MA 8019: Numerical Analysis I
Solution of Nonlinear Equations

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Introduction

- **A nonlinear equation:**
  Let $f : \varnothing \neq A \subseteq \mathbb{R} \to \mathbb{R}$ be a nonlinear real-valued function in a single variable x. *We are interested in finding the roots (solutions) of the equation $f(x) = 0$, i.e., zeros of the function $f(x)$.*
- **A system of nonlinear equations:**
  Let $F : \varnothing \neq A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a nonlinear vector-valued function in a vector variable $X = (x_1, x_2, \cdots, x_n)^\top$. *We are interested in finding the roots (solutions) of the equation $F(X) = \mathbf{0}$, i.e., zeros of the vector-valued function $F(X)$.*

Example: zeros of polynomial

- Let us look at three functions (polynomials):
  (1) $f(x) = x^4 - 12x^3 + 47x^2 - 60x$
  (2) $f(x) = x^4 - 12x^3 + 47x^2 - 60x + 24$
  (3) $f(x) = x^4 - 12x^3 + 47x^2 - 60x + 24.1$
- Find the zeros of these polynomials is not an easy task.
  (1) The first function has *real zeros* 0, 3, 4, *and* 5.
  (2) The real zeros of the second function are 1 *and* 0.888 . . . .
  (3) The third function *has no real zeros at all.*
  (4) MATLAB: see `polyzeros.m`
- *The n roots of a polynomial of degree n depend continuously on the coefficients.* (see Complex Analysis)
  (1) This result implies that the eigenvalues of a matrix depend continuously on the matrix. (see Tyrtyshnikov's book).
  (2) However, the problem of approximating the roots given the coefficients is *ill-conditioned*, see Wilkinson's polynomial. `https://en.wikipedia.org/wiki/Wilkinson%27s_polynomial`

Objectives

Consider the nonlinear equation $f(x) = 0$ or $F(X) = \mathbf{0}$.

- The basic questions:
  (1) Does the solution exist?
  (2) Is the solution unique?
  (3) *How to find it?*
- We will mainly focus on the third question and we always assume that the problem under considered has a solution x^*.
- *We will study iterative methods for finding the solution:* first find an initial guess x_0, then a better guess $x_1, \ldots,$ in the end we hope that $\lim_{n\to\infty} x_n = x^*$.
- **Iterative methods:** bisection method; Newton's method; secant method; fixed-point method; continuation method; special methods for zeros of polynomials.

Bisection method (method of interval halving)

- **An observation:** *If $f(x)$ is a continuous function on an interval $[a, b]$, and $f(a)$ and $f(b)$ have different signs such that $f(a)f(b) < 0$, then $f(x)$ must have a zero in (a, b), i.e., a root of the equation $f(x) = 0$.*

  (ensured by the Intermediate-Value Theorem for continuous functions)

- **The basic idea:** assume that $f(a)f(b) < 0$.

  (1) compute $c = \frac{1}{2}(a + b) = a + \frac{1}{2}(b - a)$.
  (2) if $f(a)f(c) = 0$, then $f(c) = 0$ and c is a zero of $f(x)$.
  (3) if $f(a)f(c) < 0$, then the zero is in $[a, c]$; otherwise the zero is in $[c, b]$. In either case, a new interval containing the root is produced, and the size of the new interval is half of the original one.
  (4) repeat the process until the interval is very small then any point in the interval can be used as approximations of the zero.

What do we need?

- We need an initial interval $[a, b]$. This is often the hardest thing to find.
- We need some stopping criteria: given $\varepsilon > 0$ and $\delta > 0$ are tolerances, k is the number of iterations.

  (1) if $|f(c)| < \varepsilon$, we stop.
  (2) if $|b - a| < \delta$, we stop.
  (3) if $k > M$, we stop to avoid infinite loop.

A pseudocode for the bisection algorithm

input $a, b, M, \delta, \varepsilon$
$u \leftarrow f(a), \quad v \leftarrow f(b), \quad e \leftarrow b - a$
output a, b, u, v
if $sign(u) = sign(v)$ **then stop**
for $k = 1$ **to** M **do**
    $e \leftarrow e/2, \quad c \leftarrow a + e, \quad w \leftarrow f(c)$
    output k, c, w
    if $|e| < \delta$ **or** (**and**) $|w| < \varepsilon$ **then stop**
    if $sign(w) \neq sign(u)$ **then**
        $b \leftarrow c, \quad v \leftarrow w$
    else
        $a \leftarrow c, \quad u \leftarrow w$
    end if
end do

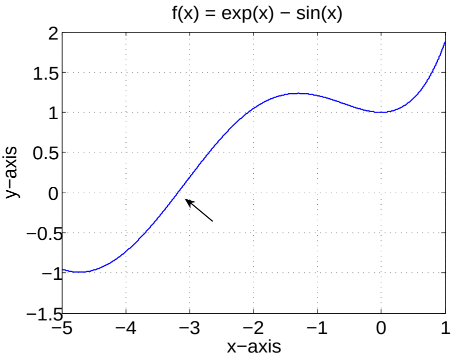
Note:

- $sign(w) \neq sign(u)$ is better than $wu < 0$. (why?)
- compute midpoint as $c = a + \frac{b-a}{2}$ rather than $c = \frac{a+b}{2}$. (why?)

An example

Use the bisection method to find the root of $e^x = \sin(x)$.

A rough plot of $f(x) = e^x - \sin(x)$ shows there are no positive zeros, and the first zero to the left of 0 is somewhere in the interval $[-4, -3]$.



see `functiongraph1.m`

Numerical results

The output obtained by bisection algorithm running a MATLAB M-file, `bisection.m`

Starting with $a = -4$ and $b = -3$:

k	c	$f(c)$
1	-3.50000000000000	-0.32058584426730
2	-3.25000000000000	-0.06942092669839
3	-3.12500000000000	0.06052882585276
4	-3.18750000000000	-0.00461629388698
$\vdots$	$\vdots$	$\vdots$
13	-3.18298339843750	0.00008284596304
14	-3.18304443359375	0.00001933261037
15	-3.18307495117188	-0.00001242395017
16	-3.18305969238281	0.00000345432045
$\vdots$	$\vdots$	$\vdots$

See the details of the M-file: `bisection.m`

Theorem (on bisection method)

Suppose that $[a_0, b_0] := [a, b], [a_1, b_1], \cdots, [a_n, b_n], \cdots$ *are the intervals in the bisection method. Then*

(1) $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *exist and the limits are equal.*

(2) *Let* $r = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$*. Then* $f(r) = 0$.

