Characteristic Polynomial of G

$$\phi(G,x) \stackrel{\text{def}}{=\!=} \det(xI - A)$$

where $A = A_G$ is the adjacency matrix of G i.e. $a_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{o.w.} \end{cases}$

Spectrum of G $\stackrel{\text{def}}{=\!=}$ the list of G's eigenvalues together with their multiplicities

$$\stackrel{\text{def}}{=\!=} \begin{pmatrix} \lambda_1, \cdots, \lambda_t \\ m_1, \cdots, m_t \end{pmatrix}$$

spec(K_n) & spec($K_{m,n}$)

Ex1 $\operatorname{spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$.

pf: $\phi(K_n, x) = \det(xI - J + I) = \det \begin{bmatrix} x & & & \\ & x & & -1 \\ -1 & & \ddots & \\ & & & x \end{bmatrix} = \det \begin{bmatrix} x-(n-1) & \cdots & \cdots & x-(n-1) \\ & x & & -1 \\ & & x & \\ -1 & & & \ddots \\ & & & x \end{bmatrix}$

$= [x-(n-1)] \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & x & & -1 \\ & & x & \\ & -1 & & \ddots \\ & & & x \end{bmatrix} = [x-(n-1)]\,(x+1)^{n-1}$.

Ex2 $\operatorname{spec}(K_{m,n}) = \begin{pmatrix} \sqrt{mn} & 0 & -\sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}$

pf: let $A = \begin{array}{c c} & \begin{array}{cc} X & Y \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right] \begin{array}{l} \}n \\ \}m \end{array} \\ & \begin{array}{cc} \underbrace{\quad}_{n} & \underbrace{\quad}_{m} \end{array} \end{array}$.

$\operatorname{rank} A = 2 \Rightarrow \operatorname{nullity} A = n+m-2$

$\Rightarrow ev(A) = \{a, b, \underbrace{0, 0, \cdots 0}_{n+m-2}\}$

$\operatorname{trace} A = 0 \Rightarrow a+b=0$. So $P_A(x) = (x-a)(x-b)\,x^{n+m-2}$

$= (x+b)(x-b)\,x^{n+m-2}$

$= x^{n+m} - b^2 x^{n+m-2}$

Pf (continued)

Let $B = [b_{ij}] = xI - A$. $t = n+m$

$$P_A(x) = \det B = \sum_{\sigma \in S_t} (\operatorname{sgn}\sigma)\, b_{1\sigma_1} b_{2\sigma_2} \cdots b_{t\sigma_t}$$

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & j & \cdots & t-1 & t \\ 1 & 2 & \cdots & j & \cdots & i & \cdots & t-1 & t \end{pmatrix} = (i\ j) \Rightarrow (\operatorname{sgn}\sigma)\, b_{1\sigma_1} b_{2\sigma_2} \cdots b_{t\sigma_t}$$

$$= (-1)\, b_{ij} b_{ji} x^{n+m-2}$$

Thus, in $P_A(x) = x^{n+m} - b^2 x^{n+m-2}$, we have

$$b^2 = mn \quad \text{and hence} \quad b = \sqrt{mn}$$

So $\operatorname{Spec}(K_{m,n}) = \begin{pmatrix} -\sqrt{mn} & 0 & \sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}$.

QED

Spectrum of Bipartite Graphs

Lemma G is bipartite graph with $\lambda \in ev(A_G)$ and $am(\lambda) = m$
algebraic multiplicity

$\Longrightarrow -\lambda \in ev(A_G)$ and $am(-\lambda) = m$.

pf: Let G have partite sets X and Y.

$\lambda \in ev(A_G) \Rightarrow \exists f : V_G \to \mathbb{R}$ s.t. $\sum_{x \sim v} f(x) = \lambda f(v)$ for $\forall v \in V_G$.

Define a function $\overset{\#}{\underset{0}{}}g : V_G \to \mathbb{R}$ s.t. $g(v) = \begin{cases} f(v) & \text{if } v \in X \\ -f(v) & \text{if } v \in Y. \end{cases}$

Note that if $v \in X$ then $\sum_{y \sim v} g(y) = -\sum_{y \sim v} f(y) = -\lambda f(v) = -\lambda g(v)$.

Clearly we also have if $v \in Y$ then $\sum_{x \sim v} g(x) = (-\lambda)\, g(v)$.

Thus $-\lambda \in ev(A_G)$.

Suppose $f_1, f_2, \cdots, f_m$ are l. indep. eigenfunctions of A_G corresponding to λ.

Construct $g_1, g_2, \cdots, g_m$ in the above way to show $am(-\lambda) = m$. QED

Eigenvalues of Bipartite Graphs

Thm

(a) G is bipartite

(b) $\lambda \in ev(A_G) \Longrightarrow -\lambda \in ev(A_G)$

(c) $\phi(G, x)$ is a product of linear factors in x^2.

