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TRAVELING WAVES IN CELLULAR NEURAL NETWORKS

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In this paper, we study the structure of traveling wave solutions of Cellular Neural Networks of the advanced type. We show the existence of monotone traveling wave, oscillating wave and eventually periodic wave solutions by using shooting method and comparison principle. In addition, we obtain the existence of periodic wave train solutions.

1. Introduction

In this paper, we are going to study the structure of traveling wave solutions of *Cellular Neural Networks* (*CNN*) which was proposed by Chua and Yang [1988], sometimes called CY-CNN, and has been studied by many authors (see [Chua & Roska, 1993; Chua & Yang, 1988; Hsu & Lin, 1998; Juang & Lin, 1998; Shih, 1998; Thiran *et al.*, 1995]). The Cellular Neural Networks on $\mathbb{Z}^2$ or $\mathbb{Z}^1$ without input terms are of the form

$$\frac{dx_{i,j}}{dt} = -x_{i,j} + z + \sum_{|k|\le d, |l|\le d} a_{k,l} f(x_{i+k,j+l}) \qquad (i, j) \in \mathbf{Z}^2 \quad (1)$$

or

$$\frac{dx_i}{dt} = -x_i + z + \sum_{|l|\le d} a_l f(x_{i+l}) \quad i \in \mathbf{Z}^1 . \quad (2)$$

Here the nonlinearity f is a piecewise-linear function (e.g. $f(x) = (1/2)(|x+1| - |x-1|)$), and called the output function. The quantity z is called threshold or biased term and the numbers $a_{k,l}$ can be arranged into a $(2d+1)\times(2d+1)$ matrix A which is called a space-invariant template.

The study of traveling wave solutions can proceed as follows. Let $\theta \in \mathbf{R}$ be given, and consider solutions of (1) or (2) of the form

$$x_{i,j}(t) = \phi(i \cos\theta + j \sin\theta - ct) \quad (3)$$

or

$$x_i = \phi(i - ct) \quad (4)$$

for some continuous function $\phi : \mathbf{R}^1 \to \mathbf{R}^1$ and some unknown real number c. A solution of form (3) (or (4)) of system (1) (or (2)) is called a *traveling wave solution* of (1) (or (2)). Denote $s = i\cos\theta + j\sin\theta - ct$ (or $s = i - ct$). Then $\phi(s)$ and c

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satisfies the equation of the form

$$-c\phi'(s) = G(\phi(s+r_0),\, \phi(s+r_1), \ldots,\, \phi(s+r_N)), \tag{5}$$

here $r_0 = 0$, r_i are real numbers for $i = 1$ to N. If Eq. (5) depends on the past and future, i.e. if

$$r_{\min} \equiv \min\{r_i\}_{i=0}^{N} < 0 < r_{\max} \equiv \max\{r_i\}_{i=0}^{N}, \tag{6}$$

then (5) is called *mixed type*. If $r_{\min} = 0$ or $r_{\max} = 0$, then (5) is called *advance* or *delay type*, respectively.

Equation (5) ((1) or (2)) is called *bistable* if it has three spatially homogeneous solutions $\phi(s) \equiv x^-$, x^0, and x^+ satisfying $x^- < x^0 < x^+$ and

$$G(x, x, \ldots, x) > 0 \text{ for } x \in (-\infty, x^-) \cup (x^0, x^+),$$
$$G(x, x, \ldots, x) < 0 \text{ for } x \in (x^-, x^0) \cup (x^+, \infty).$$

Note that if $z = 0$, $f(x) = (1/2)(|x+1| - |x-1|)$, and $\sum_{|k|\le d, |l|\le d} a_{k,l} > 1$ in (1) or $\sum_{|l|\le d} a_l > 1$ in (2), then (5) is bistable.