(3) *Let* $c_n = a_n + \frac{1}{2}(b_n - a_n)$*. Then* $\lim_{n\to\infty} c_n = r$ *and*

$$|r - c_n| \le 2^{-(n+1)}(b_0 - a_0).$$

Proof:

(1) Notice that $a_0 \le a_1 \le a_2 \le \cdots \le b_0$ and $b_0 \ge b_1 \ge b_2 \ge \cdots \ge a_0$.

$\because \{a_n\}$ is monotonically nondecreasing *(i.e., increasing, but may not be strictly increasing)* and bounded above by b_0 $\therefore \lim_{n\to\infty} a_n$ exists

$\because \{b_n\}$ is monotonically nonincreasing *(i.e., decreasing, but may not be strictly decreasing)* and bounded below by a_0 $\therefore \lim_{n\to\infty} b_n$ exists

$\because b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)\ \forall\, n \ge 0 \quad \therefore b_n - a_n = 2^{-n}(b_0 - a_0)$

$\therefore \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty}(b_n - a_n) = (b_0 - a_0)\lim_{n\to\infty} 2^{-n} = 0$

$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, say $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = r$.

Proof of the theorem

(2)

$\because f(x)$ is continuous

$\therefore \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(r)$ and $\lim_{n\to\infty} f(b_n) = f(\lim_{n\to\infty} b_n) = f(r)$

$\because f(a_n)f(b_n) < 0$

$\therefore \ 0 \geq \lim_{n\to\infty} f(a_n)f(b_n) = f(r)f(r)$

$\therefore f(r) = 0$

(3)

$\because r \in [a_n, b_n]$ and $c_n = \frac{1}{2}(a_n + b_n) = a_n + \frac{1}{2}(b_n - a_n)$

$\therefore |r - c_n| \leq \frac{1}{2}(b_n - a_n) = 2^{-(n+1)}(b_0 - a_0) \quad \square$

Note: Is it true that $|c_0 - r| \geq |c_1 - r| \geq |c_2 - r| \geq \dots$?

Answer: No! $\Rightarrow$ *not linear convergence!*

linear: if $\exists\ 0 < C < 1$ and $\exists\ n_0 \in \mathbb{N}$ s.t. $|x_{n+1} - x^*| \leq C|x_n - x^*|,\ \forall\ n \geq n_0$.

An example

If we start with the initial interval $[50, 63]$, how many steps do we need in order to have a relative accuracy less than or equal to 10^{-12}?

This is what we want

$$\frac{|r - c_n|}{|r|} \leq 10^{-12}.$$

Since we know $r \geq 50$, thus it is sufficient to have

$$\frac{|r - c_n|}{50} \leq 10^{-12}.$$

Using the above estimate, all we need is

$$2^{-(n+1)} \frac{63 - 50}{50} \leq 10^{-12}.$$

That means $n \geq 37$.

Some major problems with the bisection method

- Finding the initial interval is not easy.
- Often slow.
- Doesn't work for higher dimensional problems: $F(X) = \mathbf{0}$.

Newton's method

- **Motivation:** we know how to solve $f(x) = 0$ if f is linear. For nonlinear f, we can always approximate it with a linear function.
- Let x^* be a root of $f(x) = 0$ and x an approximation of x^*. Let $x^* = x + h$. Using Taylor's expansion, we have

  $$0 = f(x^*) = f(x+h) = f(x) + hf'(x) + O(h^2).$$

  If h is small, then we can drop the $O(h^2)$ term, $0 \approx f(x) + hf'(x)$, which means

  $$h \approx -\frac{f(x)}{f'(x)}, \quad \textit{provided } f'(x) \neq 0.$$

  Thus, if x is an approximation of $x^* = x + h$, then

  $$x^* = x + h \approx x - \frac{f(x)}{f'(x)}, \quad \textit{provided } f'(x) \neq 0.$$

- Newton's method can be defined as follows: for $n = 0, 1, \cdots$

  $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \textit{provided } f'(x_n) \neq 0.$$

An example

Find the root of $f(x) = e^x - 1.5 - \tan^{-1}(x)$.

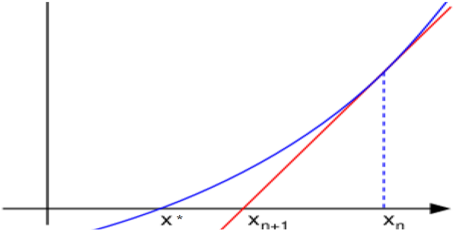
Note that $f(0) = -0.5$, $\lim_{x\to\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) > 0.07$.
Therefore, $\exists\ c^+ \in (0,\infty)$ and $c^- \in (-\infty, 0)$ are zeros of f.

Suppose we start with $x_0 = -7.0$, then the results of Newton iterations are

$$
\begin{aligned}
x_0 &= -7.0, & f(x_0) &= -0.7 \times 10^{-1} \\
x_1 &= -10.7, & f(x_1) &= -0.2 \times 10^{-1} \\
x_3 &= -14.0, & f(x_3) &= -0.2 \times 10^{-3} \\
x_4 &= -14.1, & f(x_4) &= -0.8 \times 10^{-6}
\end{aligned}
$$

The output shows rapid convergence of the iterations.

Geometrical interpretation



- This is an illustration of one iteration of Newton's method. The function f is shown in blue and the tangent line is in red. We see that x_{n+1} is a better approximation than x_n for the root x^* of the function f.
- What is the geometrical meaning of $f'(x_n) = 0$?

Some stopping criteria

- Using the residual information $f(x_n)$:

  (1) if $|f(x_n)| < \varepsilon$ then stop (absolute residual criterion).
  (2) if $|f(x_n)| < \varepsilon |f(x_0)|$ then stop (relative residual criterion).

