Fundamental Thms for Algebraic Graph Theory I

Thm Let G have the spectrum $\lambda_1 \geq \cdots \geq \lambda_n$.

Then

(1) $|\lambda_1| \leq \Delta$, where $\Delta = \Delta(G)$.

(2) If G is connected, then

$\lambda_1 = \Delta$ if and only if G is Δ-regular.

pf: (1) Let $Ax=\lambda_1 x$ (A=A(G)) and $|x_1| = \max_{1\le i\le n} |x_i| > 0$.

$$|\lambda_1||x_1| = \Big|\sum_{j\in N(1)} x_j\Big| \le \Delta |x_1|$$

$\because Ax=\lambda x$

(2) Let $x=(x_1, x_2, \cdots, x_n)^t$ be an eigenvector of λ_1.

Assume, w.l.o.g, that $x_1 = \max_{1\le i\le n} x_i$

$(\Rightarrow)$ $\lambda_1=\Delta \Rightarrow \sum_{j\in N(1)} x_j = \Delta x_1 \Rightarrow x_j = x_1 \quad \forall j\in N(1)$ & $d_{x_1}=\Delta$

$\Rightarrow x_j = x_1 \quad \forall j \in V(G)$

($\because$ G is connected)

$\Rightarrow d_v=\Delta, \ \forall v\in V(G)$

$(\Leftarrow)$ G is Δ-regular $\Rightarrow$ Δ is an eigenvalue of G

$\Rightarrow \lambda_1=\Delta$

QED

Perron-Frobenius Thm (Graph version)

Thm Let G be a connected graph with $A = A(G)$.
Then $\exists$ a vector $x > 0$ and $\lambda > 0$ s.t.

- $Ax = \lambda x$
- $\lambda \geq |\mu|$ for any eigenvalue μ of A
- λ has algebraic multiplicity 1.
- if $y \geq 0$ is an eigenvector of A, then y is a multiple of x.

Pf: Let $S = \{x \in \mathbb{R}^n : x \geq 0 \text{ and } \sum_{i=1}^n x_i = 1\}$

Define $f: S \to S$ such that $f(x) = \dfrac{1}{\sum_{i=1}^n (Ax)_i} Ax$

the denominator >0 is used to normalized the result to make sure that $f(x) \in S$

Since S is closed, convex and bounded in $\mathbb{R}^n$

and $f: S \to S$ is continuous, Brouwer's Fixed Point Thm

says that $\exists\, x \in S$ s.t. $f(x) = x$ i.e.

$$Ax = \lambda x, \quad \text{where } \lambda = \sum_{i=1}^n (Ax)_i > 0$$

Claim $x > 0$.

Pf: $x_i = 0 \Rightarrow \sum_{j \sim i} x_j = 0 \Rightarrow x_j = 0 \quad \forall j \in N(i)$

$\Rightarrow x_j = 0 \quad \forall j \in V(G)$ ($\because G$ is connected)

a contradiction to $\sum_{i=1}^n x_i = 1$.

pf (continued)

Claim $\lambda \geq |\mu|$ for any eigenvalue μ of A.

pf: let y be an eigenvector of μ and $y^+ = (|y_1|, \ldots, |y_n|)^t$.

$(Ay^+)_i = \sum_{j\in N(i)} |y_j| \geq |\sum_{j\in N(i)} y_j| = |(Ay)_i| = |\mu||y_i|$.

$\lambda x^t y^+ = (Ax)^t y^+ = x^t (Ay^+) \geq x^t |\mu| y^+ = |\mu| x^t y^+$.

$(\because x>0)$

claim λ has algebraic multiplicity 1.

eigenspace of λ

pf: Since A is diagonalizable, it suffice to show $\dim V_\lambda = 1$.

$y \in V_\lambda \Rightarrow \exists$ a small enough α s.t. $z = x - \alpha y \geq 0$ and $z_i = 0$ for some i.

$\Rightarrow Az = \lambda z$ and hence $\sum_{j\in N(i)} z_j = 0$ i.e. $z_j = 0$ $\forall j \in N(i)$.

G is connected $\Rightarrow z_j = 0$ $\forall j \in V(G) \Rightarrow x = \alpha y$

pf (continued)

Claim If y is an eigenvector of $\mu \neq \lambda$ then y has both positive and negative components.

pf: Assume $y \geq 0$. Then

$$\lambda x^t y = (Ax)^t y = x^t A y = x^t \mu y = \mu x^t y.$$

Hence $\lambda = \mu$ $\quad (\because x > 0,\ y \geq 0,\ y \neq 0)$

QED

Independent Sets in Graphs

<u>Thm</u> For a d-regular connected graph G with n vertices and spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$,

$$\alpha(G) \leq \frac{-n\lambda_n}{d-\lambda_n}$$

pf: Let $A=A(G)$ and S be a maximum independent set of G with characteristic vector $\underset{\sim}{z}$.

Let $M=A-\lambda_n I-\frac{d-\lambda_n}{n}J$

$A=A^t \Rightarrow \exists$ linear independent eigenvectors $\{\underset{\sim}{1}, v_2, v_3, \cdots, v_n\}$ corresponding to eigenvalues $d, \lambda_2, \lambda_3, \cdots, \lambda_n$ respectively.