Suppose that Eq. (5) is bistable. Recently, Mallet-Paret [1999] showed that (5) has a unique monotone solution satisfying the boundary conditions,

$$\lim_{s\to-\infty} \phi(s) = x^- \quad \text{and} \quad \lim_{s\to\infty} \phi(s) = x^+. \tag{7}$$

More precisely, it is proved in [Mallet-Paret, 1997] that under some assumptions, there is a unique c^* such that (5) has a monotone solution satisfying (7) iff $c = c^*$, and such solution is also unique up to a phase shift if $c = c^* \neq 0$. When G is quasi-monotone and satisfies a set of conditions, Hsu and Lin [1998] proved that there are a family of monotone solutions of (5) satisfying the boundary conditions

$$\lim_{s\to-\infty} \phi(s) = x^0 \quad \text{and} \quad \lim_{s\to\infty} \phi(s) = x^+. \tag{8}$$

The method used in [Hsu & Lin, 1998] is a monotone iteration scheme and the results are recalled as follows

(i) Assume that $r_{\min} < 0 < r_{\max}$ and G satisfies certain conditions (see [Hsu & Lin, 1998]). Then there exists $c_* < 0$ such that for any $c < c_*$, (5) has a nondecreasing solution satisfying the boundary conditions (8).
(ii) Assume that $r_{\max} = 0$, then for any $c < 0$ there exists a nondecreasing solution of (5) satisfying the boundary conditions (8).

We point out that (i) above also holds for the advance case $r_{\min} = 0$. In the case of advance (or delay) type, Hsu and Lin [1998] also studied (5) with the "initial" condition $\lim_{s\to\infty} \phi(s) = x^+$ (or $\lim_{s\to-\infty} \phi(s) = x^-$) and proved the local existence. Indeed, the solution $\phi(s)$ of (5) with $r_{\min} = 0$ and $\lim_{s\to\infty} \phi(s) = x^+$ can be represented as

$$\phi(s) = x^+ - \gamma e^{\sigma^+ s} - \tilde{\phi}(s) e^{2\sigma^+ s} \quad \text{for } s \gg 1, \tag{9}$$

where $\sigma^+ < 0$, $\gamma > 0$, and $\tilde{\phi}(s)$ is a bounded and C^1-function.

Our objective in this paper is to study the structure of traveling wave solutions of one-dimensional CNN of the advanced type. Namely, we consider solution $\phi(s; c)$ of

$$\begin{aligned} -c\phi'(s; c) = -\phi(s; c) + af(\phi(s; c)) \\ + \beta f(\phi(s+1; c)) \end{aligned} \tag{10}$$

satisfying

$$\lim_{s\to\infty} \phi(s; c) = x^+ \tag{11}$$

for any $c < 0$, where $x^+ = a + \beta$, and $f(x) \equiv f_\varepsilon(x)$ $(\varepsilon > 0)$,

$$f_\varepsilon(x) = \begin{cases} 1 & \text{if } x \ge \varepsilon, \\ \dfrac{x}{\varepsilon} & \text{if } |x| \le \varepsilon, \\ -1 & \text{if } x \le -\varepsilon, \end{cases} \tag{12}$$

or $f(x) \equiv f_0(x)$, f_0 is the set-valued function given by

$$f_0(x) = \begin{cases} \{1\} & \text{if } x > 0, \\ [-1, 1] & \text{if } x = 0, \\ \{-1\} & \text{if } x < 0. \end{cases} \tag{13}$$

The following is one of our main results.

Theorem A. *Suppose that $f = f_1$, $a > 0$, $\beta > 0$ and $a + \beta > 1$. Let $x^\pm = \pm(a+\beta)$, $x^0 = 0$, and $\phi(s; c)$ be the solution of (10) and (11).*

(1) *Assume that $a \ge 1 + \beta$. There is $c_* = c_*(a, \beta) < 0$ such that*
   (i) *if $c \le c_*$, then $\phi(s; c)$ is nondecreasing and satisfies (8),*
   (ii) *if $c_* < c < 0$, then $\phi(s; c)$ is oscillating (see Definition 2.2) and $|\phi(s; c)| < 1$ for $s < 0$.*

(2) *Assume that* $a < 1 + \beta$. *There exist* $c_* = c_*(a, \beta)$, $c_p = c_p(a, \beta)$, *and* $c^* = c^*(a, \beta)$ *with* $c_* \le c_p \le c^* < 0$ *such that*