- Using the step size information $|x_{n+1} - x_n|$:

  (1) if $|x_{n+1} - x_n| < \delta$ then stop (approximate absolute error criterion).
  (2) if $\dfrac{|x_{n+1} - x_n|}{|x_{n+1}|} < \delta$ then stop (approximate relative error criterion).

- Maximum number of iterations M.

Newton's algorithm including stopping criteria

input $x_0, M, \varepsilon, \delta$
$v \leftarrow f(x_0)$
if $|v| < \varepsilon$ **then stop**
for $k = 1$ **to** M **do**
  $x_1 = x_0 - v/f'(x_0)$
  $v \leftarrow f(x_1)$
  if $|x_1 - x_0| < \delta$ **or** $|v| < \varepsilon$ **then stop**
  $x_0 \leftarrow x_1$
end do

See the details of the M-file `newton.m` for $f(x) = e^x - \sin(x)$

Note: if $f'(x_0)$ is too small, then $1/f'(x_0)$ may overflow.

Convergence analysis

Assume that f'' is continuous and x^* is a simple zero of f, i.e., $f(x^*) = 0$ and $f'(x^*) \neq 0$. Define the error as $e_n = x_n - x^*$. Then

$$
\begin{aligned}
e_{n+1} &= x_{n+1} - x^* = x_n - \frac{f(x_n)}{f'(x_n)} - x^* \\
&= e_n - \frac{f(x_n)}{f'(x_n)} = \frac{e_n f'(x_n) - f(x_n)}{f'(x_n)}.
\end{aligned}
$$

Using Taylor's expansion,

$$0 = f(x^*) = f(x_n - e_n) = f(x_n) - e_n f'(x_n) + \frac{1}{2} e_n^2 f''(\xi_n),$$

for some ξ_n between x_n and x^*. Therefore, we have

$$(\star) \qquad e_{n+1} = \frac{1}{2}\frac{f''(\xi_n)}{f'(x_n)} e_n^2 \ \left(\approx \frac{1}{2}\frac{f''(x^*)}{f'(x^*)} e_n^2 := C e_n^2, \textit{ provided } x_n \approx x^*\right).$$

Define a quantity c_δ for $\delta > 0$ by

$$c_\delta := \frac{1}{2}\left(\max_{|x-x^*|\le\delta} |f''(x)|\right) \Big/ \left(\min_{|x-x^*|\le\delta} |f'(x)|\right) \ge 0.$$

We can select $\delta > 0$ such that $\rho := \delta c_\delta < 1$. (why?)

Theorem on Newton's method

Assume that $|e_0| = |x_0 - x^*| < \delta$. Then $|\xi_0 - x^*| < \delta$ and we have $\frac{1}{2}|f''(\xi_0)/f'(x_0)| \le c_\delta$. Therefore,

$$|x_1 - x^*| = |e_1| \le e_0^2 c_\delta = |e_0||e_0|c_\delta < |e_0|\delta c_\delta = |e_0|\rho < |e_0| < \delta.$$

Repeating this argument, we have

$$|e_1| < \rho|e_0|, \; |e_2| < \rho|e_1| < \rho^2|e_0|, \; \cdots, \; |e_n| < \rho^n|e_0|.$$

Since $0 \le \rho < 1$, we have $\lim_{n\to\infty} \rho^n = 0$ which implies that $\lim_{n\to\infty} e_n = 0$.

Finally, since $|e_n| = |x_n - x^*| < \delta$ and $|\xi_n - x^*| < \delta$, we have from ($\star$) that

$$|e_{n+1}| = \frac{1}{2}\frac{|f''(\xi_n)|}{|f'(x_n)|}|e_n|^2 \le \frac{1}{2}c_\delta|e_n|^2 \le \frac{1}{2}(c_\delta + 1)|e_n|^2 := C|e_n|^2,$$

which implies the quadratic convergence. □

Theorem on Newton's method

Theorem on Newton's method: *Let f'' be continuous and let x^* be a simple zero of f. Then there exist $\delta > 0$ and $C > 0$ such that if the initial guess $x_0 \in N(x^*, \delta)$ (i.e., $|x_0 - x^*| < \delta$) then Newton's method converges and satisfies*

$$|x_{n+1} - x^*| \leq C|x_n - x^*|^2 \quad (\forall\, n \geq 0).$$

Good: *the convergence is quadratic.*
Bad: *the initial guess x_0 has to be close to the solution x^*.*

Example

Find the root of $f(x) = \alpha - 1/x$, for any given $\alpha > 0$ (we know the exact solution is $x^* = 1/\alpha$). Using Newton's method, we have

$$x_{n+1} = x_n - \frac{\alpha - \frac{1}{x_n}}{1/x_n^2},$$

which is same as

$$x_{n+1} = 2x_n - \alpha x_n^2, \quad n = 0, 1, 2, \cdots$$

Questions:

- Does the sequence $x_0, x_1, x_2, \ldots$ converge? ($\Longleftrightarrow 0 < x_0 < \frac{2}{\alpha}$)
- How fast? (quadratic)
- Does the convergence depend on the initial guess x_0? (Yes)

Example (cont'd)

Let us define the error $e_n = x^* - x_n = \frac{1}{\alpha} - x_n$. Then

$$e_{n+1} = \frac{1}{\alpha} - x_{n+1} = \frac{1}{\alpha} - 2x_n + \alpha x_n^2 = \alpha(\frac{1}{\alpha} - x_n)^2 = \alpha e_n^2.$$

Thus, if it converges, then the rate is quadratic. We now have

$$\begin{aligned} e_{n+1} &= \alpha e_n^2 = \alpha(\alpha e_{n-1}^2)^2 = \alpha^3(e_{n-1}^2)^2 = \frac{1}{\alpha}(\alpha^2 e_{n-1}^2)^2 = \frac{1}{\alpha}(\alpha e_{n-1})^{2^2} \\ &= \frac{1}{\alpha}(\alpha\alpha e_{n-2}^2)^{2^2} = \frac{1}{\alpha}(\alpha^2 e_{n-2}^2)^{2^2} = \frac{1}{\alpha}(\alpha e_{n-2})^{2^3} = \cdots = \frac{1}{\alpha}(\alpha e_0)^{2^{n+1}}, \end{aligned}$$

which implies that

$$\begin{aligned} x_n \textit{ converges to } x^* &\iff \lim_{n\to\infty} e_n = 0 \iff |\alpha e_0| < 1 \iff |e_0| < \frac{1}{\alpha} \\ &\iff |\frac{1}{\alpha} - x_0| < \frac{1}{\alpha} \iff -\frac{1}{\alpha} < \frac{1}{\alpha} - x_0 < \frac{1}{\alpha} \\ &\iff 0 < x_0 < \frac{2}{\alpha}. \end{aligned}$$

Some remarks on Newton's method

Advantages:

- The convergence is quadratic.
- Newton's method works for higher dimensional problems.

