Tournaments

- $x \to y$ iff player x beats player y.
- property S_k: A tournament T_n on n players has property S_k if for every k players there is some other player who beats all of them.

Thm For each k, $\exists\, T_n$ with property S_k.

pf: let $\Omega_n =$ all tournaments of order n with $P_r(x \to y) = P_r(y \to x) = 1/2$.

$$P_r(\forall\ k\text{-set } S\ \exists\, x \text{ s.t. } x \to S) = P_r\Big(\bigcap_{S \in \binom{[n]}{k}} \{\exists\, x \text{ s.t. } x \to S\}\Big)$$

$$= 1 - P_r\Big(\bigcup_{S \in \binom{[n]}{k}} \{\nexists\, x \text{ s.t. } x \to S\}\Big) \ge 1 - \sum_{S \in \binom{[n]}{k}} P_r\Big(\bigcap_{x \in \bar{S}} \{\neg\, x \to S\}\Big)$$

$$= 1 - \binom{n}{k}(1-2^{-k})^{n-k} > 1 - \left(\frac{ne}{k}\right)^k (1-2^{-k})^{n-k} = 1 - n^k (1-2^{-k})^n \underbrace{\left[\left(\frac{e}{k}\right)^k (1-2^{-k})^{-k}\right]}_{\le 1 \text{ as } k \ge 4}$$

$$\ge 1 - n^k (1-2^{-k})^n \ge 1 - n^k e^{-2^{-k} n} > 0,$$ which can be done by choosing

$n = 2^k k^2 2 \ln 2$. $(k \ge 6)$ R.L. Graham & Spencer (1971) "A constructive solution to a tournament problem" Canad. Math. Bull 14 P45~48. **QED**

note: Let $f(k) =$ the minimum possible number of vertices of a tournament that has the property S_k. Then $f(1) = 3$ and $f(2) = 7$.

Probabilistic proof vs Algorithmic Proof

Thm Suppose $\nu_G = n$ and G has minimum degree $\delta > 1$. Then $\gamma(G) \leq n \frac{1+\ln(\delta+1)}{\delta+1}$.

pf: let D be a random vertex subset with $\Pr(v \in D) = p \ \forall v \in V_G$.

let $X_v = \begin{cases} 1 & \text{if } v \in D \\ 0 & \text{o.w.} \end{cases}$ $\quad Y_v = \begin{cases} 1 & \text{if } N[v] \cap D = \phi \\ 0 & \text{o.w.} \end{cases}$

$\mathcal{E} \sum_{v \in V} (X_v + Y_v) \leq np + n(1-p)^{(\delta+1)}$. set $p = \frac{\ln(\delta+1)}{\delta+1}$ to get

$\mathcal{E} \sum_{v \in V} (X_v + Y_v) \leq np + n e^{-p(\delta+1)}$ $\quad (\because e^x \geq 1+x \ \forall x \in \mathbb{R})$

$$= np + \frac{n}{\delta+1} = n \frac{1+\ln(\delta+1)}{\delta+1}$$

QED

Algorithmic proof: $V_G = \overline{U} + U$ where $\overline{U}$ = covered vertices, U = uncovered vertices. 可能重覆

Rule: In each step a vertex that covers the maximum number of yet uncovered vertices is picked.

Let $t_v = |N[v] \cap U|$. Note $\sum_{v \in V_G} t_v \geq \sum_{u \in U} |N[u]| \geq |U|(\delta+1)$, and hence $\exists v$ s.t. $t_v \geq \frac{|U|(\delta+1)}{n}$

If time $= t$ we have $|U|$ uncovered vertices then at time $= t+1$, the # of uncovered vertices $\leq |U|\left(1 - \frac{\delta+1}{n}\right)$

Note that $\left(1 - \frac{\delta+1}{n}\right)^{\frac{n \ln(\delta+1)}{\delta+1}} \leq e^{-\frac{\delta+1}{n} \frac{n\ln(\delta+1)}{\delta+1}} = \frac{1}{\delta+1}$.

After time $= \frac{n \ln(\delta+1)}{\delta+1}$ we have $|\text{uncovered vertices}| \leq \frac{n}{\delta+1}$.

