MA2007B: LINEAR ALGEBRA I

Final Exam/January 09, 2020

Please show all your work clearly for full credit! (each problem 15 points, total 120 points)

(1) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be an invertible matrix. Show that the columns of $\boldsymbol{A}$ are linearly independent and span $\mathbb{R}^n$.

Proof: Let $\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n]$.

- claim: $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ are linearly independent
  Assume that $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{0}$.
  Then $\boldsymbol{0} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]^\top$.
  $\therefore \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{0} = \boldsymbol{0}$
- claim: $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ span $\mathbb{R}^n$
  Let $\boldsymbol{b} \in \mathbb{R}^n$.
  $\because \boldsymbol{A}$ is invertible
  $\therefore \exists\, \boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$
  That is, $\exists\, \boldsymbol{x} := [x_1, x_2, \cdots, x_n]^\top \in \mathbb{R}^n$ such that $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.
  $\therefore \boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ span $\mathbb{R}^n$

(2) Which of the following are bases for $\mathbb{R}^3$?

(a) $[1\ \ 2\ \ 0]^\top$ and $[0\ \ 1\ \ -1]^\top$
(b) $[1\ \ 1\ \ -1]^\top, [2\ \ 3\ \ 4]^\top, [4\ \ 1\ \ -1]^\top, [0\ \ 1\ \ -1]^\top$
(c) $[1\ \ 2\ \ 2]^\top, [-1\ \ 2\ \ 1]^\top, [0\ \ 8\ \ 0]^\top$
(d) $[1\ \ 2\ \ 2]^\top, [-1\ \ 2\ \ 1]^\top, [0\ \ 8\ \ 6]^\top$

Solution:

(a) No! $\because$ there are only two vectors
(b) No! $\because$ four vectors must be linearly dependent in $\mathbb{R}^3$
(c) Yes! one can check that if $\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2 + \gamma \boldsymbol{v}_3 = \boldsymbol{0}$ then $\alpha = \beta = \gamma = 0$ $\cdots\cdots$
(d) No! $\because$ they are linearly dependent, $[0\ \ 8\ \ 6]^\top = 2 \times [1\ \ 2\ \ 2]^\top + 2 \times [-1\ \ 2\ \ 1]^\top$

(3) Let $\boldsymbol{V}$ be a subspace of $\mathbb{R}^n$ and $\boldsymbol{V}^\perp := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x}\cdot\boldsymbol{v} = 0,\ \forall \boldsymbol{v} \in \boldsymbol{V}\}$ be the orthogonal complement of $\boldsymbol{V}$. Show that $\boldsymbol{V}^\perp$ is a subspace of $\mathbb{R}^n$.

Proof:

- Let $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}^\perp$. Then $\boldsymbol{x}\cdot\boldsymbol{v} = 0$ and $\boldsymbol{y}\cdot\boldsymbol{v} = 0,\ \forall\, \boldsymbol{v} \in \boldsymbol{V}$.
  $\therefore\ (\boldsymbol{x}+\boldsymbol{y})\cdot\boldsymbol{v} = \boldsymbol{x}\cdot\boldsymbol{v} + \boldsymbol{y}\cdot\boldsymbol{v} = 0,\ \forall\, \boldsymbol{v} \in \boldsymbol{V}$
  $\therefore\ \boldsymbol{x}+\boldsymbol{y} \in \boldsymbol{V}^\perp$
- Let $\boldsymbol{x} \in \boldsymbol{V}^\perp$ and $\alpha \in \mathbb{R}$. Then $\boldsymbol{x}\cdot\boldsymbol{v} = 0,\ \forall\, \boldsymbol{v} \in \boldsymbol{V}$.
  $\therefore\ (\alpha\boldsymbol{x})\cdot\boldsymbol{v} = \alpha(\boldsymbol{x}\cdot\boldsymbol{v}) = 0,\ \forall\, \boldsymbol{v} \in \boldsymbol{V}$
  $\therefore\ \alpha\boldsymbol{x} \in \boldsymbol{V}^\perp$

$\therefore\ \boldsymbol{V}^\perp$ is a subspace of $\mathbb{R}^n$

(4) Consider the 2×3 real matrix,

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}.$$

Find the bases and dimensions for the four subspaces: $C(A)$, $C(A^\top)$, $N(A)$, $N(A^\top)$.