Disadvantages:

- Newton's method converges only locally; i.e., the initial guess x_0 has to be close enough to the solution x^*.
- It needs the first derivative of $f(x)$.

Newton's method for systems of nonlinear equations

- We wish to solve

$$\begin{cases} f_1(x_1,x_2) &= 0, \\ f_2(x_1,x_2) &= 0, \end{cases}$$

where f_1 and f_2 are nonlinear functions of x_1 and x_2.

- Assume that (x_1+h_1, x_2+h_2) is a solution of the nonlinear system of equations. Applying Taylor's expansion in two variables around (x_1,x_2), we obtain

$$\begin{cases} 0 = f_1(x_1+h_1, x_2+h_2) & \approx f_1(x_1,x_2) + h_1 \frac{\partial f_1(x_1,x_2)}{\partial x_1} + h_2 \frac{\partial f_1(x_1,x_2)}{\partial x_2}, \\ 0 = f_2(x_1+h_1, x_2+h_2) & \approx f_2(x_1,x_2) + h_1 \frac{\partial f_2(x_1,x_2)}{\partial x_1} + h_2 \frac{\partial f_2(x_1,x_2)}{\partial x_2}. \end{cases}$$

- Putting it into the matrix form, we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \approx \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1(x_1,x_2)}{\partial x_1} & \frac{\partial f_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial f_2(x_1,x_2)}{\partial x_1} & \frac{\partial f_2(x_1,x_2)}{\partial x_2} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}.$$

Newton's method for systems of nonlinear equations (cont'd)

- To simplify the notation we introduce the Jacobian matrix:

$$J(x_1,x_2) = \begin{bmatrix} \frac{\partial f_1(x_1,x_2)}{\partial x_1} & \frac{\partial f_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial f_2(x_1,x_2)}{\partial x_1} & \frac{\partial f_2(x_1,x_2)}{\partial x_2} \end{bmatrix}.$$

- Then we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \approx \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix} + J(x_1,x_2) \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}.$$

- If $J(x_1,x_2)$ is nonsingular then we can solve for $[h_1,h_2]^\top$:

$$J(x_1,x_2) \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \approx - \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix}.$$

Newton's method for systems of nonlinear equations (cont'd)

- Newton's method for the system of nonlinear equations is defined as follows: for $k = 0, 1, \cdots,$

$$\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{bmatrix} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} + \begin{bmatrix} h_1^{(k)} \\ h_2^{(k)} \end{bmatrix}$$

with

$$J(x_1^{(k)}, x_2^{(k)}) \begin{bmatrix} h_1^{(k)} \\ h_2^{(k)} \end{bmatrix} = - \begin{bmatrix} f_1(x_1^{(k)}, x_2^{(k)}) \\ f_2(x_1^{(k)}, x_2^{(k)}) \end{bmatrix}.$$

- **Exercise:**

Solve the following nonlinear system by using Newton's method with the initial guess $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)})^\top = (0, 1)^\top$. Perform two iterations.

$$\begin{cases} 4{x_1}^2 - {x_2}^2 & = 0, \\ 4x_1{x_2}^2 - x_1 & = 1. \end{cases}$$

Newton's method for higher dimensional problems

- In general, we can use Newton's method for $F(X) = \mathbf{0}$, where $X = (x_1, x_2, \ldots, x_n)^\top$ and $F = (f_1, f_2, \ldots, f_n)^\top$.
- For higher dimensional problem, the first derivative is defined as a matrix (the Jacobian matrix)

$$DF(X) := \begin{bmatrix} \frac{\partial f_1(X)}{\partial x_1} & \frac{\partial f_1(X)}{\partial x_2} & \cdots & \frac{\partial f_1(X)}{\partial x_n} \\ \frac{\partial f_2(X)}{\partial x_1} & \frac{\partial f_2(X)}{\partial x_2} & \cdots & \frac{\partial f_2(X)}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n(X)}{\partial x_1} & \frac{\partial f_n(X)}{\partial x_2} & \cdots & \frac{\partial f_n(X)}{\partial x_n} \end{bmatrix}.$$

- Newton's method: given $X^{(0)} = [x_1^{(0)}, \cdots, x_n^{(0)}]^\top$, define

$$X^{(k+1)} = X^{(k)} + H^{(k)},$$

where

$$DF(X^{(k)})H^{(k)} = -F(X^{(k)}),$$

which requires the solving of a large linear system of equations at every iteration.

Operations involved in Newton's method

- vector operations: not expensive.
- function evaluations: can be expensive.
- compute the Jacobian: can be expensive.
- solving matrix equations (linear system): very expensive – topic of the next chapter!

Methods without using derivatives

- "Finite difference Newton's method" and "secant method."
- **Basic idea:**

$$x \leftarrow x - \frac{f(x)}{f'(x)}.$$

  If $f'(x)$ is too hard or too expensive to compute, we can use an approximation.

- **Questions:** how to obtain an approximation? Do we lose the fast convergence?

Finite difference Newton's method

- Let h be a small nonzero parameter, then

$$a := \frac{f(x_n + h) - f(x_n)}{h}$$

can be a good approximation of $f'(x_n)$.

- **FD-Newton's method:**

(1) compute $a = \dfrac{f(x_n + h) - f(x_n)}{h}$.

(2) compute $x_{n+1} = x_n - \dfrac{f(x_n)}{a}$.

- **Remarks:**

(1) the method needs an extra parameter h. What shall we use?
(2) the method needs two function evaluations per iteration.
(3) what is the convergence rate?