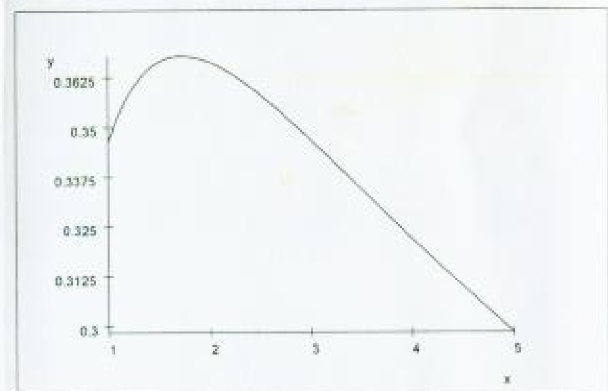
QED

Total Domination Number

Thm $\gamma_t(G) \leq n\left(\frac{2.5218 + \ln(\delta+1)}{\delta+1}\right).$

Pf: let D be a random vertex subset s.t. $\Pr(v \in D) = p$. $\forall v$.

let $X_v = \begin{cases} 1 & \text{if } v \in D \\ 0 & \text{o.w.} \end{cases}$ $\quad Y_v = \begin{cases} 1 & \text{if } N[v] \cap D = \phi \\ 0 & \text{o.w.} \end{cases}$



$p(x) = \frac{\ln(x+1)}{x+1}$, $p(e-1) = e^{-1}$

$2 + \frac{1}{4}\exp(2e^{-1}) = 2.5218$

$Z_v = \begin{cases} 1 & \text{if } v \in D \text{ and } N(v) \cap D = \phi \\ 0 & \text{o.w.} \end{cases}$

$$\gamma_t(G) \leq E\left(\sum_v X_v + \sum_v 2Y_v + \sum_v Z_v\right)$$

$$\leq np + 2n(1-p)^{\delta+1} + np(1-p)^{\delta}, \quad \text{set } p = \frac{\ln(\delta+1)}{\delta+1}$$

$$\leq np + 2n(1-p)^{\delta+1} + \frac{n}{4}(1-p)^{\delta-1}$$

$$\leq np + (2 + \tfrac{1}{4}e^{2p})ne^{-p(\delta+1)} \leq \text{RHS}$$

QED

2-coloring of a hypergraph

- Hypergraph $H=(\mathcal{V},\mathcal{E})$ is n-uniform if $|E|=n \ \forall E\in\mathcal{E}$. ← multiset
- H is 2-colorable if $\exists$ a $\exists$ a 2-coloring of V s.t no monochromatic edge.
- $m(n) \stackrel{\text{def}}{=} \min\{|\mathcal{E}| : H=(\mathcal{V},\mathcal{E})$ is an n-uniform hypergraph which is not 2-colorable$\}$

Thm $m(n) \geq 2^{n-1}$

pf: To show $\forall$ n-uniform hypergraph with $< 2^{n-1}$ edges is 2-colorable.

Let $H=(V,\mathcal{E})$ be an n-uniform hypergraph with $|\mathcal{E}| < 2^{n-1}$.

Let $c: V \to \{0,1\}$ be a random 2-coloring s.t $P(c(v)=1)=\frac{1}{2}$.

$P($c is a proper 2-coloring of H$) = 1 - P($c is NOT a proper 2-coloring of H$)$

$= 1 - P\left(\bigcup_{E\in\mathcal{E}} \{E \text{ is monochromatic under } c\}\right) \geq 1 - \sum_{E} P\{E \text{ is monochromatic under } c\}$

$\geq 1 - |\mathcal{E}|\, 2\left(\frac{1}{2}\right)^n > 0.$

QED

The best known upper bound for $m(n)$

Thm (Erdös 1964) $m(n) < (1+o(1))\dfrac{e\ln 2}{4} n^2 2^n$

Pf: To construct a n-uniform hypergraph $\mathcal{H}=(\mathcal{V},\mathcal{E})$ which is not 2-colorable.

Fix $\mathcal{V}$, say $|\mathcal{V}|=v$. Let (Ω, P) be a pr. space with $P(\omega)=\dfrac{1}{\binom{v}{n}^m}$ $\forall \omega \in \Omega \stackrel{def}{=} \left[\binom{\mathcal{V}}{n}\right]^m$, $\omega = (\omega_1,\dots,\omega_m)$.