$Av_i=\lambda_i v_i \Rightarrow \underset{\sim}{1}^t A v_i=\lambda_i \underset{\sim}{1}^t v_i \Rightarrow d\,\underset{\sim}{1}^t v_i=\lambda_i \underset{\sim}{1}^t v_i \Rightarrow \underset{\sim}{1}^t v_i=0$

($\because$ Perron's Thm says that $d=\lambda_1>\lambda_2\geq\cdots\geq\lambda_n$). Thus $Jv_i=0$

$$\left(A-\lambda_n I-\frac{d-\lambda_n}{n}J\right)\underset{\sim}{1}=0\,\underset{\sim}{1}$$

$$\left(A-\lambda_n I-\frac{d-\lambda_n}{n}J\right)v_i=(\lambda_i-\lambda_n)v_i \quad \text{foreach } 2\leq i\leq n$$

pf:

It follows that the spectrum of $(A-\lambda_n I-\frac{d-\lambda_n}{n}J)$ is $\{\lambda_2-\lambda_n, \lambda_3-\lambda_n, \lambda_4-\lambda_n, \cdots, \lambda_{n-1}-\lambda_n, 0, 0\}$, since their eigenvectors $\{u_2, u_3, \ldots, u_{n-1}, u_n, \underset{\sim}{1}\}$ are l.i.

M is positive semidefinite

$$\Rightarrow 0 \leq z^t M z = z^t A z - \lambda_n z^t z - \frac{d-\lambda_n}{n} z^t J z$$

$$= 0 - \lambda_n |S| - \frac{d-\lambda_n}{n} |S|^2$$

Therefore $\alpha(G) \leq \dfrac{-n\lambda_n}{d-\lambda_n}$ $(\because d-\lambda_n > 0)$ QED

By-product

Corollary: For a d-regular connected graph G on n vertices

$$\chi(G) \geq 1 - \frac{\theta_1(G)}{\theta_n(G)}$$

Remark: In fact this bound is also true for non-regular graph.

Lemma ☆

Let A be the adjacency matrix of G.

Let $ev(A) = \{\theta_1(A) \geq \theta_2(A) \geq \cdots \geq \theta_n(A)\}$ be the eigenvalues of A.

If G is k-colorable then

$$\theta_1(A) + \sum_{i=1}^{k-1} \theta_{n-i+1}(A) \leq 0.$$

proof: $n \overset{\text{def}}{=} |V(G)|$. Let $(V_1, V_2, \cdots, V_k)$ be a proper k-coloring of G. View V_i as a characteristic column vector.

Let $\underset{\sim}{z}$ be an eigenvector for A with $A\underset{\sim}{z} = \theta_1(A)\underset{\sim}{z}$, say $\underset{\sim}{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$

Let $D \overset{\text{def}}{=} \begin{bmatrix} z_1 & & 0 \\ & z_2 & \\ 0 & & \ddots \; z_n \end{bmatrix}$. $S \overset{\text{def}}{=} \underbrace{\left[\frac{DV_1}{|DV_1|}, \frac{DV_2}{|DV_2|}, \cdots, \frac{DV_k}{|DV_k|} \right]}_{} \in M_{n \times k}$

note: $|v| = \sqrt{v_1^2 + \cdots + v_n^2}$

$\underset{\sim}{y} = [\,|DV_1|, |DV_2|, \cdots, |DV_k|\,]^T$

note: It is possible that $|DV_i| = 0$, if it is the case we delete $\frac{DV_i}{|DV_i|}$ to get $S \in M_{n \times k'}$, $k' < k$.

Claim 1. $S^T S = I \in M_{k \times k}$ & $S\underset{\sim}{y} = \underset{\sim}{z}$

2. $S^T A S \underset{\sim}{y} = \theta_1(A)\underset{\sim}{y}$

3. $\theta_1(A) = \theta_1(S^T A S)$

$S^t A S \underset{\sim}{y}$

$= S^t A \underset{\sim}{z}$

$= S^t \theta_1(A) \underset{\sim}{z}$

$= \theta_1(A) S^t S \underset{\sim}{y}$

$= \theta_1(A) \underset{\sim}{y}$

pf (3) $\theta_1(A) \underset{\text{Thm}^{\Delta}}{\geq} \theta_1(S^T A S) \underset{\text{part 2.}}{\geq} \theta_1(A)$

proof: (continued)

$$\theta_1(A) + \sum_{i=1}^{k-1} \theta_{n-i+1}(A)$$

$$= \theta_1(S^T A S) + \sum_{i=2}^{k} \theta_{n-k+i}(A)$$

Thm$^{\Delta}$

$$\leq \theta_1(S^T A S) + \sum_{i=2}^{k} \theta_i(S^T A S)$$

$$= \operatorname{trace}(S^T A S) = \sum_{i=1}^{k} (S^T A S)_{ii}$$

$$= \sum_{i=1}^{k} \frac{(DV_i)^T A (DV_i)}{|DV_i||DV_i|} = 0$$

$\because$ the support of DV_i is an independent set.

QED

Interlacing Inequalities

Thm$^{\Delta}$ If $A = A^t \in M_{n\times n}(\mathbb{R})$, $S \in M_{n\times k}(\mathbb{R})$ and $S^tS = I$. Then

$$\theta_i(A) \geq \theta_i(S^tAS) \geq \theta_{n-k+i}(A)$$

Thm: Let

$$\phi(x) = \det(xI - A_{n\times n})$$
$$= b_0 x^n - b_1 x^{n-1} + b_2 x^{n-2} - b_3 x^{n-3} + \cdots$$

Then

b_k = the sum of the principal $k\times k$ subdeterminants of A

$$= \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \det A\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix}$$

proof: Let $A=[A_1, A_2, \cdots A_n]$. $I=[E_1, E_2, \cdots, E_n]$

$$\phi(x) = \det(xI - A) = \det[xE_1 - A_1, xE_2 - A_2, \cdots, xE_n - A_n]$$

$$= \det[xE_1, xE_2 - A_2, \cdots, xE_n - A_n] +$$

$$\det[-A_1, xE_2 - A, \cdots, xE_n - A]$$

$$= \det[xE_1, xE_2, xE_3 - A_3, \cdots, xE_n - A_n] +$$

$$\det[xE_1, -A_2, xE_3 - A_3, \cdots, xE_n - A_n] +$$

$$\det[-A_1, xE_2, xE_3 - A_3, \cdots, xE_n - A_n] +$$

$$\det[-A_1, -A_2, xE_3 - A_3, \cdots, xE_n - A_n]$$

$= \cdots = 2^n$ 個 determint 之和. 每一个 det 之第 j column 不是 xE_j 就是 $-A_j$.