Secant method

- Since h can be any small number in the FD-Newton's method, why don't we simply use $h = x_n - x_{n-1}$, which may be positive or negative, but usually not zero.
- **Secant method:**

  (1) compute $a = \dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$.

  (2) compute $x_{n+1} = x_n - \dfrac{f(x_n)}{a}$.
- **Remarks:**

  (1) now we need only one function evaluation per iteration.

  (2) x_{n+1} depends on two previous iterations. For example, to compute x_2, we need both x_1 and x_0.

  (3) how do we obtain x_1? We need to use FD-Newton: pick a small parameter h, compute $a_0 = (f(x_0 + h) - f(x_0))/h$, then $x_1 = x_0 - f(x_0)/a_0$.

Which of the three methods is better?

An example: $f(x) = x^2 - 1$, and we take $x_0 = 2.0$.

Stopping parameters: $\delta = 10^{-10}$, $\varepsilon = 10^{-10}$.

$h = 10^{-7}$ in FD-Newton method.

Iter.	Newton	FD-Newton	Secant
x_0	2.0	2.0	2.0
x_1	1.25000000000000	1.25000001709125	1.25000001709125
x_2	1.02500000000000	1.02500001222170	1.07692308177740
x_3	1.00030487804878	1.00030487955710	1.00826446381851
x_4	1.00000004646115	1.00000004647732	1.00030487810437
x_5	1.00000000000000	1.00000000000000	1.00000125445212
x_6			1.00000000019120
x_7			1.00000000000000

See the details of the M-files: `comparisonnewton.m`, `comparisonFDnewton.m`, `comparisonsecant.m`

Convergence rates

- If $|h_n| \leq C|x_n - x^*|$, then the convergence of FD-Newton is quadratic.
- *The convergence of secant method is superlinear (i.e., better than linear).* More precisely, we have (see Textbook, pp. 96-97)

$$|e_{n+1}| \leq C|e_n|^{(1+\sqrt{5})/2}, \quad (1+\sqrt{5})/2 \approx 1.62 < 2.$$

- **Remark:** when selecting algorithms for a particular problem, one should consider not only the rate (order) of convergence, but also the cost of computing $f(x_n)$ and $f'(x_n)$.

An informal convergence analysis of the secant method

Let $e_n := x_n - x^*$. Under suitable assumptions, it can be shown that $e_{n+1} \approx C e_n e_{n-1}$ (Textbook, p. 96) and $\lim_{n\to\infty} e_n = 0$ (cf. analysis for Newton's method).

To discover the order of convergence, we assume that for large n, $|e_{n+1}| \approx \lambda |e_n|^\alpha$. Thus, $|e_n| \approx \lambda |e_{n-1}|^\alpha \Rightarrow |e_{n-1}| \approx \lambda^{-1/\alpha} |e_n|^{1/\alpha}$.

$\therefore \lambda |e_n|^\alpha \approx |e_{n+1}| \approx |C||e_n|\lambda^{-1/\alpha}|e_n|^{1/\alpha}$

$\therefore |e_n|^\alpha \approx |C|\lambda^{-1/\alpha-1}|e_n|^{1+1/\alpha}$

$\therefore |e_n|^{\alpha-1-1/\alpha} \approx |C|\lambda^{-1/\alpha-1}$

$\because$ the right side of this relation is a nonzero constant while $e_n \to 0$

$\therefore \alpha - 1 - 1/\alpha = 0$

$\therefore \alpha^2 - \alpha - 1 = 0$

$\therefore \alpha = \frac{1+\sqrt{5}}{2} \approx 1.62 > 0 \quad \square$

Steffensen's method – method without using derivative

Steffensen's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}, \quad where\ g(x_n) := \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}.$$

Under suitable hypotheses, the method is quadratically convergent (p. 90, # 4).

An informal convergence analysis: Assume that $f \in C^2$. By Taylor expansion, we have

$$f(x + f(x)) = f(x) + f(x)f'(x) + \frac{f(x)^2}{2} f''(\xi),$$

for some ξ between x and $x + f(x)$. Therefore,

$$g(x) := \frac{1}{f(x)} \{f(x + f(x)) - f(x)\} = f'(x) + \frac{f(x)}{2} f''(\xi) \approx f'(x), \text{ if } f(x) \approx 0.$$

Let $e_n := x_n - x^*$. Then, $e_{n+1} = e_n - \frac{f(x_n)}{g(x_n)} = \frac{1}{g(x_n)} \left\{ e_n g(x_n) - f(x_n) \right\}$.

Steffensen's method (cont'd)

$\because\ 0 = f(x^*) = f(x_n - e_n) = f(x_n) - e_n f'(x_n) + \frac{e_n^2}{2} f''(\xi_n),$
for some ξ_n between x_n and $x_n - e_n$

$$\therefore\ f(x_n) - e_n g(x_n) \approx -\frac{e_n^2}{2} f''(\xi_n)$$

$$\therefore\ e_{n+1} \approx \frac{e_n^2}{2}\frac{f''(\xi_n)}{g(x_n)} \left(\approx \frac{f''(x^*)}{2f'(x^*)} e_n^2,\ \textit{provided}\ x_n \approx x^*\right)$$

(cf. analysis of Newton's method). □

Remarks:

- Bisection algorithms is global, and all the other Newton-type algorithms are local.
- Local algorithms are often fast, and global algorithms are often slow.

Fixed points

- A function $F : x \mapsto F(x)$ is often called a mapping from x to $F(x)$ (F takes an input value x and generates an output value $F(x)$).

  If there is a point p, at which the output is the same as the input, then that point is called a fixed point of F, i.e., $p = F(p)$.

- Finding the fixed points of F has many applications. For example, if

$$F(x) := x - \frac{f(x)}{f'(x)},$$

  then the fixed point of F is simply the root of $f(x) = 0$.

  "root-finding problem" $\Longrightarrow$ "fixed point problem"

Fixed point iterations

- Fixed point iterations:

$$x_{n+1} = F(x_n), \quad n = 0, 1, \cdots$$

Assume that F is continuous and $\lim_{n\to\infty} x_n = p$. Then

$$F(p) = F(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} x_{n+1} = p.$$

Therefore, p is a fixed point of the function F.