(i) *if* $c \le c_*$, *then* $\phi(s;\, c)$ *is nondecreasing and satisfies* (*8*),

(ii) *if* $c_* < c < c_p$, *then* $\phi(s;\, c)$ *is oscillating*,

(iii) *if* $c_p \le c < c^*$, *then* $\phi(s;\, c)$ *is eventually periodic* (*see Definition 2.2*) *and* $\sup_{s<0} |\phi(s;\, c)| > 1$,

(iv) *if* $c = c^* > c_*$, *then* $\phi(s;\, c^*)$ *is nondecreasing and satisfies* (*7*),

(v) *if* $c^* < c < 0$, *then* $\phi(s;\, c)$ *is nondecreasing and unbounded.*

Clearly, (10) with $f = f_\varepsilon$ ($\varepsilon > 0$) and that with $f = f_1$ have the same dynamics. In fact, if $\phi_\varepsilon(s;\, c)$ is a solution of (10) with $f = f_\varepsilon$, then $\phi(s;\, c) = (1/\varepsilon)\phi_\varepsilon(s;\, c)$ is a solution of (10) with $f = f_1$ and a, β being replaced by a/ε, β/ε. When $f = f_0$, we have,

Theorem B. *Suppose that* $a > 0$, $\beta > 0$. *Let* $\phi_0(s;\, c)$ *be the solution of* (*10*) *and* (*11*) *with* $f = f_0$.

(1) *Assume that* $a \ge \beta$. *Then for any* $c < 0$, $\phi_0(s;\, c)$ *is nondecreasing and satisfies* (*8*).

(2) *Assume that* $a < \beta$. *There exists* $c^* < 0$ *such that*

(i) *if* $c < c^*$, *then* $\phi_0(s;\, c)$ *is eventually periodic*,

(ii) *if* $c = c^*$, *then* $\phi_0(s;\, c)$ *is nondecreasing and satisfies* (*7*),

(iii) *if* $c^* < c < 0$, *then* $\phi_0(s;\, c)$ *is nondecreasing and unbounded.*

We remark that when $f = f_0$ and $a < \beta$, nondecreasing solutions of (10) satisfying (8) no longer exist. Roughly, this is due to the fact that $c_p(a/\varepsilon,\, \beta/\varepsilon) \to -\infty$ as $\varepsilon \to 0$ (see Remark 4.1). The idealized nonlinearity $f = f_0$ is used in many problems to provide an insight into the dynamics of the problems (see [Cahn *et al.*, 1998] and references therein). Comparing Theorems A and B above, we see that the dynamics of (10) and (11) with $f = f_0$ carry over most but not all basic features of that with $f = f_1$.

Note that the eventual periodicity of $\phi(s;\, c)$ in Theorems A and B is due to the piecewise linearity and symmetry of f. On reflection, eventually periodic solutions result in periodic solutions of (10) (such solutions are called *periodic traveling wave* or *periodic wave train solutions* of (3), see Definition 2.2). Equation (10) has also periodic solutions with arbitrary small magnitude resulting from the linear part of the output function.

There has been many studies on traveling wave solutions of spatially discrete or both spatially and time discrete systems (see [Afraimovich & Nekorkin, 1994; Hankerson & Zinner, 1993; Keener, 1987; Mallet-Paret, 1995; Shen, 1996], etc). However, as far as we know, oscillating traveling wave solutions have been hardly studied in such discrete systems. Though f in Theorem A is piecewise linear, we believe that similar results to Theorem A hold for more general f with bistable properties, that is, (10) is bistable.

The paper is organized as follows. In Sec. 2, we introduce definitions and present basic results of later use. We consider (10) with piecewise-linear nonlinearity and prove Theorem A and the existence of eventually periodic and periodic solutions in Sec. 3. In Sec. 4, we study (10) with the idealized nonlinearity $f = f_0$ and prove Theorem B.

2. Preliminary Results

In this section, we introduce definitions and present basic results for later use.

