MA2007B: LINEAR ALGEBRA I

Midterm1/October 24, 2019

Please show all your work clearly for full credit!

(1) (15 pts) State the Cauchy-Schwarz-Buniakowsky inequality for vectors in $\mathbb{R}^n$ and then use the Cauchy-Schwarz-Buniakowsky inequality to prove the triangle inequality,

$$\|\boldsymbol{v}+\boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|, \quad \forall\, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n.$$

Solution:

(i). Cauchy-Schwarz-Buniakowsky inequality: Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$. Then $|\boldsymbol{v}\cdot\boldsymbol{w}| \leq \|\boldsymbol{v}\|\|\boldsymbol{w}\|$.

(ii). **Proof:** For any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, we have

$$\begin{aligned} \|\boldsymbol{v}+\boldsymbol{w}\|^2 &= (\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}+\boldsymbol{w}) \\ &= \boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{w} \\ &= \boldsymbol{v}\cdot\boldsymbol{v} + 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w} \\ &= \|\boldsymbol{v}\|^2 + 2\boldsymbol{v}\cdot\boldsymbol{w} + \|\boldsymbol{w}\|^2. \end{aligned}$$

By the Cauchy-Schwarz-Buniakowsky inequality, we have

$$\begin{aligned} \|\boldsymbol{v}+\boldsymbol{w}\|^2 &\leq \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{v}\|\|\boldsymbol{w}\| + \|\boldsymbol{w}\|^2 \\ &= (\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2. \end{aligned}$$

Therefore, we obtain $\|\boldsymbol{v}+\boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$. $\square$

(2) (10 pts) Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be two nonzero vectors in $\mathbb{R}^2$ and $\boldsymbol{v} \neq \alpha\boldsymbol{w}$, $\forall\, \alpha \in \mathbb{R}$. Let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. Show that $0 < \theta < (\pi/2)$ if and only if $\|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 > \|\boldsymbol{v}-\boldsymbol{w}\|^2$.

Proof: First, we note that

$$(\boldsymbol{v}-\boldsymbol{w})\cdot(\boldsymbol{v}-\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} - 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w}.$$

By the cosine formula, we have

$$\|\boldsymbol{v}-\boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 - 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta + \|\boldsymbol{w}\|^2. \qquad (\star)$$

($\Rightarrow$): If $0 < \theta < (\pi/2)$, then $\cos\theta > 0$. By $(\star)$, we have $\|\boldsymbol{v}-\boldsymbol{w}\|^2 < \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2$

($\Leftarrow$): If $\|\boldsymbol{v}-\boldsymbol{w}\|^2 < \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2$, by $(\star)$, we obtain $\cos\theta > 0$. Therefore, $0 < \theta < (\pi/2)$. $\square$

(3) (10 pts) Can four vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ in $\mathbb{R}^2$ have $\boldsymbol{u}_i\cdot\boldsymbol{u}_j < 0$ for all $i \neq j$?

Solution:

No, it is impossible!

Suppose that there are four vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ in $\mathbb{R}^2$ distributed in counterclockwise such that $\boldsymbol{u}_i\cdot\boldsymbol{u}_j < 0$ for all $i \neq j$, then $\angle\boldsymbol{u}_1\boldsymbol{u}_2 > \pi/2$, $\angle\boldsymbol{u}_2\boldsymbol{u}_3 > \pi/2$, $\angle\boldsymbol{u}_3\boldsymbol{u}_4 > \pi/2$, and $\angle\boldsymbol{u}_4\boldsymbol{u}_1 > \pi/2$.

$\therefore$ The total angle $> 4 \times \pi/2 = 2\pi$. This is a contradiction!

$\therefore$ It is impossible that there are four vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ in $\mathbb{R}^2$ such that $\boldsymbol{u}_i\cdot\boldsymbol{u}_j < 0$ for all $i \neq j$.