(continued)

proof: 收集其中含恰好 $n-k$ 个 $x E_j$ 的 det. 得

$$(-1)^k b_k = \sum_{1 \le j_1 < j_2 < \cdots < j_{n-k} \le n} \det\left[-A_1, \cdots, E_{j_1}, \cdots E_{j_2}, \cdots, E_{j_{n-k}}, \cdots -A_n\right]$$

將 I 之第 $i_1, i_2, \cdots, i_k$ columns 换成 $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$, 而其餘不變

$$= \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \det\left[E_1, \cdots, \underbrace{-A_{i_1}, \cdots, -A_{i_2}, \cdots, -A_{i_k}, \cdots, E_n}\right]$$

$$= (-1)^k \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \det\left[E_1, \cdots, A_{i_1}, \cdots, A_{i_2}, \cdots, A_{i_k}, \cdots, E_n\right]$$

$$= (-1)^k \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \prod_{j \in [n] - \{i_1, i_2, \cdots, i_k\}} (-1)^{2j} \det A \begin{pmatrix} i_1, & i_2 \cdots & i_k \\ i_1, & i_2 \cdots & i_k \end{pmatrix}$$

$$= (-1)^k \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \det A \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix}$$

Fundamental Thms of Algebraic Graph Theory II

Thm Let $\phi(G,x) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n$ be the characteristic polynomial of G. Then

(1) $c_1 = 0$

(2) $-c_2$ is the number of edges of G.

(3) $-c_3$ is twice the number of triangles in G

(4) $c_i = (-1)^i \sum_{|S|=i} \det \underbrace{A(G[S])}$

adjacency matrices of induced subgraphs

proof:

(1) $$C_1 = -\sum_{1 \le i_1 \le n} \det A\begin{pmatrix} i_1 \\ i_1 \end{pmatrix} = -\operatorname{trace} A = 0$$

(2) $$C_2 = \sum_{1 \le i_1 < i_2 \le n} \det A\begin{pmatrix} i_1 & i_2 \\ i_1 & i_2 \end{pmatrix} = \sum_{1 \le i < j \le n} \det \begin{array}{c c} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{bmatrix} 0 & a_{ij} \\ a_{ji} & 0 \end{bmatrix} \end{array}$$ where $a_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{o.w.} \end{cases}$

$$= -|E(G)|$$

(3) $$C_3 = -\sum_{1 \le i < j < k \le n} \det A\begin{pmatrix} i & j & k \\ i & j & k \end{pmatrix} = -\sum_{1 \le i < j < k \le n} \det \begin{array}{c c} & \begin{array}{ccc} i & j & k \end{array} \\ \begin{array}{c} i \\ j \\ k \end{array} & \begin{bmatrix} 0 & a_{ij} & a_{ik} \\ a_{ji} & 0 & a_{jk} \\ a_{ki} & a_{kj} & 0 \end{bmatrix} \end{array}$$

$$= -\ 2(\text{the number of 3-cycles in } G)$$

three possibilities for 3x3 non-trivial principal minors

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(4) $$C_i = (-1)^i \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n-1} \det A\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix}$$

A. J. Hoffman's bound

Thm (Hoffman's ratio bound on $\chi(G)$)

For a graph on n vertices,

$$\chi(G) \geqslant 1 - \frac{\theta_1(G)}{\theta_n(G)}$$

If equality holds, the multiplicity of $\theta_n(G)$ is at least $\chi(G)-1$.

proof let $\phi(x)$ be the characteristic polynomial of G.

$\phi(x) = x^n - |E(G)|x^{n-2} + \cdots$ and trace $A(G) = 0$

imply $\theta_1(G) > 0 > \theta_n(G)$

$$\text{Thm}☆ \Rightarrow \theta_1 + (\theta_n + \theta_{n-1} + \theta_{n-2} + \cdots + \theta_{n-k+2}) \leq 0$$

$$\Rightarrow \theta_1 + (k-1)\theta_n \leq 0$$

$$\Rightarrow \frac{\theta_1}{\theta_n} + (k-1) \geq 0 \quad (\because 0 > \theta_n)$$

$$\Rightarrow k \geq 1 - \frac{\theta_1}{\theta_n}$$

QED

Lemma ☆

Let A be the adjacency matrix of G.

Let $ev(A) = \{\theta_1(A) \geq \theta_2(A) \geq \cdots \geq \theta_n(A)\}$

be the eigenvalues of A.

If G is **c-colorable**

then $\theta_{n-c+1}(A) + (c-1)\,\theta_2(A) \geq 0$.

proof[a]**:** $A = A^T$ implies $\exists$ orthonormal matrix

$P = [\, z_1 \;\; z_2 \;\; z_3 \; \dots \; z_n \,] \in M_{n\times n}$ s.t.

square matrix, so $PP^T = I$.

$$P^TAP = \begin{bmatrix} \theta_1 & & & 0 \\ & \theta_2 & & \\ & & \ddots & \\ 0 & & & \theta_n \end{bmatrix}, \quad P^TP = I \quad \text{where } \theta_i = \theta_i(A).$$

Let $(V_1, V_2, \cdots, V_c)$ be a proper c-coloring of G.