Consider the following one-dimensional CNN with zero biased term,

$$\frac{dx_i}{dt} = -x_i + af(x_i) + \beta f(x_{i+1})\,, \quad i \in \mathbb{Z}\,, \tag{14}$$

where a, β are constants and f is the output function. We first assume that $f = f_1$, where f_1 is as in (12), and $a > 0$, $\beta > 0$, $a + \beta > 1$. Then (14) has three spatially homogeneous stationary solutions $x^- < x^0 < x^+$, where

$$x^+ = a + \beta\,,\ x^0 = 0\,, \quad \text{and} \quad x^- = -a - \beta\,, \tag{15}$$

and is of the bistable type.

Suppose that $x_i(t) = \phi(i - ct;\, c)$ is a traveling wave solution of (14) with $\phi \in C^1(\mathbb{R}^1, \mathbb{R}^1)$. Then c and $\phi(s;\, c)$ satisfy

$$\begin{aligned} -c\phi'(s;\, c) = -\phi(s;\, c) &+ af(\phi(s;\, c)) \\ &+ \beta f(\phi(s+1;\, c))\,. \end{aligned} \tag{16}$$

We investigate basic properties about solutions of (16) with $c < 0$ and

$$\lim_{s\to\infty} \phi(s;\, c) = x^+\,. \tag{17}$$

First of all, by direct computation, we have,

Lemma 2.1. *For any $c < 0$,*

$$\phi(s;\, c) = (1 - a - \beta)e^{\frac{1}{c}s} + a + \beta \tag{18}$$

is a solution of (16) and (17) for $s \in [0,\, \infty)$.

For the rest of this section, we assume that $c < 0$ and $\phi(s;\, c)$ for $s \in [0,\, \infty)$ is given by (18), unless specified otherwise. Note that solution $\phi(s;\, c)$ in (18) has been normalized such that $\phi(0;\, c) = 1$. We say $\phi(s;\, c)$ *monotone* if it is nondecreasing or nonincreasing.

Lemma 2.2. *If there is $s_0 < 0$ such that $1 \le \phi(s;\, c) < x^+$ or $x^- < \phi(s;\, c) \le -1$ for $s \in [s_0 - 1,\, s_0]$, then $\phi(s;\, c)$ is not monotone for $s \in [s_0 - 1,\, 0]$ and is periodic for $s < 0$.*

Proof. For simplicity, we denote $\phi(s;\, c)$ by $\phi(s)$.

Case 1. $1 \le \phi(s) < x^+$ for $s \in [s_0 - 1,\, s_0]$ and some $s_0 < 0$.

First, since $\phi(0) = 1$ and $\phi'(0) = (1/c)(1 - a - \beta) > 0$, we have $\phi(-s) < 1$ for $0 < s \ll 1$. Hence $\phi(s)$ is not monotone for $s \in [s_0 - 1,\, 0]$.

Next, note that

$$\phi'(s_0 - 1) = \frac{1}{c}(\phi(s_0 - 1) - a - \beta) > 0\,.$$

Hence, for any $s_1 \le s_0 - 1$ with $\phi(s) \ge 1$ for $s \in [s_1,\, s_0 - 1]$,

$$\phi'(s) \ge \frac{1}{c}(\phi(s_0 - 1) - a - \beta) > 0\,.$$

This implies that there is $s_2 \le s_0 - 1$ such that $\phi(s_2) = 1$ and $1 \le \phi(s) < x^+$ for $s \in [s_2,\, s_0]$.