(4) (15 pts) Any real-valued function $f: \mathbb{R}^n \to \mathbb{R}$ is called a norm on $\mathbb{R}^n$ if it satisfies the following three conditions: (i) $f(\boldsymbol{v}) \geq 0, \ \forall\, \boldsymbol{v} \in \mathbb{R}^n$; $f(\boldsymbol{v}) = 0$ if and only if $\boldsymbol{v} = \mathbf{0}$; (ii) $f(\alpha \boldsymbol{v}) = |\alpha| f(\boldsymbol{v}), \ \forall\, \boldsymbol{v} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$; (iii) $f(\boldsymbol{v} + \boldsymbol{w}) \leq f(\boldsymbol{v}) + f(\boldsymbol{w}), \forall\, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$.

Define

$$f(\boldsymbol{v}) := \|\boldsymbol{v}\|_1 := |v_1| + |v_2| + \cdots + |v_n|, \quad \boldsymbol{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$$

and

$$f(\boldsymbol{v}) := \|\boldsymbol{v}\|_\infty := \max\{|v_1|, |v_2|, \cdots, |v_n|\}, \quad \boldsymbol{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n.$$

Show that both $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms on $\mathbb{R}^n$.

Proof:

- Claim: $\|\cdot\|_1$ is a norm on $\mathbb{R}^n$:

  (i) $\forall\, \boldsymbol{v} \in \mathbb{R}^n$, we have $\|\boldsymbol{v}\|_1 = |v_1| + |v_2| + \cdots + |v_n| \geq 0$, since $|v_i| \geq 0\ \forall\, i$.
  $\|\boldsymbol{v}\|_1 = |v_1| + |v_2| + \cdots + |v_n| = 0$ if and only if $|v_i| = 0, 1 \leq i \leq n$, if and only if $\boldsymbol{v} = \mathbf{0}$.

  (ii) Let $\boldsymbol{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then $\alpha \boldsymbol{v} = (\alpha v_1, \alpha v_2, \cdots, \alpha v_n)$ and

$$\begin{aligned}
\|\alpha \boldsymbol{v}\|_1 &= |\alpha v_1| + |\alpha v_2| + \cdots + |\alpha v_n| \\
&= |\alpha|(|v_1| + |v_2| + \cdots + |v_n|) = |\alpha| \|\boldsymbol{v}\|_1.
\end{aligned}$$

  (iii) Let $\boldsymbol{v} = (v_1, v_2, \cdots, v_n)$, $\boldsymbol{w} = (w_1, w_2, \cdots, w_n) \in \mathbb{R}^n$. Then

$$\begin{aligned}
\|\boldsymbol{v} + \boldsymbol{w}\|_1 &= \|(v_1, v_2, \cdots, v_n) + (w_1, w_2, \cdots, w_n)\|_1 \\
&= \|(v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n)\|_1 \\
&= |v_1 + w_1| + |v_2 + w_2| + \cdots + |v_n + w_n| \\
&\leq |v_1| + |w_1| + |v_2| + |w_2| + \cdots + |v_n| + |w_n| \\
&= (|v_1| + |v_2| + \cdots + |v_n|) + (|w_1| + |w_2| + \cdots + |w_n|) \\
&= \|\boldsymbol{v}\|_1 + \|\boldsymbol{w}\|_1.
\end{aligned}$$

- Claim: $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^n$:

  (i) $\forall\, \boldsymbol{v} \in \mathbb{R}^n$, we have $\|\boldsymbol{v}\|_\infty = \max\{|v_1|, |v_2|, \cdots, |v_n|\} \geq 0$, since $|v_i| \geq 0\ \forall\, i$.
  $\|\boldsymbol{v}\|_\infty = \max\{|v_1|, |v_2|, \cdots, |v_n|\} = 0$ if and only if $|v_i| = 0, 1 \leq i \leq n$, if and only if $\boldsymbol{v} = \mathbf{0}$.