View V_i as a characteristic column vector

$$V_i = \begin{bmatrix} U_{1i} \\ U_{2i} \\ \vdots \\ U_{ni} \end{bmatrix} \in M_{n\times 1}$$

Let $D_i = \begin{bmatrix} U_{1i} & & & 0 \\ & U_{2i} & & \\ & & \ddots & \\ 0 & & & U_{ni} \end{bmatrix} \in M_{n\times n}$. $i = 1, 2, 3, \cdots, c$

(continued)

proof[b]

$U_1 \overset{\text{def}}{=} \operatorname{span}\{D_1 z_1,\ D_2 z_1,\ \cdots\ D_c z_1\} \subseteq M_{n\times 1}$

$U_0 \overset{\text{def}}{=} \operatorname{span}\{\underbrace{z_{n-c+1},\ z_{n-c+2},\ \cdots,\ z_n}_{c}\} \subseteq M_{n\times 1}$

Claim ① $z_1 \in U_0^\perp$ ② $z_1 \in (U_1^\perp)^\perp$

③ $U_0 \subseteq (\operatorname{span}\{z_1\})^\perp$ & $U_1^\perp \subseteq (\operatorname{span}\{z_1\})^\perp$

④ $\exists$ non-zero vector $y \in U_0 \cap U_1^\perp \subseteq M_{n\times 1}$

pf[claim] ② Note that

$(U_1^\perp)^\perp = U_1$ & $(D_1 z_1 + D_2 z_1 + \cdots + D_c z_1) = (D_1 + D_2 + \cdots + D_c) z_1 = I z_1 = z_1$

④ $(D_i z_1 | D_j z_1) = z_1^t D_i^t D_j z_1 = \begin{cases} 0 & \text{if } i \neq j \\ |D_i z_1|^2 & \text{if } i = j \end{cases}$ note: it is possible that $|D_i z_1|^2 = 0$.

so $\dim U_1 \leq c$ and hence $\dim U_1^\perp \geq n-c$.

However $\dim U_0 + \dim U_1^\perp \geq c + (n-c) = n$ & $\dim(\operatorname{span}\{z_1\})^\perp = n-1$,

Therefore such a y exists.

(continued)

proof: $S \overset{\text{def}}{=} \left[\frac{D_1 y}{|D_1 y|}, \frac{D_2 y}{|D_2 y|}, \cdots, \frac{D_c y}{|D_c y|} \right] \in M_{n\times c}$

note: Here we assume $|D_i y| \neq 0 \ \forall i = 1, 2, \cdots, c$. It is possible that $|D_i y| = 0$ for some i, In this case we delete it from S and proceed similarly to get $S \in M_{n\times c'}$ where $c' < c$.

$$\frac{y^T D_i D_j y}{|D_i y||D_j y|} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \quad \text{so} \quad S^T S = I \in M_{c\times c}$$

not square matrix

$A' \overset{\text{def}}{=} A - (\theta_1 - \theta_2) z_1 z_1^T \in M_{n\times n}$. $ev(A') \overset{\text{def}}{=} \{\theta_1' \geq \theta_2' \geq \cdots \geq \theta_n'\}$

Claim* $\theta_1' = \theta_2,\ \theta_2' = \theta_2,\ \theta_3' = \theta_3,\ \theta_4' = \theta_4,\ \cdots,\ \theta_n' = \theta_n$

pf of claim: if $i \neq 1$ then
$A' z_i = A z_i - (\theta_1 - \theta_2) z_1 z_1^T z_i = A z_i = \theta_i z_i$

if $i = 1$ then $A' z_1 = A z_1 - (\theta_1 - \theta_2) z_1 z_1^T z_1 = A z_1 - (\theta_1 - \theta_2) z_1 = \theta_1 z_1 - (\theta_1 - \theta_2) z_1 = \theta_2 z_1$ ∎

$B \overset{\text{def}}{=} S^T A' S \in M_{c\times c}$

note: The following argument is valid even if S has fewer than c columns.

(continued)

proof^d claim$*$ shows that $A'\overbrace{[z_1\ z_2\ z_3 \cdots z_n]}^{P} = [\theta_2 z_1,\ \theta_2 z_2,\ \theta_3 z_3, \ldots, \theta_n z_n]$

and hence $A'P = P \begin{bmatrix} \theta_2 & & & & & 0 \\ & \theta_2 & & & & \\ & & \theta_3 & & & \\ & & & \theta_4 & & \\ & & & & \ddots & \\ 0 & & & & & \theta_n \end{bmatrix}$

$$\mathcal{X} \overset{\text{def}}{=} \begin{bmatrix} |D_1 y| \\ \vdots \\ |D_c y| \end{bmatrix} \neq \text{zero vector.}$$

$$\theta_c(B) = \min_{x \in \mathbb{R}^c \setminus 0} \frac{x^T B x}{x^T x} \leq \frac{\mathcal{X}^T B \mathcal{X}}{\mathcal{X}^T \mathcal{X}} = \frac{(S\mathcal{X})^T A' (S\mathcal{X})}{(S\mathcal{X})^T (S\mathcal{X})}$$

Rayleigh's inequality

$$= \frac{y^T A' y}{y^T y} = \frac{(P^T y)^T \begin{bmatrix} \theta_2 & & & & \\ & \theta_2 & & & \\ & & \theta_3 & & \\ & & & \ddots & \\ & & & & \theta_n \end{bmatrix} (P^T y)}{(P^T y)^T (P^T y)}$$

$S\mathcal{X}$
$= D_1 y + D_2 y + \cdots + D_c y$
$= I y = y$

$\because y \in U_0$ and $z_i^T z_j = 0$ if $i \neq j$

$$= \frac{\omega^T \begin{bmatrix} \theta_2 & & & & 0 \\ & \theta_2 & & & \\ & & \theta_3 & & \\ & & & \ddots & \\ 0 & & & & \theta_n \end{bmatrix} \omega}{\omega^T \omega}, \text{ where } \omega = \begin{bmatrix} z_1^T y \\ z_2^T y \\ \vdots \\ z_n^T y \end{bmatrix} = \begin{bmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} c \text{ terms} \\ \omega_{n-c+1} \\ \omega_{n-c+2} \\ \vdots \\ \omega_n \end{bmatrix}$$

$$= \frac{\sum_{i=n-c+1}^{n} \theta_i \omega_i^2}{\sum_{i=n-c+1}^{n} \omega_i^2} \leq \theta_{n-c+1}$$