Solution:

- $C(A)$: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ are a basis, since $\begin{bmatrix} 4 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\cdots\cdots$ $\quad \therefore \dim C(A) = 2$
- $C(A^\top)$: $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$ are a basis, since $\cdots\cdots$ $\quad \therefore \dim C(A^\top) = 2$
- $N(A)$: One can check that $\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$ is a basis $\cdots\cdots$ $\quad \therefore \dim N(A) = 1$
- $N(A^\top)$: One can check that $N(A^\top) = \{\mathbf{0}\}$, so $N(A^\top)$ has no basis. $\quad \therefore \dim N(A^\top) = 0$

(5) Please state the Fundamental Theorem of Linear Algebra, Part I and Part II.

Solution: Let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ real matrix. Then

- **Part I:**
  $\dim C(A^\top) + \dim N(A) = r + (n - r) = n = \dim \mathbb{R}^n$.
  The row space $C(A^\top)$ has dimension r and the nullspce $N(A)$ has dimension $n - r$.

  $\dim C(A) + \dim N(A^\top) = r + (m - r) = m = \dim \mathbb{R}^m$.
  The column space $C(A)$ has dimension r and the left nullspce $N(A^\top)$ has dimension $m - r$.
- **Part II:**
  $N(A)$ is the orthogonal complement of the row space $C(A^\top)$ in $\mathbb{R}^n$.
  $N(A^\top)$ is the orthogonal complement of the column space $C(A)$ in $\mathbb{R}^m$.

(6) Let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ real matrix. Show that the left nullspace $N(A^\top)$ and the column space $C(A)$ are orthogonal in $\mathbb{R}^m$.

Proof:

Let $y := [y_1, y_2, \cdots, y_m]^\top \in N(A^\top) \subseteq \mathbb{R}^m$.

Denote $A = [a_1, a_2, \cdots, a_n]$, where a_i is the i-th column vector of A.

Then we have $A^\top y = \mathbf{0}$, i.e., $\begin{bmatrix} a_1^\top \\ \vdots \\ a_n^\top \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

$\therefore a_1^\top y = 0, a_2^\top y = 0, \cdots, a_n^\top y = 0$

$\therefore a_1 \cdot y = 0, a_2 \cdot y = 0, \cdots, a_n \cdot y = 0$

$\because C(A) = \{c_1 a_1 + \cdots + c_n a_n | \, c_i \in \mathbb{R}\}$

$\therefore$ For any vector in $C(A)$, we have $(c_1 a_1 + \cdots + c_n a_n) \cdot y = c_1 a_1 \cdot y + \cdots + c_n a_n \cdot y = 0$

$\therefore N(A^\top) \perp C(A)$

(7) Let $b \in \mathbb{R}^m$ and $A := [a_1 \; a_2 \; \cdots \; a_n]$ be an $m \times n$ matrix, where $a_1, a_2, \cdots, a_n \in \mathbb{R}^m$ are linearly independent.

(a) Find the projection $p := A\hat{x}$ of vector b onto the column space $C(A)$ and also find the projection matrix P.

(b) Let us consider a concrete example with $n = 2$, where $\boldsymbol{a}_1 = [1,0,0]^\top$, $\boldsymbol{a}_2 = [1,1,0]^\top$, and $\boldsymbol{b} = [2,3,4]^\top$. Find $\widehat{\boldsymbol{x}}$, $\boldsymbol{p}$ and $\boldsymbol{P}$.