We prove that $\phi(s) = \phi(s - s_2)$ for $s \le s_2$. To do so, let $\psi(s) = \phi(s - s_2)$. Then

$$\psi(s_2) = \phi(s_2)\,, \quad f(\psi(s + 1)) = f(\phi(s + 1)) = 1 \quad \text{for} \quad s \in [s_2 - 1,\, s_2]\,. \tag{19}$$

By (16),

$$\psi'(s) = \frac{1}{c}(\psi(s) - af(\psi(s)) - \beta f(\psi(s + 1)))\,, \tag{20}$$

and

$$\phi'(s) = \frac{1}{c}(\phi(s) - af(\phi(s)) - \beta f(\phi(s + 1)))\,. \tag{21}$$

Following from (19)–(21), we have

$$\phi(s) = \psi(s) \quad \text{for} \quad s \in [s_2 - 1,\, s_2]\,. \tag{22}$$

Now, by (22),

$$\begin{aligned} \psi(s_2 - 1) &= \phi(s_2 - 1)\,, \\ f(\psi(s + 1)) &= f(\phi(s + 1))\,, \\ & s \in [s_2 - 2,\, s_2 - 1]\,. \end{aligned} \tag{23}$$

Then following from (20), (21), and (23),

$$\phi(s) = \psi(s) \quad \text{for} \quad s \in [s_2 - 2,\, s_2 - 1]\,. \tag{24}$$

Continuing this process, we have $\phi(s) = \psi(s) = \phi(s - s_2)$ for $s \le s_2$. Therefore $\phi(s)$ is periodic with period $\omega = -s_2$ for $s < 0$.

Case 2. $x^- < \phi(s) \le -1$ for $s \in [s_0 - 1,\, s_0]$ and some $s_0 < 0$.

Since $\phi'(0) = (1/c)(1 - a - \beta) > 0$ and $\phi'(s_0 - 1) = (1/c)(\phi(s_0 - 1) + a + \beta) < 0$, $\phi(s)$ is not a monotone for $s \in [s_0 - 1,\, 0]$. Moreover, there is $s_2 < s_0 - 1$ such that

$$\phi(s_2) = -1 \quad \text{and} \quad x^- < \phi(s) \le -1 \quad \text{for} \quad s \in [s_2,\, s_0]\,. \tag{25}$$

Let $\psi(s) = -\phi(s - s_2)$. Then we have

$$\phi(s_2) = \psi(s_2) \text{ and } f(\psi(s + 1)) = f(\phi(s + 1)) \quad \text{for} \quad s \in [s_2 - 1,\, s_2] \tag{26}$$

and $\psi(s)$, $\phi(s)$ satisfy (20) and (21) respectively. By similar arguments as in Case 1, $\phi(s) = \psi(s)$ for all $s \le s_2$. This implies that $\phi(s) = \phi(s - 2s_2)$ for $s \le 2s_2$, that is, $\phi(s)$ is periodic for period $\omega = -2s_2$ for $s \le 0$. ■

Lemma 2.3. *Suppose that there is $s_* < 0$ such that $\phi(s_*;\, c) = x^-$, and $\phi(s;\, c) > x^-$ and is monotone for $s > s_*$. Then $s_* \ge s^* - 1$, where $s^* > s_*$ is such that $\phi(s^*;\, c) = -1$. Moreover, $\phi(s;\, c) = x^-$ for $s \le s_*$ if $s_* = s^* - 1$, and $\phi(s;\, c)$ is monotone and unbounded if $s_* > s^* - 1$.*

Proof. For simplicity, we denote $\phi(s;\, c)$ by $\phi(s)$.

First, note that $x^- < \phi(s) \le -1$ for $s_* < s < s^*$. If $s_* < s^* - 1$, then by Lemma 2.2, $\phi(s)$ is not a monotone on $(s_*, 0)$, a contradiction. Hence, $s_* \geq s^* - 1$.

Now, clearly, $\#\{s > s_* | \phi'(s) = 0\} < \infty$. Hence if $s_* > s^* - 1$, then

$$\phi'(s_*) = \frac{1}{c}(x^- + a - \beta f(\phi(s_* + 1))) > \frac{1}{c}(x^- + a + \beta) = 0 .$$

This implies that

$$\begin{aligned}\phi'(s) &= \frac{1}{c}(\phi(s) - af(\phi(s) - \beta f(\phi(s+1))) \\ &\geq \frac{1}{c}(\phi(s) + a + \beta) > \frac{1}{c}(x^- + a + \beta) = 0\end{aligned}$$

for all $s \geq s_*$. Therefore, $\phi(s)$ is monotone and unbounded for $s < 0$. If $s_* = s^* - 1$, then it is easy to see that

$$\phi(s) = x^- \quad \text{for} \quad s \leq s_* . \quad \blacksquare$$

Lemma 2.4. *If there is $s_0 < 0$ such that $\phi(s; c)$ is not a monotone for $s_0 < s < 0$, then $\phi(s; c)$ lies in (x^-, x^+) for all $s \leq 0$.*

Proof. For simplicity again, we denote $\phi(s; c)$ by $\phi(s)$.