(continued)

proof

$$y \in U_1^{\perp} \Rightarrow y^T(D_i z_1) = 0 \quad i=1,2,\cdots,c$$

$$\Rightarrow y^T D_i^T z_1 = 0 \quad i=1,2,\cdots,c \qquad (\because D_i^T = D_i)$$

$$\Rightarrow (D_i y)^T z_1 = 0 \quad i=1,2,\cdots,c$$

$$\Rightarrow \underbrace{\begin{bmatrix} \frac{(D_1 y)^T}{|D_1 y|} \\ \frac{(D_2 y)^T}{|D_2 y|} \\ \vdots \\ \frac{(D_c y)^T}{|D_c y|} \end{bmatrix}}_{S^T} z_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \Big[\overbrace{\frac{D_1 y}{|D_1 y|}, \frac{D_2 y}{|D_2 y|}, \cdots, \frac{D_c y}{|D_c y|}}^{S}\Big]^T z_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

That is $S^T z_1 = \underset{\sim}{0} \in M_{c\times 1}$

$$\text{trace}\,(S^T A S) = \sum_{i=1}^{c} \frac{(D_i y)^T A (D_i y)}{|D_i y|^2} = \sum_{i=1}^{c} \frac{y^T \overbrace{D_i A D_i}^{\in M_{n\times n}} y}{|D_i y|} = 0,$$

since
$$D_i A D_i = \begin{bmatrix} v_{1i} & & & 0 \\ & v_{2i} & & \\ & & v_{3i} & \\ 0 & & & \ddots \; v_{ni} \end{bmatrix} \begin{bmatrix} a_{st} \end{bmatrix} \begin{bmatrix} v_{1i} & & & 0 \\ & v_{2i} & & \\ & & v_{3i} & \\ 0 & & & \ddots \; v_{ni} \end{bmatrix} = \begin{bmatrix} a_{st} v_{si} v_{ti} \end{bmatrix} = \text{zero matrix}$$

$\because V_i$ is an independent set and $a_{ss} = 0 \ \forall s$.

(continued)

proof^{f} Note that
$$\begin{cases} S^T z_1 = \text{zero matrix} \in M_{c\times 1}. \\ S^T A S \in M_{c\times c} \ \text{such that} \ \operatorname{trace}(S^T A S) = 0 \end{cases}$$

So
$$\begin{aligned}
0 &= \operatorname{trace}\left(S^T A S - (\theta_1-\theta_2)\, S^T z_1 (S^T z_1)^T \right) \\
&= \operatorname{trace}\left[S^T \left(A - (\theta_1-\theta_2) z_1 z_1^T \right) S \right] \\
&= \operatorname{trace} S^T A' S, \quad \text{where } S \in M_{n\times c} \\
&= \sum_{i=1}^{c} \theta_i (S^T A' S), \quad \text{note that } B \overset{\text{def}}{=} S^T A' S \\
&= \theta_c (S^T A' S) + \sum_{i=1}^{c-1} \theta_i (S^T A' S) \\
&\leq \theta_{n-c+1} + \sum_{i=1}^{c-1} \theta_i (S^T A' S) \\
\text{Thm}^4\ &\leq \theta_{n-c+1} + \sum_{i=1}^{c-1} \theta_i (A') = \theta_{n-c+1} + \overbrace{\theta_2 + \theta_2 + \theta_3 + \cdots + \theta_{c-1}}^{\text{Assume } c \geq 2} \leq \theta_{n-c+1} + (c-1)\theta_2
\end{aligned}$$

QED

Thm (Haemers)

If $\theta_2(G) > 0$ and $\chi(G) \leq m_n$, where m_n is the multiplicity of $\theta_n(G)$ as an eigenvalue

Then
$$\chi(G) \geq 1 - \frac{\theta_n(G)}{\theta_2(G)}.$$

proof Let $c = \chi(G)$

$$\chi(G) \leq m_n \Longrightarrow \theta_{n-c+1}(G) = \theta_n(G)$$

Note that Lemma ☆ says that

$$\theta_{n-c+1}(G) + (c-1)\,\theta_2(G) \geq 0$$

$$\Longrightarrow (c-1)\,\theta_2(G) \geq -\theta_{n-c+1}(G) = -\theta_n(G)$$

$\because \theta_2 > 0$

$$\Longrightarrow c \geq 1 - \frac{\theta_n(G)}{\theta_2(G)}$$

QED

Tips on how to calculate eigenvalues of a graph

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These tips are based on the following book

Algebraic Graph Theory,
by Chriss Godsil and Gordon Royle

Notation and basic facts

- $A_{ij} \overset{\text{def}}{=}$ the submatrix of A resulting from the deletion of row i and column j.

  $A(\alpha,\beta) \overset{\text{def}}{=}$ the submatrix of A that lies in the rows of A indexed by α and the columns indexed by β.

  $A(\alpha) \overset{\text{def}}{=} A(\alpha,\alpha)$

  $A(\alpha',\beta') \overset{\text{def}}{=}$ the submatrix of A obtained by deleting the rows indicated by α and the column indicated by β.

  cofactor of $A \overset{\text{def}}{=} (-1)^{i+j}\det A_{ij}$

  $\operatorname{adj} A \overset{\text{def}}{=} [(-1)^{i+j}\det A_{ji}]$

<u>Fact:</u> $A^{-1} = \dfrac{\operatorname{adj} A}{\det A}$, provided $\det A \neq 0$.