$\forall \omega\in\Omega,\ X_c(\omega) \stackrel{def}{=} \begin{cases} 1 & \text{if none of the } \omega_i\text{'s are monochromatic under 2-coloring } c \text{ (given)} \\ 0 & \text{o.w.} \end{cases}$

$$P(\omega_1 \text{ is monochromatic under } c) = \frac{\binom{a}{n}+\binom{b}{n}}{\binom{v}{n}} \geq \frac{2\binom{\frac{a+b}{2}}{n}}{\binom{v}{n}} = \frac{2\binom{v/2}{n}}{\binom{v}{n}} \quad \left(\begin{array}{l}\text{let } a=c^{-1}(1)\\ \quad\ \ b=c^{-1}(0)\\ f(x)=\binom{x}{n} \text{ is convex}\end{array}\right)$$

$$=2^{1-n}\prod_{j=0}^{n-1}\frac{v-2j}{v-j}=2^{1-n}\prod_{j=0}^{n-1}\left(1-\frac{j}{v}+O\left(\frac{j^2}{v^2}\right)\right) \quad \left(\because \frac{v-2j}{v-j}=\frac{1-2\frac{j}{v}}{1-\frac{j}{v}},\ \ \frac{1-2x}{1-x}=2-\frac{1}{1-x}\right)$$

$$\sim 2^{1-n}e^{-\frac{n(n-1)}{2v}}+O\left(\frac{n(n-1)^2}{v^2}\right) > 2^{1-n}e^{-1}(1+o(1)) \quad (\text{set } v=n^2/2)$$

$$\Pr\Big(\bigcap_{c \text{ is a 2-coloring of } \mathcal{V}} \{X_c=0\}\Big) \geq 1-\sum_c P(X_c=1) = 1-\sum_c\left[1-P(\omega_1 \text{ is monochromatic under } c)\right]^m$$

$$> 1-\underbrace{2^v\left[1-2^{1-n}e^{-1}(1+o(1))\right]^m}_{\star}.$$ Note that $\star \leq 1 \Leftarrow 2^v\exp\{m[-2^{1-n}e^{-1}(1+o(1))]\}\leq 1$

$$\Leftarrow \frac{n^2}{2}\ln 2-2^{1-n}e^{-1}(1+o(1))m\leq 0 \quad \left(\because v=\frac{n^2}{2}\right) \Leftarrow m\geq\frac{n^2}{2}\ln 2\, 2^{n-1}e(1+o(1))$$

QED

(k, ℓ)-system

Thm Let $\mathcal{F} = \{(A_i, B_i)\}_{i=1}^{h}$ be a family of pairs of subsets of an arbitrary set. If $\mathcal{F}$ has the following properties

(a) $|A_i| = k$, $|B_i| = \ell$, $i = 1, \cdots, h$ (b) $A_i \cap B_j = \phi$ if $i = j$, $\neq \phi$ if $i \neq j$

then $h \leq \binom{k+\ell}{k}$.

Pf: $X \stackrel{\text{def}}{=} \bigcup_{i=1}^{h} (A_i \cup B_i)$ and $\Omega \stackrel{\text{def}}{=}$ all permutations of X.

Let (Ω, P_r) be a pr. space s.t. $P_r(\omega) = \frac{1}{|X|!}$ $\forall \omega \in \Omega$.

$E_i \stackrel{\text{def}}{=} \{\omega \in \Omega :$ all the elements of A_i precede all those of B_i in $\omega\}$

Note that $P_r(E_i) = \dfrac{\binom{|X|}{k+\ell} k! \, \ell! \, (|X| - k - \ell)!}{|X|!} = 1 / \binom{k+\ell}{k}$.

Since $\underset{i \neq j}{E_i \cap E_j} = \phi$, $1 \geq P(\bigcup_{i=1}^{h} E_i) = \sum_{i=1}^{h} P_r(E_i) = h / \binom{k+\ell}{k}$ and hence done!

(Assume $E_i \cap E_j \ni \omega$ then ω looks like $-A_i - B_i -$ and hence $\underset{B_j}{\cup\!\!\!\!|} \; \underset{A_j}{\cup\!\!\!\!|}$ contradiction)

QED

Sum-Free Sets

- A subset A of an abelian group is sum-free if the equation $a+b=c$ with $a, b, c \in A$ has no solution.
- The set of all odd numbers is a sum-free set in the integers.
- In the cyclic group $\mathbb{Z}_{3k+2}$, the set $I = \{k+1, k+2, \cdots, 2k+1\}$ is sum-free.

Thm (Erdös 1965) Let B be a set of positive integers. Then B contains a sum-free set of size $> \frac{|B|}{3}$.

Pf: choose a prime $p = 3k+2 > \max_{b \in B} b$. Let $\Omega = \{1, 2, \cdots, 3k+1\}$ and $(\Omega, \mathbb{P})$ be a prob. space with $\mathbb{P}(\omega) = 1/|\Omega|$. Define rvs $I_b : \Omega \to \{0,1\}$ s.t.

$$I_b(\omega) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } k+1 \le \omega b \pmod p \le 2k+1 \\ 0 & \text{o.w.} \end{cases}$$

Let $X = \sum_{b \in B} I_b$.