  (ii) Let $\boldsymbol{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then $\alpha \boldsymbol{v} = (\alpha v_1, \alpha v_2, \cdots, \alpha v_n)$ and

$$\begin{aligned}
\|\alpha \boldsymbol{v}\|_\infty &= \max\{|\alpha v_1|, |\alpha v_2|, \cdots, |\alpha v_n|\} \\
&= \max\{|\alpha||v_1|, |\alpha||v_2|, \cdots, |\alpha||v_n|\} \\
&= |\alpha| \max\{|v_1|, |v_2|, \cdots, |v_n|\} = |\alpha| \|\boldsymbol{v}\|_\infty.
\end{aligned}$$

  (iii) Let $\boldsymbol{v} = (v_1, v_2, \cdots, v_n)$, $\boldsymbol{w} = (w_1, w_2, \cdots, w_n) \in \mathbb{R}^n$. Then

$$\begin{aligned}
\|\boldsymbol{v} + \boldsymbol{w}\|_\infty &= \|(v_1, v_2, \cdots, v_n) + (w_1, w_2, \cdots, w_n)\|_\infty \\
&= \|(v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n)\|_\infty \\
&= \max\{|v_1 + w_1|, |v_2 + w_2|, \cdots, |v_n + w_n|\} \\
&\leq \max\{|v_1| + |w_1|, |v_2| + |w_2|, \cdots, |v_n| + |w_n|\} \quad (\text{since } |v_i + w_i| \leq |v_i| + |w_i|\ \forall\, i) \\
&\leq \max\{|v_1|, |v_2|, \cdots, |v_n|\} + \max\{|w_1|, |w_2|, \cdots, |w_n|\} \\
&= \|\boldsymbol{v}\|_\infty + \|\boldsymbol{w}\|_\infty. \qquad \square
\end{aligned}$$

(5) (10 pts) Is the following matrix $\boldsymbol{C}$ invertible? Please give your reason without using Gaussian elimination or determinant.

$$\boldsymbol{C} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

Consider the linear system $\boldsymbol{Cx} = \boldsymbol{b}$. Find a condition on $\boldsymbol{b}$ such that the linear system has no solution.

Solution: No! $\boldsymbol{C}$ is not invertible. Because there exists a nonzero vector $\boldsymbol{x}^* = (1, 1, \cdots, 1) \in \mathbb{R}^n$ such that $\boldsymbol{Cx}^* = \boldsymbol{0}$:

$$\boldsymbol{Cx}^* = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

If $\boldsymbol{C}$ is invertible, then $\boldsymbol{x}^* = \boldsymbol{C}^{-1}(\boldsymbol{Cx}^*) = \boldsymbol{C}^{-1}\boldsymbol{0} = \boldsymbol{0}$. This is a contradiction!

Consider the linear system $\boldsymbol{Cx} = \boldsymbol{b}$. Adding all rows of the linear system, we have

$$0 = b_1 + b_2 + b_3 + b_4.$$

Therefore, if $b_1 + b_2 + b_3 + b_4 \neq 0$, then the linear system $\boldsymbol{Cx} = \boldsymbol{b}$ has no solution.

(6) (15 pts) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two $n \times n$ matrices. Prove that if the product $\boldsymbol{C} = \boldsymbol{AB}$ is invertible, then $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible. Find a formula for $\boldsymbol{A}^{-1}$ that involves $\boldsymbol{C}^{-1}$ and $\boldsymbol{B}$.

Proof:

(i) If the product $\boldsymbol{C} = \boldsymbol{AB}$ is invertible, then $\boldsymbol{CC}^{-1} = (\boldsymbol{AB})\boldsymbol{C}^{-1}$.

$\therefore\ \boldsymbol{I} = \boldsymbol{CC}^{-1} = (\boldsymbol{AB})\boldsymbol{C}^{-1} = \boldsymbol{A}(\boldsymbol{BC}^{-1})$

$\therefore\ \boldsymbol{A}(\boldsymbol{BC}^{-1}) = \boldsymbol{I}$

$\therefore\ \boldsymbol{A}$ is invertible, i.e., $\boldsymbol{A}^{-1}$ exists.

$\therefore\ \boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{I} = \boldsymbol{A}^{-1}\boldsymbol{A}(\boldsymbol{BC}^{-1}) = \boldsymbol{I}(\boldsymbol{BC}^{-1}) = \boldsymbol{BC}^{-1}$

(ii) Claim: $\boldsymbol{B}$ is invertible.