<u>Fact:</u> Let $A \in M_{n\times n}(\mathbb{R})$, nonsingular. Let $\alpha = \{i\}$. Then $A^{-1}(\alpha) = \dfrac{\det A(\alpha')}{\det A}$

<u>pf:</u>

$(\operatorname{adj} A)(\alpha) = (-1)^{i+i}\det A_{ii}$ $\quad$ ($\because$ definition of $\operatorname{adj} A$)

$\Rightarrow ((\det A)A^{-1})(\alpha) = \det A_{ii}$ $\quad$ ($\because A^{-1} = \dfrac{\operatorname{adj} A}{\det A}$)

$\Rightarrow \quad A^{-1}(\alpha) = \dfrac{\det A(\alpha')}{\det A}$

- **Lemma 8.13.1** P187 Let $A \in M_{n\times n}(\mathbb{R})$, and B obtained by deleting the ith row and column of A. Then $\dfrac{\phi(B,x)}{\phi(A,x)} = e_i^T(xI-A)^{-1}e_i$, where e_i is the ith standard basis vector.

pf: Let $\alpha=\{i\}$ $[\operatorname{adj}(xI-A)](\alpha) = (-1)^{i+i}\det[(xI-A)(\alpha)] = \det(xI-B)$ ($\because$ definition of $\operatorname{adj}(xI-A)$)

$$\Rightarrow [\det(xI-A)](xI-A)^{-1}(\alpha) = [\operatorname{adj}(xI-A)](\alpha) = \det(xI-B) \quad \left(\because (xI-A)^{-1} = \frac{\operatorname{adj}(xI-A)}{\det(xI-A)}\right)$$

$$\Rightarrow \frac{\det(xI-B)}{\det(xI-A)} = (xI-A)^{-1}(\alpha) = e_i^T(xI-A)^{-1}e_i$$

QED

Corollary: 8.13.2 p187 For any graph G we have

$$\phi'(G,x) = \sum_{u \in V_G} \phi(G\backslash u, x)$$

graph obtained from G by deleting u

pf: Let $V_G = \{1, 2, \cdots, n\}$

$$\text{RHS} = \phi(G,x) \sum_{i \in V_G} \frac{\phi(G\backslash i, x)}{\phi(G,x)}$$

characteristic polynomial of G

adjacency matrix of G.

$$= \phi(G,x) \sum_{i=1}^{n} e_i^T (xI - A(G))^{-1} e_i$$

$$= \phi(G,x) \,\mathrm{trace}\, (xI - A(G))^{-1}$$

$$= \phi(G,x) \,\mathrm{trace}\left(P^T \begin{bmatrix} \frac{1}{x-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{x-\lambda_n} \end{bmatrix} P\right)$$

where $A(G)$ is symmetric and hence $\exists$ orthogonal matrix P s.t. $A(G) = P^T \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P$.

$$= \phi(G,x) \sum_{i=1}^{n} \frac{1}{x - \lambda_i}$$

$$= \phi(G,x) \frac{\phi'(G,x)}{\phi(G,x)} \quad \text{since } \phi(G,x) = \prod_{i=1}^{n} (x - \lambda_i)$$

$$= \phi'(G,x)$$

Remark: $\phi(G,x) = \int \sum_{u \in V_G} \phi(G\backslash u, x)\, dx + \det(-A(G))$

QED

E

incidence matrix of $G = B(G) \overset{def}{=} V\begin{bmatrix} & ue & \end{bmatrix}$ s.t. $ue = \begin{cases}1 & \text{if } u \in \text{edge } e\\ 0 & \text{o.w.}\end{cases}$

Lemma 8.2.2. / 8.2.3 Let B be the incidence matrix of G, and let $L = L(G)$ be the line graph of G. Then ① $B^TB = 2I + A(L)$, ② $BB^T = D + A(G)$

Where $A(G) =$ adjacency matrix of $G = V\begin{bmatrix} & \overset{v}{uv} & \end{bmatrix}$ s.t. $uv = \begin{cases}1 & \text{if } uv \in \text{edge}\\ 0 & \text{o.w.}\end{cases}$

proof: ① $e_s^T\begin{bmatrix} & B^T & \end{bmatrix}\begin{bmatrix} \overset{e_t}{|} & B & \end{bmatrix} = 2I + A(L)$ since $e_s^Te_t = \begin{cases}2 & \text{if } s = t\\ 1 & \text{if } s \neq t\\ 0 & \text{if } s \neq t\end{cases}$

② $v_s\begin{bmatrix} & B & \end{bmatrix}\begin{bmatrix} \overset{v_t^T}{|} & B^T & \end{bmatrix} = D + A(G)$ since $v_sv_t^T = \begin{cases}1 & \text{if } s \neq t \quad v_s \bullet\!\!-\!\!\bullet v_t\\ 0 & \text{if } s \neq t \quad v_s \bullet \cdots \bullet v_t\\ d_s & \text{if } s = t \quad v_s\end{cases}$

If $C \in M_{n\times m}$, $D \in M_{m\times n}$ then

Fact* CD & DC have the same nonzero eigenvalues with the same multiplicities.

proof: let $X = \begin{pmatrix} xI_n & C\\ D & I_m\end{pmatrix}$, $Y = \begin{pmatrix} I_n & O\\ -D & xI_m\end{pmatrix}$. Then $XY = \begin{pmatrix} xI_n - CD & xC\\ O & xI_m\end{pmatrix}$ & $YX = \begin{pmatrix} xI_n & C\\ O & xI_m - DC\end{pmatrix}$

$\det(XY) = \det(YX) \Longrightarrow x^m\det(xI_n - CD) = x^n\det(xI_m - DC)$

Lemma 8.2.5 p167 Let G be a k-regular graph with n vertices and m edges and let L be the line graph of G. Then the characteristic polynomial $\phi(L,x)$ has

$$\phi(L,x) = (x+2)^{m-n}\,\phi(G, x-k+2)$$

adjacency matrix of L

proof: $\phi(L,x) = \det(xI_m - A(L))$

Lemma 8.2.2 p166

$= \det(xI_m - (B^TB - 2I_m))$, B is the incidence matrix of G. $B \in M_{n\times m}$