Solution:

(a) $\because\ (\boldsymbol{b} - \boldsymbol{A}\widehat{\boldsymbol{x}}) \perp C(\boldsymbol{A})$ and $C(\boldsymbol{A}) = \text{span}\{\boldsymbol{a}_1\ \boldsymbol{a}_2\ \cdots\ \boldsymbol{a}_n\}$

$\therefore\ \boldsymbol{a}_i \cdot (\boldsymbol{b} - \boldsymbol{A}\widehat{\boldsymbol{x}}) = 0, \quad \forall\, i = 1,2,\cdots,n$

$\therefore\ \boldsymbol{a}_i^\top (\boldsymbol{b} - \boldsymbol{A}\widehat{\boldsymbol{x}}) = 0, \quad \forall\, i = 1,2,\cdots,n$

$$\therefore\ \begin{bmatrix} \boldsymbol{a}_1^\top \\ \vdots \\ \boldsymbol{a}_n^\top \end{bmatrix} (\boldsymbol{b} - \boldsymbol{A}\widehat{\boldsymbol{x}}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\therefore\ \boldsymbol{A}^\top(\boldsymbol{b} - \boldsymbol{A}\widehat{\boldsymbol{x}}) = \boldsymbol{0}$

$\therefore\ \boldsymbol{A}^\top\boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{A}^\top\boldsymbol{b}$

$\because\ \boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ are linearly independent

$\therefore\ \boldsymbol{A}^\top\boldsymbol{A}$ is invertible

$\therefore\ \widehat{\boldsymbol{x}} = (\boldsymbol{A}^\top\boldsymbol{A})^{-1}\boldsymbol{A}^\top\boldsymbol{b}$, $\boldsymbol{p} = \boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{A}^\top\boldsymbol{A})^{-1}\boldsymbol{A}^\top\boldsymbol{b}$, and $\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^\top\boldsymbol{A})^{-1}\boldsymbol{A}^\top$

(b) $n = 2$, $\boldsymbol{a}_1 = [1,0,0]^\top$ and $\boldsymbol{a}_2 = [1,1,0]^\top$

$$\because\ \boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{A}^\top = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{A}^\top\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad (\boldsymbol{A}^\top\boldsymbol{A})^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } \boldsymbol{A}^\top\boldsymbol{b} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\therefore\ \widehat{\boldsymbol{x}} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \boldsymbol{p} = \boldsymbol{A}\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

$$\text{and } \boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(8) Let $(t_1, b_1), (t_2, b_2), \cdots, (t_m, b_m)$ be m distinct points on the plane and $m > 2$. Assume that $[1,1,\cdots,1]^\top$ and $[t_1, t_2, \cdots, t_m]^\top$ are orthogonal.

(a) Derive the closest line $b = C + Dt$ to these points by a geometry approach.

(b) Let us consider a concrete example with the three points $(-2,1), (0,2), (2,4)$. Find C and D.

Solution:

(a) $C + Dt_1 = b_1, C + Dt_2 = b_2, \cdots, C + Dt_m = b_m$

$$\Longrightarrow \quad \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

The closest line $b = C + Dt$ to these points has heights p_1, p_2, $\cdots$, p_m, where $\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix}$ is the projection of $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ onto the column space of $\boldsymbol{A}$.

Therefore, we solve $\boldsymbol{A}^\top \boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{A}^\top \boldsymbol{b}$ for $\widehat{\boldsymbol{x}} = \begin{bmatrix} C \\ D \end{bmatrix}$.

$\because$ $[1, 1, \cdots, 1]^\top$ and $[t_1, t_2, \cdots, t_m]^\top$ are orthogonal

$$\therefore \boldsymbol{A}^\top \boldsymbol{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_m \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & \sum_{i=1}^{m} t_i^2 \end{bmatrix}$$

$$\boldsymbol{A}^\top \boldsymbol{b} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_m \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} b_i \\ \sum_{i=1}^{m} t_i b_i \end{bmatrix}$$

$$\therefore \widehat{\boldsymbol{x}} = \begin{bmatrix} \frac{1}{m} \sum_{i=1}^{m} b_i \\ \left(\sum_{i=1}^{m} t_i b_i \right) \Big/ \left(\sum_{i=1}^{m} t_i^2 \right) \end{bmatrix}$$

(b) Consider the three points $(-2, 1), (0, 2), (2, 4)$. Then $[1, 1, 1]^\top$ and $[-2, 0, 2]^\top$ are orthogonal.

$$\therefore \widehat{\boldsymbol{x}} = \begin{bmatrix} \frac{1}{3}(1 + 2 + 4) \\ \frac{1}{8}(-2 + 0 + 8) \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{6}{8} \end{bmatrix}$$