(d) $\sum_{i=1}^{n} \lambda_i^{2t-1} = 0$ for any $t \in \mathbb{N}$, where $\lambda_1, \cdots, \lambda_n$ are eigenvalues of A_G. $\leftarrow |V_G| = n$

Then we have (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d)

pf: To show

$$\begin{tikzcd} a \arrow[r] & b \arrow[r] \arrow[d, Leftrightarrow] & d \arrow[ll, bend right] \\ & c & \end{tikzcd}$$

It suffices to show (d) $\Longrightarrow$ (a)

Let $A = A_G$. Then $(A^k)_{ii}$ = the # of closed k-walk passing vertex i.

The # of closed $(2t-1)$-walk in $G \leq \sum_{i=1}^{n} (A^{2t-1})_{ii} = \mathrm{trace}(A^{2t-1})$

$$= \sum_{i=1}^{n} \lambda_i^{2t+1} = 0$$

QED

Diametr(G) & Spec(G)

Thm Suppose $spec(G) = \begin{pmatrix} \lambda_1 > \lambda_2 > \dots > \lambda_t \\ m_1 & m_2 & & m_t \end{pmatrix}$. Then $diam(G) < t$.

Pf: Let $A = A_G$ and $m_A(x)$ be the minimal Polynomial of A.

$A = A^t \implies m_A(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_t)$.

Assume $diam(G) \geq t$. Thus $\exists\ i, j \in V(G)$ s.t. $distance(i, j) = t$.

and hence $(A^t)_{ij} \neq 0$ and $(A^k)_{ij} = 0$ for $k = 0, 1, 2, \dots t-1$.

However $m_A(A) = 0$ implies A^t is a linear combination of

$I = A^0, A^1, A^2, \dots, A^{t-1}$ a contradiction.

QED

Courant – Fisher Min-Max Thm

Thm: If $A = A^t \in M_{n\times n}(\mathbb{R})$ and $\sigma(A) = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$.

Then $\lambda_k = \max\limits_{\dim(U)=k} \min\limits_{x\in U\setminus 0} \dfrac{x^tAx}{x^tx}$, $\lambda_{n-k+1} = \min\limits_{\dim(U)=k} \max\limits_{x\in U\setminus 0} \dfrac{x^tAx}{x^tx}$

pf: $\exists$ orthogonal matrix $P = [u_1, \dots, u_n]$ s.t. $P^tAP = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$.

Let $V_{k-1} = \mathrm{span}(u_1, \dots u_{k-1})$. If $\dim(U) = k$ then $\exists$ nonzero vector $y \in U \cap V_{k-1}^{\perp}$.

$$\frac{y^tAy}{y^ty} = \frac{y^tA\left(\sum_{i=k}^{n} c_iu_i\right)}{y^ty} = \left(\sum_{i=k}^{n} \lambda_i c_i^2 u_i^tu_i\right) \Big/ \left(\sum_{i=k}^{n} c_i^2 u_i^tu_i\right) \leq \lambda_k.$$

Thus RHS $\leq \lambda_k$. We also have $\dfrac{z^tAz}{z^tz} \geq \lambda_k$ for any $z \in V_k \setminus 0$, and hence

RHS $\geq \lambda_k$. Next,

$$\min_{\dim(U)=k} \max_{x\in U\setminus 0} \frac{x^tAx}{x^tx} = -\max_{\dim(U)=k} \min_{x\in U\setminus 0} \frac{x^t(-A)x}{x^tx}$$

$$= -\lambda_k(-A) = \lambda_{n-k+1}(A).$$

QED

Rayleigh-Ritz Thm

Thm If $A = A^*_{n\times n}$ then $\lambda_{max} = \max_{x\neq 0} \dfrac{x^tAx}{x^tx}$ and $\lambda_{min} = \min_{x\neq 0} \dfrac{x^tAx}{x^tx}$.

where $ev(A) = \{\lambda_{min} \leq \cdots \leq \lambda_{max}\}$.

Pf: (Method 1) Courant-Fisher Min-Max Thm says that

$$\lambda_{max} = \max_{\dim(U)=1} \min_{x\in U\setminus 0} (x^tAx)/(x^tx) = \max_{x\neq 0} \frac{x^tAx}{x^tx}.$$

$$\lambda_{min} = \min_{\dim(U)=1} \max_{x\in U\setminus 0} (x^tAx)/x^tx = \min_{x\neq 0} (x^tAx)/(x^tx).$$

(Method 2) $x^tAx = x^tP^t\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}\overbrace{Px}^{y} = \sum_{i=1}^n \lambda_i y_i^2 \leq \lambda_{max}\sum_{i=1}^n y_i^2 = \lambda_{max}x^tx.$

Suppose $A\mathfrak{X} = \lambda_{max}\mathfrak{X}$. We have $\mathfrak{X}^tA\mathfrak{X} = \lambda_{max}\mathfrak{X}^t\mathfrak{X}$.

Therefore $\lambda_{max} = \max_{x\neq 0} x^tAx/x^tx$.

QED

$|\lambda(G)| \leq \Delta_G$

Thm If $\lambda \in ev(A_G)$ then $|\lambda| \leq \Delta_G$.

Pf: Let f be an eigenfunction of A_G corresponding to λ.

Let $|f(v)| = \max_{u \in V} |f(u)|$. Note that $\sum_{x \in N(v)} f(x) = \lambda f(v)$.