$= \det((x+2)I_m - B^TB)$

see the proof of Fact *

$= (x+2)^{m-n}\det((x+2)I_n - BB^T)$

$= (x+2)^{m-n}\det((x+2)I_n - kI_n - A(G))$

Lemma 8.2.3

$= (x+2)^{m-n}\det((x+2-k)I_n - A(G))$

$= (x+2)^{m-n}\,\phi(G, x-k+2)$

where m = the number of edges

n = the number of vertices

QED

Fact: $\underset{\sim}{1}$ is an eigenvector of a graph G with eigenvalue $k \Leftrightarrow G$ is k-regular

pf: $(\Rightarrow)$ let $A = A(G) = [uv]$, $uv = \begin{cases} 1 & \text{if } uv \in E_G \\ 0 & \text{o.w.} \end{cases}$. Clearly $A\underset{\sim}{1} = k\underset{\sim}{1} \Rightarrow$ for any $u \in V$ $|N(u)| = k$.

Lemma 8.5.1 p172: Let G be a k-regular graph on n-vertices with eigenvalues $k, \lambda_2, \lambda_3, \cdots, \lambda_n$. Then G and $\overline{G}$ have the same eigenvectors and the eigenvalues of $\overline{G}$ are $n-k-1, -1-\lambda_2, \cdots\cdots, -1-\lambda_n$.

proof: Note that $A(\overline{G}) + A(G) + I = J$. ← all-ones matrix

$\exists$ orthogonal matrix $P = [\frac{\underset{\sim}{1}}{\sqrt{n}}, P_2, P_3, \cdots, P_n]$ such that

$$\Rightarrow P^T A(G) P = \begin{bmatrix} k & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \ \lambda_n \end{bmatrix}, \quad \text{Note that } \underset{\sim}{1}^t P_i = 0,\ i = 2, 3, \ldots, n.\ P_i^t P_i = 1$$

$$\Rightarrow P^T A(\overline{G}) P = \begin{bmatrix} -k & & & \\ & -\lambda_2 & & \\ & & -\lambda_3 & \\ & & & \ddots \ -\lambda_n \end{bmatrix} - I + P^T J P$$

$$= \begin{bmatrix} -k & & & \\ & -\lambda_2 & & \\ & & -\lambda_3 & \\ & & & \ddots \ -\lambda_n \end{bmatrix} - I + P^T \begin{bmatrix} \frac{n}{\sqrt{n}} & 0 & 0 & \cdots & 0 \\ \frac{n}{\sqrt{n}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{n}{\sqrt{n}} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-k & & & \\ & -1-\lambda_2 & & \\ & & -1-\lambda_3 & \\ & & & \ddots \ -1-\lambda_n \end{bmatrix} + \begin{bmatrix} n & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & \cdots & 0 \end{bmatrix} = \begin{bmatrix} n-k-1 & & & \\ & -1-\lambda_2 & & \\ & & -1-\lambda_3 & \\ & & & \ddots \ -1-\lambda_n \end{bmatrix}$$

QED

Ex P169 (a) Determine the eigenvalues of K_5

(b) Find the eigenvalue of $L(K_5)$

(c) Find the eigenvalues of $P = \overline{L(K_5)}$

(d) Find the eigenvalues of $L(P)$.

Sol: (a) Using Lemma 8.13.2 P187. $\phi'(K_5, x) = \sum_{u \in V(K_5)} \phi(K_5 \setminus u, x) = 5\phi(K_4, x)$

$\phi'(K_4, x) = \sum_{u \in V(K_4)} \phi(K_4 \setminus u, x) = 4\phi(K_3, x)$, $\phi'(K_3, x) = 3\phi(K_2, x) = 3(x^2-1)$

So $\phi(K_3, x) = \int 3(x^2-1)\,dx + \det(-A(K_3)) = \left[x^3 - 3x + \det\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}\right] = x^3 - 3x - 2 = (x-2)(x+1)^2$

$\phi(K_4, x) = \int 4(x^3 - 3x - 2)\,dx + \det(-A(K_4)) = -3 - 8x - 6x^2 + x^4$

$\phi(K_5, x) = \int 5(-3 - 8x - 6x^2 + x^4)\,dx + \det(-A(K_5)) = (x-4)(x+1)^4$

(b) $e(K_5) = m = 10$, $\nu(K_5) = n = 5$, Lemma 8.2.5 P167 says that

K_5 is 4-regular

$\phi(L(K_5), x) = (x+2)^{m-n}\phi(K_5, x-4+2) = (x+2)^5\phi(K_5, x-2) = (x+2)^5(x-6)(x-1)^4$

(c) Lemma 8.5.1 P172 & $L(K_5)$ is a 6-regular graph on 10 vertices with eigenvalues $6, 1, 1, 1, 1, -2, -2, -2, -2, -2$. $\Longrightarrow$ $\overline{L(K_5)}$ has eigenvalues

$$\begin{array}{lcccccccccc} & 10-6-1, & -1-1, & -1-1, & -1-1, & -1-1, & -1+2, & -1+2, & -1+2, & -1+2, & -1+2 \\ \text{i.e} & \| & \| & \| & \| & \| & \| & \| & \| & \| & \| \\ & 3 & -2 & -2 & -2 & -2 & 1 & 1 & 1 & 1 & 1 \end{array}$$

(d) Lemma 8.2.5 p167 says that

P is 3-regular

$$\phi(L(P), x) = (x+2)^{e(P)-v(P)} \phi(P, x-3+2)$$

$$= (x+2)^{15-10} \phi(P, x-1).$$

$$= (x+2)^{15-10} (x-4)(x+1)^4(x-2)^5 \qquad \because \phi(P, y) = (x-3)(x+2)^4(x-1)$$

$$= (x-4)(x-2)^5(x+1)^4(x+2)^5$$

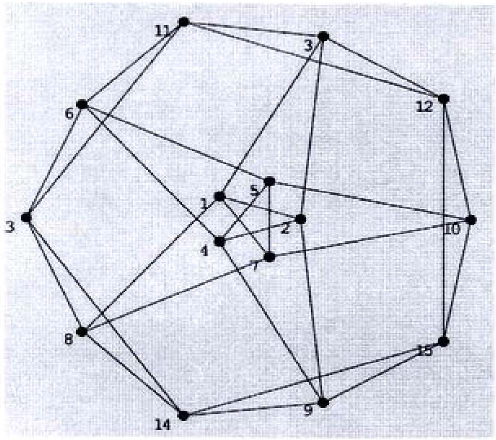
So $L(P)$'s eigenvalues can be found easily!