$\because\ \boldsymbol{A}$ is invertible

$\therefore\ \boldsymbol{A}^{-1}$ is invertible

$\because\ \boldsymbol{C} = \boldsymbol{AB}$

$\therefore\ \boldsymbol{A}^{-1}\boldsymbol{C} = \boldsymbol{A}^{-1}(\boldsymbol{AB}) = (\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{B} = \boldsymbol{IB} = \boldsymbol{B}$

$\therefore\ \boldsymbol{B} = \boldsymbol{A}^{-1}\boldsymbol{C}$, a product of two invertible matrices $\boldsymbol{A}^{-1}$ and $\boldsymbol{C}$

$\therefore\ \boldsymbol{B}$ is invertible $\quad\square$

(7) (10 pts) Consider the 4×4 matrix,

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

What three elimination matrices $\boldsymbol{E}_{21}, \boldsymbol{E}_{32}, \boldsymbol{E}_{43}$ put $\boldsymbol{A}$ into upper triangular form $\boldsymbol{E}_{43}\boldsymbol{E}_{32}\boldsymbol{E}_{21}\boldsymbol{A} = \boldsymbol{U}$. Multiply by $\boldsymbol{E}_{43}^{-1}$, $\boldsymbol{E}_{32}^{-1}$ and $\boldsymbol{E}_{21}^{-1}$ to factor $\boldsymbol{A}$ into $\boldsymbol{LU}$.

Solution:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{\ell_{21}=\frac{-1}{2}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{\ell_{32}=\frac{-2}{3}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{\ell_{43}=\frac{-3}{4}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} := \boldsymbol{U}.$$

Therefore, we have the following three elimination matrices:

$$\boldsymbol{E}_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{E}_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{E}_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{bmatrix},$$

and $\boldsymbol{E}_{43}\boldsymbol{E}_{32}\boldsymbol{E}_{21}\boldsymbol{A} = \boldsymbol{U}$. The three inverses of $\boldsymbol{E}_{21}, \boldsymbol{E}_{32}, \boldsymbol{E}_{43}$ are, respectively,

$$\boldsymbol{E}_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{E}_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{E}_{43}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix},$$

and

$$\boldsymbol{E}_{21}^{-1}\boldsymbol{E}_{32}^{-1}\boldsymbol{E}_{43}^{-1}(\boldsymbol{E}_{43}\boldsymbol{E}_{32}\boldsymbol{E}_{21}\boldsymbol{A}) = \boldsymbol{E}_{21}^{-1}\boldsymbol{E}_{32}^{-1}\boldsymbol{E}_{43}^{-1}\boldsymbol{U}.$$

Therefore,

$$\boldsymbol{A} = \underbrace{\boldsymbol{E}_{21}^{-1}\boldsymbol{E}_{32}^{-1}\boldsymbol{E}_{43}^{-1}}_{\equiv \boldsymbol{L}}\boldsymbol{U} = \boldsymbol{L}\boldsymbol{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}.$$

(8) (15 pts) Find the solution of the following linear system by solving two triangular systems, one with the lower triangular matrix $\boldsymbol{L}$ and the other with the upper triangular matrix $\boldsymbol{U}$, both derived in problem (7):

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Solution: From problem (7), we have

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}, \quad \boldsymbol{U} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}.$$

Since $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$, where $\boldsymbol{L}$ is a lower triangular matrix and $\boldsymbol{U}$ is a upper triangular matrix, we solve first

$$\boldsymbol{L}\boldsymbol{c} = \boldsymbol{b} := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and then solve

$$\boldsymbol{U}\boldsymbol{x} = \boldsymbol{c}.$$

- $\boldsymbol{L}\boldsymbol{c} = \boldsymbol{b}$: Let

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}.$$

Then we solve

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

By forward substitution, we have

$$c_1 = 1 \Longrightarrow c_2 = \frac{1}{2}c_1 = \frac{1}{2} \Longrightarrow c_3 = \frac{2}{3}c_2 = \frac{1}{3} \Longrightarrow c_4 = 1 + \frac{3}{4}c_3 = \frac{5}{4}.$$

- $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{c}$:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 5/4 \end{bmatrix}$$

By backward substitution, we have

$$x_4 = 1 \Longrightarrow x_3 = 1 \Longrightarrow x_2 = 1 \Longrightarrow x_1 = 1.$$