- The following three fixed point iterations can be considered for solving $x^3 - x - 5 = 0$:

$x_{n+1} = F(x_n), n = 0, 1, \cdots,$ where

(1) $F(x) = x^3 - 5.$

(2) $F(x) = (x + 5)^{1/3}.$

(3) $F(x) = \dfrac{5}{x^2 - 1}.$

Do the iterations converge?

A fixed point theorem

- *If $F \in C[a,b]$ and $F(x) \in [a,b]$, $\forall\, x \in [a,b]$, then F has a fixed point in $[a,b]$.*
- *If, in addition, F' exists on (a,b) and $\exists\, 0 < k < 1$ such that $|F'(x)| \le k$, $\forall\, x \in (a,b)$, then the fixed point is unique in $[a,b]$.*
- *Then, for any $x_0 \in [a,b]$ and $x_{n+1} := F(x_n)$, $n \ge 0$, the sequence converges to the unique fixed point $p \in [a,b]$ and*
  (1) $|x_n - p| \le k^n \max\{x_0 - a, b - x_0\}, \forall\, n \ge 1;$
  (2) $|x_n - p| \le \frac{k^n}{1-k}|x_1 - x_0|, \forall\, n \ge 1.$

Proof.

- If $F(a) = a$ or $F(b) = b$ then F has a fixed point in $[a,b]$. Suppose not, then $a < F(a) \le b$ and $a \le F(b) < b$. Define $H(x) := F(x) - x$. Then H is continuous on $[a,b]$ and $H(a) > 0$, $H(b) < 0$. By the Intermediate Value Theorem, $\exists\, p \in (a,b)$ such that $H(p) = 0$, i.e., $F(p) = p$. $\square$
- Suppose that $\exists\, p < q \in [a,b]$ are fixed points of F. Then $F(p) = p$ and $F(q) = q$. By the Mean Value Theorem, $\exists\, \xi \in (p,q)$ such that $\frac{F(q)-F(p)}{q-p} = F'(\xi) \Longrightarrow$ $\frac{|F(q)-F(p)|}{|q-p|} = |F'(\xi)| \le k < 1 \Longrightarrow 1 = \frac{|q-p|}{|q-p|} \le k < 1$. This is a contradiction. Therefore, the fixed point is unique. $\square$

Proof of the fixed point theorem (cont'd)

- For $n \geq 1$, by the Mean Value Theorem, $\exists\, \xi \in (a, b)$ such that
$0 \leq |x_n - p| = |F(x_{n-1}) - F(p)| = |F'(\xi)||x_{n-1} - p| \leq k|x_{n-1} - p|$.
$\Longrightarrow 0 \leq |x_n - p| \leq k|x_{n-1} - p| \leq k^2|x_{n-2} - p| \leq \cdots \leq k^n|x_0 - p|$.
$\Longrightarrow \lim_{n\to\infty} |x_n - p| = 0 \Leftrightarrow \lim_{n\to\infty} x_n - p = 0 \Leftrightarrow \lim_{n\to\infty} x_n = p$.

(1) $\because |x_n - p| \leq k^n|x_0 - p|$ and $p \in [a, b]$
$\therefore |x_n - p| \leq k^n \max\{x_0 - a, b - x_0\}, \forall\, n \geq 1$

(2) For $n \geq 1$,
$|x_{n+1} - x_n| = |F(x_n) - F(x_{n-1})| \leq k|x_n - x_{n-1}| \leq \cdots \leq k^n|x_1 - x_0|$.
$\therefore$ For $m > n \geq 1$, we have

$$\begin{aligned}
|x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots + x_{n+1} - x_n| \\
&\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\
&\leq k^{m-1}|x_1 - x_0| + k^{m-2}|x_1 - x_0| + \cdots + k^n|x_1 - x_0| \\
&= k^n(1 + k + \cdots + k^{m-n-1})|x_1 - x_0|.
\end{aligned}$$

$\because \lim_{n\to\infty} x_n = p$

$$\therefore |p - x_n| = \lim_{m\to\infty} |x_m - x_n| \leq k^n|x_1 - x_0| \sum_{i=0}^{\infty} k^i = k^n|x_1 - x_0| \frac{1}{1-k}$$

($\because$ geometric series with $0 < k < 1$)

$\therefore |p - x_n| \leq \frac{k^n}{1-k}|x_1 - x_0| \quad \square$

Contractive mappings

- **Definition:** *A mapping (function) F is said to be contractive if $\exists\, 0 < \lambda < 1$ such that $|F(x) - F(y)| \le \lambda |x - y|$, for all x, y in the domain of F.*
- **Note:** In the above theorem, F is contractive on $[a, b]$.
- **Example:** $F(x) = 4 + \frac{1}{3}\sin(2x)$ is contractive on $\mathbb{R}$.

$$
\begin{aligned}
|F(x) - F(y)| &= \frac{1}{3}|\sin(2x) - \sin(2y)| \\
&= \frac{2}{3}|\cos(2\xi)||x - y| \\
&\le \frac{2}{3}|x - y|.
\end{aligned}
$$

Contraction mapping principle

Let F be a contractive mapping from a complete metric space $X \subseteq \mathbb{R}$ into itself. Then F has a unique fixed point p and the sequence $\{x_n\}$ generated by $x_{n+1} := F(x_n)$, $n \geq 0$, converges to p for any $x_0 \in X$.

Proof:

- show that $\{x_n\}$ converges;
- let $\lim_{n\to\infty} x_n = p$. Then $F(p) = p$;
- show that p is unique. $\square$

Note: *Let X be a closed subset of $\mathbb{R}$. Then X is a complete metric space.*

Example: closed subsets of $\mathbb{R}$: $[a, b]$, $\mathbb{R}$, etc.