Thus $|f(v)| \Delta_G \geq \sum_{x \in N(v)} |f(v)| \geq \sum_{x \in N(v)} |f(x)| \geq \left| \sum_{x \in N(v)} f(x) \right| = |\lambda f(v)|$.

QED

Key Lemma

<u>Lemma*</u> If $A = A^t \in M_{n\times n}(\mathbb{R})$.

Then for $1 \le i \le n$,

$$\theta_i(A) = \max_{\substack{R \in M_{n\times i}(\mathbb{R}) \\ R^t R = I_i}} \theta_i(R^t A R)$$

pf: **Let** $R \in M_{n\times i}$ and $R^t R = I$.

Let $U_R =$ column space of R. Note that $\dim U_R = i$.

$x \in U_R \setminus \{0\} \Rightarrow \exists\, y \in \mathbb{R}^i$ s.t. $x = Ry$

Thus $$\min_{x \in U_R \setminus 0} \frac{x^t A x}{x^t x} = \min_{y \in \mathbb{R}^i \setminus 0} \frac{y^t R^t A R y}{y^t y} = \theta_i(R^t A R)$$

Thus $$\theta_i(A) = \max_{\dim(U)=i} \min_{x \in U \setminus 0} \frac{x^t A x}{x^t x} \geq \max_{\substack{R \in M_{n\times i} \\ R^t R = I}} \theta_i(R^t A R)$$

Moreover there exists $\hat{R} \in M_{n\times i}$

such that $\hat{R}^t \hat{R} = I$ and $\hat{R}^t A R = \mathrm{diag}[\theta_1(A), \theta_2(A), \ldots \theta_i(A)]$

We are done.

QED

Interlacing Inequalities

Thm: If $A=A^t\in M_{n\times n}(\mathbb{R})$, $S\in M_{n\times k}(\mathbb{R})$ and $S^tS=I$.

Then $\theta_i(A)\geq\theta_i(S^tAS)\geq\theta_{n-k+i}(A)$, where $ev(A)=\{\theta_1(A)\geq\cdots\geq\theta_n(A)\}$.

pf: Key Lemma implies

$$\theta_i(A)=\max_{\substack{R\in M_{n\times i}\\ R^tR=I}}\theta_i(R^tAR)\geq\max_{\substack{Q\in M_{k\times i}\\ Q^tQ=I}}\theta_i\Big((SQ)^tA(SQ)\Big)$$

$$=\max_{\substack{Q\in M_{k\times i}\\ Q^tQ=I}}\theta_i(Q^tS^tASQ)=\theta_i(S^tAS) \text{ ---------- first inequality.}$$

$$=-\theta_{k-i+1}(S^t(-A)S)$$

$$\geq-\theta_{k-i+1}(-A) \text{ ----- by the first inequality.}$$

$$=-\left[-\theta_{n-(k-i+1)+1}(A)\right]$$

$$=\theta_{n-k+i}(A)$$

QED

Corollary: let $A=A^t\in M_{n\times n}(\mathbb{R})$ and B be a principal submatrix of A (i.e. $B=A(\alpha,\alpha)$) of order $m\times m$. Then for $i=1,2,\cdots,m$.

$\theta_i(A)\geq\theta_i(B)\geq\theta_{n-m+i}(A)$.

Interlacing Theorem (Graph Version)

Thm Let G be a graph with adjacency matrix A having spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $H = G - v$ with adjacency matrix B having spectrum $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$. Then $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$.

Corollary: Let $A = A^t \in M_{n\times n}(\mathbb{R})$ with spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and B any $m\times m$ principal minor of A $(m<n)$ with spectrum $\mu_1 \geq \cdots \geq \mu_m$. Then for $\forall\, \alpha$, A has at least as many eigenvalues less (greater) than α as B does, i.e. $\#\{i : \lambda_i \leq \alpha\} \geq \#\{i : \mu_i \leq \alpha\}$ and $\#\{j : \lambda_j \geq \alpha\} \geq \#\{j : \mu_j \geq \alpha\}$.

Graph Version Let H be a vertex-induced subgraph of G. Then for $\forall\, \alpha$, G has at least as many eigenvalues less (greater) than α as H does.

Cvetković's bound

Lemma Let G be a graph on n vertices.

Let $A \in M_{n\times n}(\mathbb{R})$, $A = A^t$ and $A_{uv} = 0$ if $u \not\sim v$ in G.

Then

$$\alpha(G) \leq \min\left\{ n - n^+(A),\ n - n^-(A) \right\}$$

where $n^{\pm}(A) \overset{\text{def}}{=}$ the number of positive eigenvalues of A.
negative

pf: Let α be the subgraph of G induced by a maximum independent set.

Let B be the principle submatrix $A(\alpha,\alpha)$ i.e. a submatrix of A induced by the rows & columns in α.

Let $m=\alpha(G)$. Clearly B is a zero matrix.

We have $\theta_m(A) \geq \theta_m(B)=0 \Rightarrow n^-(A) \leq n-m$

By interlacing inequalities $\quad \theta_m(-A) \geq \theta_m(B)=0 \Rightarrow n^+(A) \leq n-m$