_2, \cdots, d_M \in \mathbb{F}. \end{aligned}$$

We obtain

- $\boldsymbol{v} + \boldsymbol{w} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_N\boldsymbol{v}_N + d_1\boldsymbol{w}_1 + d_2\boldsymbol{w}_2 + \cdots + d_M\boldsymbol{w}_M$ is a linear combination of vectors in $\boldsymbol{K} \Longrightarrow \boldsymbol{v} + \boldsymbol{w} \in \boldsymbol{S}$.
- $c\boldsymbol{v} = c(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_N\boldsymbol{v}_N) = cc_1\boldsymbol{v}_1 + cc_2\boldsymbol{v}_2 + \cdots + cc_N\boldsymbol{v}_N$ is a linear combination of vectors in $\boldsymbol{K} \Longrightarrow c\boldsymbol{v} \in \boldsymbol{S}$.

Therefore, $\boldsymbol{S}$ is a subspace of $\boldsymbol{V}$.

(ii) Assume that $\boldsymbol{T}$ is a subspace of $\boldsymbol{V}$ that contains $\boldsymbol{K}$. We want to show that $\boldsymbol{S} \subseteq \boldsymbol{T}$.
Let $\boldsymbol{v} \in \boldsymbol{S}$. Then $\exists\ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_N \in \boldsymbol{K}$ and $c_1, c_2, \cdots, c_N \in \mathbb{F}$ s.t. $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_N\boldsymbol{v}_N$.
$\because$ $\boldsymbol{T}$ is a subspace of $\boldsymbol{V}$ and $\boldsymbol{K} \subseteq \boldsymbol{T}$ and $\boldsymbol{v}$ is a linear combination of vectors in $\boldsymbol{K}$
$\therefore$ $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_N\boldsymbol{v}_N \in \boldsymbol{T}$
$\therefore$ $\boldsymbol{S} \subseteq \boldsymbol{T}$

(5) Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times p}$. Show that $\text{rank}(\boldsymbol{AB}) \le \text{rank}(\boldsymbol{B})$.

Proof: Let $\boldsymbol{B} = [B_1, B_2, \cdots, B_p]$. Then $\boldsymbol{AB} = \boldsymbol{A}[B_1, B_2, \cdots, B_p] = [\boldsymbol{A}B_1, \boldsymbol{A}B_2, \cdots, \boldsymbol{A}B_p]$.
If the j-th column B_j of $\boldsymbol{B}$ is a linear combination of previous columns of $\boldsymbol{B}$, i.e.,

$$B_j = c_1B_1 + c_2B_2 + \cdots + c_{j-1}B_{j-1}, \text{ for some } c_1, c_2, \cdots, c_{j-1} \in \mathbb{R},$$

then we have

- B_j is a free column, i.e., B_j is not a pivot column.
- The j-th column of $\boldsymbol{AB}$ is

$$\boldsymbol{A}B_j = \boldsymbol{A}(c_1B_1 + c_2B_2 + \cdots + c_{j-1}B_{j-1}) = c_1\boldsymbol{A}B_1 + c_2\boldsymbol{A}B_2 + \cdots + c_{j-1}\boldsymbol{A}B_{j-1}.$$

  i.e., the j-th column $\boldsymbol{A}B_j$ of $\boldsymbol{AB}$ is a linear combination of previous columns of $\boldsymbol{AB}$. Therefore, $\boldsymbol{A}B_j$ is not a pivot column of $\boldsymbol{AB}$.

We can conclude that

$$\text{rank}(\boldsymbol{AB}) = \text{number of pivot columns of } \boldsymbol{AB} \le \text{number of pivot columns of } \boldsymbol{B} = \text{rank}(\boldsymbol{B}).$$

(6) Consider the linear system $\boldsymbol{Ax} = \boldsymbol{b}$,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Reduce the augmented matrix $[\boldsymbol{A}\ \ \boldsymbol{b}]$ to $[\boldsymbol{U}\ \ \boldsymbol{c}]$ so that $\boldsymbol{Ax} = \boldsymbol{b}$ becomes a triangular system $\boldsymbol{Ux} = \boldsymbol{c}$ and find the condition on b_1, b_2, b_3 for $\boldsymbol{Ax} = \boldsymbol{b}$ solvable.

Solution:

$$[\boldsymbol{A}\ \ \boldsymbol{b}] = \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & -2 & -2 & b_3 - 3b_1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{bmatrix} = [\boldsymbol{U}\ \ \boldsymbol{c}]$$

Therefore, the solvability condition of $\boldsymbol{Ax} = \boldsymbol{b}$ is $b_3 + b_2 - 5b_1 = 0$.