Error analysis

- Assume that F' exists and continuous. Consider the fixed point iterations,
$$x_{n+1} = F(x_n), \quad n \geq 0.$$
Assume that $\{x_n\}$ converges to p (p is a fixed point). Let $e_n := x_n - p$. Then, by MVT, we have
$$e_{n+1} = x_{n+1} - p = F(x_n) - F(p) = F'(\xi_n)(x_n - p) = F'(\xi_n)e_n,$$
for some ξ_n between x_n and p. The condition $|F'(x)| < 1$ for all x ensures that the errors decrease in magnitude. If e_n is small then ξ_n is near p, and $F'(\xi_n) \approx F'(p)$.
- One would expect rapid convergence if $F'(p)$ is small. Ideally, $F'(p) = 0$.

Error analysis (cont'd)

- Assume that $F^{(k)}(p) = 0$ for $1 \le k < r$ but $F^{(r)}(p) \neq 0$. Then

$$
\begin{aligned}
e_{n+1} &= x_{n+1} - p = F(x_n) - F(p) = F(p + e_n) - F(p) \\
&= \left\{ F(p) + e_n F'(p) + \frac{e_n^2}{2} F''(p) + \cdots + \frac{1}{r!} e_n^r F^{(r)}(\xi_n) \right\} - F(p) \\
&= e_n F'(p) + \frac{e_n^2}{2} F''(p) + \cdots + \frac{e_n^{r-1}}{(r-1)!} F^{(r-1)}(p) + \frac{e_n^r}{r!} F^{(r)}(\xi_n) \\
&= \frac{e_n^r}{r!} F^{(r)}(\xi_n).
\end{aligned}
$$

- If we know that the method converges and $F^{(r)}$ is continuous then

$$
\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^r} = \frac{1}{r!} |F^{(r)}(p)|
$$

and the method converges with order r.

Newton' method

Newton' method: $F(x) = x - \dfrac{f(x)}{f'(x)}, f(p) = 0$ and $f'(p) \neq 0, F(p) = p.$

$$\because F'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}$$

$$\therefore F'(p) = 0$$

$$\because$$
$$F''(x) = \frac{(f'(x))^2\{f(x)f'''(x) + f''(x)f'(x)\} - (f(x)f''(x))(2f'(x)f''(x))}{(f'(x))^4}$$

$\therefore$ we usually have $F''(p) = \dfrac{f''(p)}{f'(p)} \neq 0$

$\therefore$ under suitable assumptions,
the order (rate) of convergence of Newton's method is 2

Roots of polynomials

- A general polynomial: $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z^1 + a_0$, where coefficients $a_i \in \mathbb{C}$, $i = 0, 1, \cdots, n$. If $a_n \neq 0$ then we say *$degree(p) = n$.*
- **Fundamental Theorem of Algebra:** *Every nonconstant polynomial has at least one root in $\mathbb{C}$.*

  ($\Longleftrightarrow$ A polynomial of degree n has exactly n roots in $\mathbb{C}$).
- If p is a polynomial whose coefficients are all real, $a_i \in \mathbb{R}\ \forall\ i$, then its roots may be complex and if $w = w_1 + iw_2$ is a complex root then its conjugate $\overline{w} := w_1 - iw_2$ is also a root.

In what follows, we consider polynomials with real coefficients.

Horner's algorithm

Newton's method: $z_{k+1} := z_k - \frac{p(z_k)}{p'(z_k)}, k = 0, 1, 2, \ldots.$

We need function evaluations $p(z_k)$ and $p'(z_k)$ in Newton's method.

- Given a polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z^1 + a_0$ and $z_0 \in \mathbb{R}$. **Horner's algorithm** will produce the number $p(z_0)$ and the polynomial $q(z)$ such that $p(z) = (z - z_0)q(z) + p(z_0)$.
- Assume that $q(z) = b_{n-1} z^{n-1} + b_{n-2} z^{n-2} + \cdots + b_1 z^1 + b_0$. Then we have $b_{n-1} = a_n$, $b_{n-2} = a_{n-1} + z_0 b_{n-1}, \cdots, b_0 = a_1 + z_0 b_1$, $p(z_0) = p(z) - (z - z_0)q(z) = a_0 + z_0 b_0$.
- **Synthetic division:** (綜合除法)

$$\begin{array}{c|ccccc}
 & a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \\
z_0 & & z_0 b_{n-1} & z_0 b_{n-2} & \cdots & z_0 b_0 \\
\hline
 & b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_{-1} & \leftarrow p(z_0)
\end{array}$$

We have $p(z_0) = b_{-1}$.

Example

Let $p(z) = z^4 - 4z^3 + 7z^2 - 5z - 2$. Evaluate $p(3)$.

	1	-4	7	-5	-2	
3		3	-3	12	21	
	1	-1	4	7	19	$\leftarrow p(3)$

$\therefore p(3) = 19$, $q(z) = z^3 - z^2 + 4z + 7$, and

$p(z) = (z-3)(z^3 - z^2 + 4z + 7) + 19$.

Complete Horner's algorithm

Given $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z^1 + a_0$ and $z_0 \in \mathbb{R}$.

We wish to find $c_i, i = 0, 1, \cdots, n$ such that

$$p(z) = c_n (z - z_0)^n + c_{n-1} (z - z_0)^{n-1} + \cdots + c_1 (z - z_0)^1 + c_0.$$

If so, by Taylor Theorem, we know that $c_k = \dfrac{p^{(k)}(z_0)}{k!}$.

$\therefore p(z_0) = c_0$ and $p'(z_0) = c_1 = q(z_0)$

$\therefore$ We can apply Horner's algorithm again to $q(z)$ with point z_0

Repeat this process, we can obtain $c_i, i = 0, 1, \cdots, n$.

Example

Let $p(z) = z^4 - 4z^3 + 7z^2 - 5z - 2$ and $z_0 = 3$.