Find CharPoly(Line(P))

In[162]:=
```
A = ToAdjacencyMatrix[ LineGraph[PetersenGraph]]
MatrixForm[A]
ShowLabeledGraph[ LineGraph[PetersenGraph] , Background → Yellow];
Spectrum[LineGraph[PetersenGraph]]
p = Det[x * IdentityMatrix[15] - A]
Factor[p]
```

Out[163]//MatrixForm=

$$\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}$$



Out[165]= {4, -2, -2, -2, -2, -2, 2, 2, 2, 2, 2, -1, -1, -1, -1}

Out[166]= $4096 + 15360\,x + 15360\,x^2 - 8960\,x^3 - 23040\,x^4 - 4224\,x^5 + 12160\,x^6 + 5280\,x^7 - 3120\,x^8 - 1940\,x^9 + 396\,x^{10} + 345\,x^{11} - 20\,x^{12} - 30\,x^{13} + x^{15}$

Out[167]= $(-4 + x)\,(-2 + x)^5\,(1 + x)^4\,(2 + x)^5$

Combinatorial Laplacian of G

$$L \overset{\text{def}}{=} D - A = \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & d_3 & & \\ & & & \ddots & \\ & & & & d_n \end{bmatrix} - [A_{ij}]$$

d_i = the degree of vertex i

$$A_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{o.w.} \end{cases}$$

Adjacency Matrix and Laplacian

Lemma 13.1.2 Let G be a k-regular graph with adjacency matrix A having eigenvalues $\theta_1, \theta_2, \cdots, \theta_n$. Then the Laplacian L has eigenvalues $k-\theta_1, k-\theta_2, \cdots, k-\theta_n$.

pf:

$$L = D - A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} - \begin{bmatrix} & & \\ & a_{ij} & \\ & & \end{bmatrix} \quad \text{where } d_i = \text{the degree of } i$$

$$a_{ij} = \begin{cases} 1 & \text{if } ij \in E \\ 0 & \text{o.w} \end{cases}$$

That is $L = kI - A$.

$A = A^T \Rightarrow \exists$ orthogonal matrix P (i.e. $P^TP = I$)

s.t. $P^TAP = \begin{bmatrix} \theta_1 & & & \\ & \theta_2 & & \\ & & \ddots & \\ & & & \theta_n \end{bmatrix}$

$$\Rightarrow L = kPP^T - P\begin{bmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_n \end{bmatrix}P^T$$

$$= P\begin{bmatrix} k-\theta_1 & & \\ & \ddots & \\ & & k-\theta_n \end{bmatrix}P^T$$

$\Rightarrow L$ has eigenvalues $k-\theta_1, \cdots, k-\theta_n$.

QED

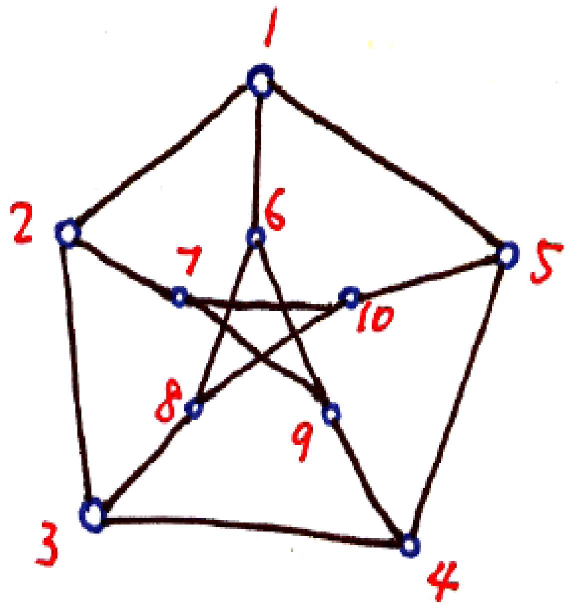
Fact p207 (1) The characteristic polynomial of the adjacency matrix of the petersen graph is

$$(x-3)(x+2)^4(x-1)^5$$

(2) The eigenvalues of the Laplacian of the Petersen graph are 0 5, 2 with multiplicities $1, 4, 2$ respectively.

pf: The adjacency matrix of the Petersen graph is

$$A=\begin{array}{cc} & \begin{array}{cccccccccc} 1&2&3&4&5&6&7&8&9&10 \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array} & \begin{bmatrix} 0&1&0&0&1&1&0&0&0&0\\ 1&0&1&0&0&0&1&0&0&0\\ 0&1&0&1&0&0&0&1&0&0\\ 0&0&1&0&1&0&0&0&1&0\\ 1&0&0&1&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&1&1&0\\ 0&1&0&0&0&0&0&0&1&1\\ 0&0&1&0&0&1&0&0&0&1\\ 0&0&0&1&0&1&1&0&0&0\\ 0&0&0&0&1&0&1&1&0&0 \end{bmatrix} \end{array},$$



The characteristic polynomial of the matrix A is $(X-3)(X+2)^4(X-1)^5$