Then $\mathcal{E}X = \sum_{b \in B} P(I_b = 1) = \sum_{b \in B} \frac{k+1}{3k+1} > \frac{|B|}{3}$. So $\exists\ (\omega, A) \in \Omega \times 2^B$ with $|A| > \frac{|B|}{3}$ and ωA (modulo p) $\subseteq \{k+1, k+2, \cdots, 2k+1\}$. Assume $\exists\, a_1, a_2, a_3 \in A$ s.t. $a_1 + a_2 = a_3$, then $wa_1 + wa_2 \equiv wa_3 \pmod p$ contradicts to $\{k+1, k+2, \cdots, 2k+1\}$ is sum-free in $\mathbb{Z}_p$. Therefore A is sum-free.

QED

Verstraete said ...

One may ask for the largest c for which every set B of positive integers contains a sum-free set of size at least $c|B|$. The theorem above shows that $c \geq \frac{1}{3}$. On the other hand, it is remarkable that the value of c is not known. Alon and Kleitman gave the upper bound $c < \frac{12}{29}$, and these are the best bounds on c which are known to date. The ideas in the proof above may also be extended to finding large subsets of a given set without a solution to a prescribed list of homogenous linear equations.

N. Alon and D. J. Kleitman, Sum-free subsets,
in : "A Tribute to Paul Erdős" (A. Baker, B. Bollobàs and A. Hajnal eds.),
Cambridge University Press, Cambridge, England 1990, 13-26

Disjoint Pairs

Thm (Alon & Frankl 1985) If $\mathcal{F}\subseteq 2^X$ having $|\mathcal{F}|=2^{(\frac{1}{2}+\delta)n}$ where $n=|X|$ and $0<\delta\le\frac{1}{2}$. Then $\#\{\{F,F'\}: F,F'\in\mathcal{F} \text{ and } F\cap F'=\phi\}<|\mathcal{F}|^{2-\frac{\delta^2}{2}}$.

Pf: Assume LHS $\ge$ RHS. $\Omega \overset{def}{=} \overbrace{\mathcal{F}\times\cdots\times\mathcal{F}}^{t} \equiv \mathcal{F}^t$. $P(\omega)=1/|\Omega|,\ \forall\omega\in\Omega$.

let $A=\{\omega\in\Omega: |\bigcup_{i=1}^{t}\omega_i|>n/2\}$ where $\omega=(\omega_1,\omega_2,\cdots,\omega_t)$.

let $B=\{\omega\in\Omega:$ $\bigcup_{i=1}^{t}\omega_i$ is disjoint to more than $2^{n/2}$ distinct subset of $X\}$

$$Pr(A)=1-P\left(|\textstyle\bigcup_{i=1}^{t}\omega_i|\le\frac{n}{2}\right)\ge 1-P\left(\bigcup_{S\in\binom{X}{n/2}}\{\omega:\bigcup_{i=1}^{t}\omega_i\subseteq S\}\right)\ge 1-\sum_S P\left(\bigcup_{i=1}^{t}\omega_i\subseteq S\right)$$

$$=1-\sum_{S\in\binom{X}{n/2}}\prod_{i=1}^{t} P(\omega_i\subseteq S)\ge 1-2^n\left(2^{n/2}/|\mathcal{F}|\right)^t=1-2^{n(1-\delta t)}.\quad (\because |\mathcal{F}|=2^{(\frac{1}{2}+\delta)n})$$

Let $t=\lceil 1+\frac{1}{\delta}\rceil$, then we have $|\mathcal{F}|^{1-\frac{t\delta^2}{2}}\overset{(*)}{>}2^{n/2}$. $Y\overset{def}{=}\sum_{S\in\mathcal{F}} I_{\{(\bigcup_{i=1}^{t}\omega_i)\cap S=\phi\}}$

$*$ $\mathcal{E}Y=\int_{Y>\ell} Y\,dP+\int_{Y\le\ell} Y\,dP\le|\mathcal{F}|\,P(Y>\ell)+\ell \quad (\because Y\le|\mathcal{F}|)$

$*$ $\mathcal{E}Y=\sum_{S\in\mathcal{F}}\prod_{i=1}^{t} P(\omega_i\cap S=\phi)=\frac{1}{m^{t-1}}\sum_{S\in\mathcal{F}}\frac{g_S^t}{m}$, where $m=|\mathcal{F}|$

