MA8019: ADVANCED NUMERICAL ANALYSIS I
Midterm/November 8, 2023

Please show all your work clearly for full credit!

(1) (10 pts) Assume that we are considering a convergent real sequence $\{x_n\}$ with the limit x^*, and for a sufficiently large n, we have

$$\begin{aligned} x_n - x^* &= -8 \times 10^{-2}, \\ x_{n+1} - x^* &= 4 \times 10^{-4}, \\ x_{n+2} - x^* &= -2 \times 10^{-8}. \end{aligned}$$

Please estimate the order of convergence of the sequence. (You may need $\log_{10} 2 \approx 0.301$)

(2) (10 pts) Consider the hypothetical computer `Marc-32`. Suppose that we are given a single precision floating-point word,

$x =$

1	0000 1110	1010 0000 0000 0000 0000 000

What is the number x represented in the decimal number system?

(3) Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a smooth real-valued function.

(a) (5 pts) What is the definition of the condition number of f at x?

(b) (5 pts) Use the mean-value theorem to derive an approximation of the condition number of f at x.

(c) (5 pts) Assume that $f(x) = \dfrac{10}{4 - x^2}$. Is f well-conditioned for $|x| \approx 2$? Explain why.

(4) The method of fixed point iterations for finding a fixed point of $F : \mathbb{R} \to \mathbb{R}$ is given by

$$x_{n+1} = F(x_n), \quad n = 0, 1, \cdots$$

(a) (5 pts) Show that if F is continuous and $\lim_{n\to\infty} x_n = p$ then p is a fixed point of F.

(b) (5 pts) Assume that we have the fixed point iterations: $x_{n+1} = F(x_n)$, where

$$F(x) = x - \frac{x^2 - 2}{2x}, \quad n \geq 0.$$

What is the root-finding method for f associated with this fixed point iteration method?

(5) (10 pts) Consider the fixed point iterations: $x_{n+1} = F(x_n)$, $n \geq 0$. Assume that $\{x_n\}$ converges to p, $F^{(r)}$ is continuous, and $F^{(k)}(p) = 0$ for $1 \leq k < r$ but $F^{(r)}(p) \neq 0$. Show that the order of convergence of the fixed point method is r.

Hint: $e_{n+1} = x_{n+1} - p = F(x_n) - F(p)$.

(6) Let f be a real-valued function in $C^2(\mathbb{R})$ and let x^* be a simple zero of f. Define a method for approximating the zero x^* by

$$x_{n+1} = x_n - \frac{f(x_n)}{A}, \; n \geq 0.$$

(a) (5 pts) What is A for the finite difference Newton method? What is A for the secant method?

(b) (10 pts) What is A for Steffensen's method? Explain why it is quadratically convergent when it converges.

(7) (15 pts) Consider the following system of nonlinear equations:

$$\begin{cases} f_1(x,y) &= 0, \\ f_2(x,y) &= 0, \end{cases}$$

where f_1 and f_2 are nonlinear smooth real-valued functions of x and y. Please derive Newton's method for solving the system of nonlinear equations.

(8) (15 pts) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector-valued function given by

$$f(x,y) = \begin{bmatrix} x^2 - 3y^2 + 3 \\ xy + 8 \end{bmatrix}, \quad \forall \, (x,y) \in \mathbb{R}^2.$$

A homotopy $h(t,(x,y))$ is defined by

$$h(t,(x,y)) = tf(x,y) + (1-t)\Big(f(x,y) - f(1,2)\Big).$$

Suppose that $h(t,x,y) = 0$ has a unique solution for each $t \in [0,1]$ and we write the solution curve as $\boldsymbol{x}(t) = (x(t), y(t))$ for $t \in [0,1]$. Moreover, assume that $\boldsymbol{x}(t)$ and $h(t,x,y)$ are differentiable functions. Show that $\boldsymbol{x}(t)$ satisfies an initial value problem of system of ordinary differential equations with initial condition $(x(0), y(0))^\top = (1,2)^\top$.