	1	-4	7	-5	-2	
3		3	-3	12	21	
	1	-1	4	7	19	$\leftarrow p(3)$
3		3	6	30		
	1	2	10	37	$\leftarrow p'(3)$	
3		3	15			
	1	5	25			
3		3				
	1	8				

$\therefore p(3) = 19, p'(3) = 37$ and

$$p(z) = 1(z-3)^4 + 8(z-3)^3 + 25(z-3)^2 + 37(z-3)^1 + 19$$

Newton's method with Horner's algorithm

program `horner` $(n, (a_i : 0 \le i \le n), z_0, \alpha, \beta)$

$\alpha \leftarrow a_n$

$\beta \leftarrow 0$

for $k = n-1 : -1 : 0$ **do**

  $\beta \leftarrow \alpha + z_0\beta$

  $\alpha \leftarrow a_k + z_0\alpha$

end do

output $\alpha (= p(z_0)), \beta (= p'(z_0))$

---

program `newton` $(n, (a_i : 0 \le i \le n), z_0, M, \delta)$

for $k = 1 : 1 : M$ **do**

  call `horner` $(n, (a_i : 0 \le i \le n), z_0, \alpha, \beta)$

  $z_1 \leftarrow z_0 - \alpha/\beta$

  output α, β, z_1

  if $|z_1 - z_0| < \delta$ **then stop**

  $z_0 \leftarrow z_1$

end do

Basic idea of continuation method (延拓法)

The basic idea of the continuation method is to embed the given problem in a one-parameter family of problems, using a parameter t that runs over $[0,1]$, such that for $t=1$ we have the original problem, while for $t=0$ we have another problem with known solution.

Below is an example:

- Consider a root-finding problem: $f(x)=0$. We extend the problem to a one-parameter family of problems:

  $$h(t,x) = tf(x) + (1-t)g(x),$$

  where $t \in [0,1]$ and $g(x)$ is given and have a known zero, say x_0.
- Select points $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$. We then solve each equation $h(t_i,x)=0$, $i=0,1,\cdots,m$. We say each solution x_i, $i=0,1,\cdots,m$.
- Assume that some iterative method such as Newton's method is used to solve $h(t_i,x)=0$, we use the solution x_{i-1} of $h(t_{i-1},x)=0$ as the starting point.

Homotopy (同倫)

Definition: *Let X and Y be two topological spaces and $f, g : X \to Y$ be two continuous functions. A homotopy between f and g is defined to be a continuous function $h : [0,1] \times X \to Y$ such that, for all points $x \in X$, $h(0,x) = g(x)$ and $h(1,x) = f(x)$. If such a map exists, we say that f is homotopic to g.*

A simple example that is often used in continuation method is

$$h(t,x) = tf(x) + (1-t)\underbrace{\left(f(x) - f(x_0)\right)}_{:=g(x)},$$

where x_0 can be any point in X.

Homotopy continuation method

- If $h(t,x)=0$ has a unique solution for each $t\in[0,1]$, then the solution is a function of t, and we write $x(t)\in X$. The set $\{x(t):0\le t\le 1\}$ can be interpreted as a curve in X. *The continuation method attempts to determine this curve by computing points on it, $x(t_0),x(t_1),\cdots,x(t_m)$.*
- **Homotopy continuation method:** Assume that $x(t)$ and $h(t,x)$ are differentiable functions. Then

$$0=h(t,x(t))\Longrightarrow 0=h_t(t,x(t))+h_x(t,x(t))x'(t)$$

$$\Longrightarrow x'(t)=-\Big(h_x(t,x(t))\Big)^{-1}h_t(t,x(t)).$$

  This is an ODE with a known initial value $x(0)$, it can be solved using numerical methods (cf. Chapter 8).
- If necessary, we can apply Newton's iteration starting at the point produced by the homotopy method to approximate the solution of $h(1,x)=0$ one more time.

Example

Let $X = Y = \mathbb{R}^2$ and define

$$f(x,y) = \begin{bmatrix} x^2 - 3y^2 + 3 \\ xy + 6 \end{bmatrix}, \quad (x,y) \in \mathbb{R}^2.$$

A homotopy is defined by

$$\begin{aligned} h(t,(x,y)) &= tf(x,y) + (1-t)(f(x,y) - f(1,1)) \\ &= f(x,y) + tf(1,1) - f(1,1), \quad t \in [0,1], (x,y) \in \mathbb{R}^2, \end{aligned}$$

$$\begin{aligned} h_x(t,(x,y)) &= Df(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{bmatrix} = \begin{bmatrix} 2x & -6y \\ y & x \end{bmatrix}, \\ h_t(t,(x,y)) &= f(1,1) = \begin{bmatrix} 1 \\ 7 \end{bmatrix}. \end{aligned}$$

Example (cont'd)

$$h_x^{-1}(t,(x,y)) = [Df(x,y)]^{-1} = \frac{1}{2x^2+6y^2}\begin{bmatrix} x & 6y \\ -y & 2x \end{bmatrix}.$$

The ODE is

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = -\frac{1}{2x^2+6y^2}\begin{bmatrix} x & 6y \\ -y & 2x \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix} = -\frac{1}{2x^2+6y^2}\begin{bmatrix} x+42y \\ 14x-y \end{bmatrix}.$$

with initial condition $(x(0),y(0))^\top = (1,1)^\top$. By the numerical method for initial-value problem, we have an approximation solution $(-2.961, 1.978)^\top$ of $(x(1),y(1))^\top$. We can use this approximation as the initial guess in the Newton method:

k	$(x^{(k)}, y^{(k)})$	$\|f(x^{(k)}, y^{(k)})\|_2$
0	(-2.96100000000000, 1.97800000000000)	0.14626611680427
1	(-3.00025328131376, 2.00012057060499)	0.00087135657948
2	(-3.00000001019155, 2.00000000338437)	0.00000003679978
3	(-3.00000000000000, 2.00000000000000)	0.00000000000000

See the details of the M-file: `homotopynewton.m`

Theorem on continuously differentiable solution

[Ortega and Rheinboldt, 1970]

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and if $\|[Df(x)]^{-1}\| \le M$ on $\mathbb{R}^n$, then for any $x_0 \in \mathbb{R}^n$ there is a unique curve $\{x(t) : 0 \le t \le 1\}$ in $\mathbb{R}^n$ such that $f(x(t)) + (t-1)f(x_0) = 0$, $0 \le t \le 1$. The function $t \to x(t)$ is a continuously differentiable solution of the initial-value problem $x'(t) = -[Df(x)]^{-1} f(x_0)$, where $x(0) = x_0$.

Note: $$tf(x(t)) + (1-t)\underbrace{\left(f(x(t)) - f(x_0)\right)}_{:=g(x(t))} = f(x(t)) + (t-1)f(x_0).$$