$g_S=\#\{\omega_i\in\mathcal{F}:\omega_i\cap S=\phi\}$

$$\ge\frac{1}{m^{t-1}}\left(\sum_{S\in\mathcal{F}} g_S/m\right)^t \quad (\because \text{convexity of } f(x)=x^t)$$

$$=\frac{1}{m^{t-1}}\left(\frac{2\,\text{LHS}}{m}\right)^t\ge\frac{1}{m^{t-1}}\left(\frac{2\,\text{RHS}}{m}\right)^t=2^t m^{1-\frac{t\delta^2}{2}}\ge 2m^{1-\frac{t\delta^2}{2}}$$

$$Pr(B)\ge P(Y>2^{n/2})\overset{\text{assumption}}{\ge}\frac{\mathcal{E}Y-2^{n/2}}{m}\overset{(*)}{\ge}\frac{\mathcal{E}Y-m^{1-\frac{t\delta^2}{2}}}{m}\overset{*}{\ge}m^{-\frac{t\delta^2}{2}}>2^{n(1-\delta t)}.$$

so $A\cap B\ne\phi$, since $P(A)+P(B)>1$. a contradiction

QED

Erdös-Ko-Rado Thm

- Let $\mathcal{F} \subseteq \{ S \subseteq \{1,2,\cdots,n\} : |S| = k \}$, where $n \geq 2k$.
- The family $\mathcal{F}$ is called intersecting if $A, B \in \mathcal{F} \Rightarrow A \cap B \neq \phi$

Thm (Erdös-ko-Rado Thm) $|\mathcal{F}| \leq \binom{n-1}{k-1}$ and the equality is achievable, where $\mathcal{F}$ is an intersecting family of k-subsets of $[n]$.

pf: (Method 1)

addition is modulo n

Fact. Let $A_i = \{ i, i+1, \cdots, i+k-1 \}$, $0 \leq i \leq n-1$.

Then $\mathcal{F}$ contains at most k of the sets A_i.

pf. By Pigeonhole principle !

pf (Method 1, continued)

Let a permutation σ of $\overbrace{\{0,1,2,\cdots,n-1\}}^{\text{notation } [n]}$ and $i \in [n]$ be chosen randomly, uniformly and independently.

Let $A = \{\sigma(i), \sigma(i+1), \cdots, \sigma(i+k-1)\}$. (addition modulo n)

Then
$$\frac{|\mathcal{F}|}{\binom{n}{k}} = \Pr\{A \in \mathcal{F}\}$$
$$= \sum_{\rho \text{ is a permutation of } [n]} \Pr\{A \in \mathcal{F}, \sigma = \rho\}$$
$$= \sum_{\rho \text{ is a permutation of } [n]} \Pr\{A \in \mathcal{F} \mid \sigma = \rho\} \Pr\{\sigma = \rho\}$$
$$\leq \sum_{\rho \text{ is a permutation of } [n]} \frac{k}{n} \frac{1}{n!} = \frac{k}{n}.$$
Thus $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

$\exists$ a direct interpretation!

QED

Useful Estimates

1 $n! > \left(\frac{n}{e}\right)^n$, $n > 1$.

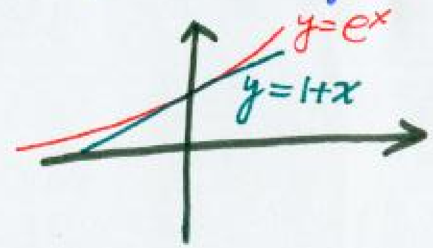
pf: By induction on n.

$$(n+1)! = n!\,(n+1) > \left(\frac{n}{e}\right)^n (n+1) = \left(\frac{n+1}{e}\right)^{n+1}\left(\frac{n}{n+1}\right)^n e$$

$$= \left(\frac{n+1}{e}\right)^{n+1} \frac{1}{\left(1+\frac{1}{n}\right)^n} e > \left(\frac{n+1}{e}\right)^{n+1}.$$ since $\left(1+\frac{1}{n}\right)^n \nearrow e$.

2. (a) $\binom{n}{k} < \left(\frac{en}{k}\right)^k$.

(b) $1+x \le e^x$, $\forall x \in \mathbb{R}$



pf: $\binom{n}{k} = \frac{n(n-1)\cdots\cdots(n-k+1)}{k!} < \frac{n^k}{\left(\frac{k}{e}\right)^k} = \left(\frac{ne}{k}\right)^k.$

3. **Golden Rule**: In the probabilistic method, we often need to estimate some complicate-looking expressions. The golden rule here is to start with the roughest estimates, and only if they don't work, one can try more refined ones.