(7) Let $\boldsymbol{b} := [0, 6, -6]^\top$ in Problem (6). Find the reduced row echelon form $[\boldsymbol{R}\ \ \boldsymbol{d}]$ and find the complete solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

Solution: First, note that $b_3 + b_2 - 5b_1 = -6 + 6 - 0 = 0$.

$$[\boldsymbol{U}\ \ \boldsymbol{c}] = \begin{bmatrix} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 2 & -9 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = [\boldsymbol{R}\ \ \boldsymbol{d}]$$

Free columns: $2, 4$ free variables: x_2, x_4

Pivot columns: $1, 3$ pivot variables: x_1, x_3

Consider $\boldsymbol{Ax} = \boldsymbol{0} \Leftrightarrow \boldsymbol{Rx} = \boldsymbol{0}$.

Set $x_2 = 1$, $x_4 = 0$: $x_1 = -2$ and $x_3 = 0$.

Set $x_2 = 0$, $x_4 = 1$: $x_1 = -2$ and $x_3 = -1$.

We have two special solutions to $\boldsymbol{Ax} = \boldsymbol{0}$:

$$\boldsymbol{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{s}_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Consider $\boldsymbol{Ax}_p = \boldsymbol{b} \Leftrightarrow \boldsymbol{Rx}_p = \boldsymbol{d}$.

Set $x_2 = 0$, $x_4 = 0$: $x_1 = -9$ and $x_3 = 3$.

We obtain a particular solution to $\boldsymbol{Ax}_p = \boldsymbol{b}$:

$$\boldsymbol{x}_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix}.$$

Therefore, the complete solution to $\boldsymbol{Ax} = \boldsymbol{b}$ is

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

(8) Suppose you know that the 3×4 real matrix $\boldsymbol{A}$ has the vector $\boldsymbol{s} = [2, 3, 1, 0]^\top$ as the only special solution to $\boldsymbol{Ax} = \boldsymbol{0}$. (a) What is the rank of $\boldsymbol{A}$ and the complete solution to $\boldsymbol{Ax} = \boldsymbol{0}$? (b) What is the exact reduced row echelon form $\boldsymbol{R}$ of A? (c) How do you know that $\boldsymbol{Ax} = \boldsymbol{b}$ can be solved for all $\boldsymbol{b} \in \mathbb{R}^3$?

Solution:

$\boldsymbol{A}$ has only one free variable x_3 and one free column, the 3rd column.

(a) rank($\boldsymbol{A}$) = number of pivot columns $= 4 - 1 = 3$. The complete solution to $\boldsymbol{Ax} = \boldsymbol{0}$ is

$$\boldsymbol{x} = c\boldsymbol{s} = c \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}.$$

(b) Since $\boldsymbol{s} = [2, 3, 1, 0]^\top$, the exact reduced row echelon form $\boldsymbol{R}$ of $\boldsymbol{A}$ is given by

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(c) $\boldsymbol{Ax} = \boldsymbol{b}$ can be solved for all $\boldsymbol{b} \in \mathbb{R}^3$ because $\boldsymbol{A}$ and $\boldsymbol{R}$ have full row rank $r = 3$.

(9) Find the largest possible number of independent vectors among

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \boldsymbol{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \boldsymbol{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Solution:

(i) Claim: $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent.
Let $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3 = \boldsymbol{0}$. Then

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow x_1 = 0, \quad x_2 = 0, \quad x_3 = 0.$$

Therefore, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent.

(ii) $\boldsymbol{v}_4 = \boldsymbol{v}_2 - \boldsymbol{v}_1 \Longrightarrow -\boldsymbol{v}_1 + \boldsymbol{v}_2 - \boldsymbol{v}_4 = \boldsymbol{0}$
$\boldsymbol{v}_5 = \boldsymbol{v}_3 - \boldsymbol{v}_1 \Longrightarrow -\boldsymbol{v}_1 + \boldsymbol{v}_3 - \boldsymbol{v}_5 = \boldsymbol{0}$
$\boldsymbol{v}_6 = \boldsymbol{v}_3 - \boldsymbol{v}_2 \Longrightarrow -\boldsymbol{v}_2 + \boldsymbol{v}_3 - \boldsymbol{v}_6 = \boldsymbol{0}$

By (i) and (ii), 3 is the largest number of independent vectors.

(10) Show that if an $n \times n$ real matrix $\boldsymbol{A}$ is invertible, then the set of all column vectors of $\boldsymbol{A}$ is linearly independent.

Proof:

Let $\boldsymbol{A} = [A_1, A_2, \cdots, A_n]$, where A_i, $1 \le i \le n$, are the column vectors of $\boldsymbol{A}$.

Assume that $x_1A_1 + x_2A_2 + \cdots + x_nA_n = \boldsymbol{0}$, i.e., $\boldsymbol{Ax} = \boldsymbol{0}$. Then

$\because$ $\boldsymbol{A}$ is invertible, so $\boldsymbol{A}^{-1}$ exists

$\therefore$ $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{Ax} = \boldsymbol{A}^{-1}\boldsymbol{0} = \boldsymbol{0}$

$\therefore$ $A_1, A_2, \cdots, A_n$ are linearly independent