Clearly, $\phi(s)$ is increasing for $s \in [-\delta, 0]$ with $0 < \delta \ll 1$. Therefore, there is s_1 with $s_0 < s_1 < 0$ such that $\phi(s)$ is nondecreasing for $s \in [s_1, 0]$, but is not monotone for $s \in (s_1 - \delta, s_1 + \delta)$ with any $\delta > 0$.

We first claim that $x^- < \phi(s_1) < 0$. In fact, if $\phi(s_1) \geq 0$, then $\#\{s \in [s_1, 0] | \phi'(s) = 0\} < \infty$, and hence

$$\begin{aligned}0 = \phi'(s_1) &= \frac{1}{c}((1-a)\phi(s_1) - \beta f(\phi(s_1 + 1))) \\ &> \frac{1}{c}((1-a)\phi(s_1) - \beta\phi(s_1)) \geq 0 ,\end{aligned}$$

a contradiction. If $\phi(s_1) \leq x^-$, then there is $s_1 \leq s_* < s^*$ such that $\phi(s_*) = x^-$, $\phi(s^*) = -1$, and $\phi(s) > x^-$ for $s > s_*$. By Lemma 2.3, $\phi(s)$ is monotone for $s < 0$, a contradiction again.

Therefore, we must have $x^- < \phi(s_1) < 0$. It then follows that

$$x^- < \phi(s_1) \leq \phi(s) < 1 \quad \text{for} \quad s_1 \leq s < 0 , \tag{27}$$

and

$$\mathcal{S} = \{s \in (s_1 - \delta, 0) | \phi'(s) = 0\}$$

is a finite set for any $0 < \delta \ll 1$. Hence, $\phi(s_1)$ must be a nontrivial local minimum.

Next we prove that

$$\phi(s_1) \leq \phi(s) \leq -\phi(s_1) \quad \text{for} \quad s < s_1 . \tag{28}$$

To do so, let $\tilde{s}_1 > s_1$ be such that $\phi(\tilde{s}_1) = -\phi(s_1)$, and $s_2 < s_1 < \tilde{s}_2 < \tilde{s}_1$ be such that $\phi(s_2) = \phi(\tilde{s}_2) > \phi(s_1)$ and $\phi(s)$ is decreasing for $s_2 < s < s_1$. Hence

$$\phi(s_1) \leq \phi(s) \leq -\phi(s_1) \quad \text{for} \quad s_2 \leq s < s_1 . \tag{29}$$

Let $\bar{s} = \min\{1, \tilde{s}_1 - s_1, \tilde{s}_2 - s_2\}$, and define $\psi(s)$ by

$$\psi(s) = -\phi(s) \quad \text{for} \quad s \in \mathbb{R} .$$

Then we have

$$\phi(s) \geq \psi(s + \tilde{s}_1 - s_1) \quad \text{for} \quad s \geq s_1 , \tag{30}$$

and

$$\phi(s) \leq \phi(s + \tilde{s}_2 - s_2) \quad \text{for} \quad s \geq s_2 . \tag{31}$$

We claim that

$$\phi(s) \leq \psi(s + \tilde{s}_1 - s_1) \leq -\phi(s_1) \quad \text{for} \quad s_1 - \bar{s} \leq s \leq s_1 , \tag{32}$$

and

$$\phi(s) \geq \phi(s + \tilde{s}_2 - s_2) \geq \phi(s_1) \quad \text{for} \quad s_2 - \bar{s} \leq s \leq s_2 . \tag{33}$$

In fact, let $\eta_1(s) = \phi(s)$, $\eta_2(s) = \psi(s + \tilde{s}_1 - s_1)$, and $\eta_3(s) = \phi(s + \tilde{s}_2 - s_2)$. By (16),

$$\dot{\eta}_1(s) = \frac{1}{c}(\eta_1(s) - af(\eta_1(s)) - \beta f(\phi(s+1))) , \tag{34}$$

$$\begin{aligned}\dot{\eta}_2(s) = \frac{1}{c}(&\eta_2(s) - af(\eta_2(s)) \\ &- \beta f(\psi(s + 1 + \tilde{s}_1 - s_1))) ,\end{aligned} \tag{35}$$

and

$$\begin{aligned}\dot{\eta}_3(s) = \frac{1}{c}(&\eta_3(s) - af(\eta_3(s)) \\ &- \beta f(\phi(s + 1 + \tilde{s}_2 - s_2))) .\end{aligned} \tag{36}$$

Note that $\eta_1(s_1) = \eta_2(s_1)$, $\eta_1(s_2) = \eta_3(s_2)$, and by (30) and (31),

$$-\frac{\beta}{c}f(\phi(s+1)) \geq -\frac{\beta}{c}f(\psi(s + 1 + \tilde{s}_1 - s_1)) \quad \text{for } s \geq s_1 - \bar{s} ,$$

$$-\frac{\beta}{c}f(\phi(s+1)) \leq -\frac{\beta}{c}f(\phi(s + 1 + \tilde{s}_2 - s_2)) \quad \text{for } s \geq s_2 - \bar{s} .$$

Then following from the comparison arguments for scalar ODE's, we have

$$\eta_1(s) \le \eta_2(s) \le -\phi(s_1) \quad \text{for} \quad s_1 - \bar{s} \le s \le s_1$$

and

$$\eta_1(s) \ge \eta_3(s) \ge \phi(s_1) \quad \text{for} \quad s_2 - \bar{s} \le s \le s_2\,,$$

that is, (32) and (33) hold.

By (32) and (33), there are $\tilde{s}_3 \in [\tilde{s}_1 - \bar{s},\, \infty)$ and $\tilde{s}_4 \in [\tilde{s}_2 - \bar{s},\, \infty)$ such that

$$\phi(s_1 - \bar{s}) = \psi(\tilde{s}_3)\,, \quad \phi(s_2 - \bar{s}) = \phi(\tilde{s}_4)\,.$$

We claim that

$$\phi(s) \ge \psi(s + \tilde{s}_3 - s_1 + \bar{s}) \quad \text{for} \quad s \ge s_1 - \bar{s}\,, \tag{37}$$

$$\phi(s) \le \phi(s + \tilde{s}_4 - s_2 + \bar{s}) \quad \text{for} \quad s \ge s_2 - \bar{s}\,, \tag{38}$$

and

$$\begin{aligned}\phi(s) &\le \psi(s + \tilde{s}_3 - s_1 + \bar{s}) \\ &\le -\phi(s_1) \quad \text{for} \quad s_1 - 2\bar{s} \le s \le s_1 - \bar{s}\,,\end{aligned} \tag{39}$$

$$\begin{aligned}\phi(s) &\ge \phi(s + \tilde{s}_4 - s_2 + \bar{s}) \\ &\ge \phi(s_1) \quad \text{for} \quad s_2 - 2\bar{s} \le s \le s_2 - \bar{s}\,.\end{aligned} \tag{40}$$

In fact, let $\eta_4(s) = \psi(s + \tilde{s}_3 - s_1 + \bar{s})$ and $\eta_5(s) = \phi(s + \tilde{s}_4 - s_2 + \bar{s})$. Then

$$\begin{aligned}\eta_4'(s) = \frac{1}{c}(\eta_4(s) &- af(\eta_4(s)) \\ &- \beta f(\psi(s + 1 + \tilde{s}_3 - s_1 + \bar{s})))\,,\end{aligned} \tag{41}$$

and

$$\begin{aligned}\eta_5'(s) = \frac{1}{c}(\eta_5(s) &- af(\eta_5(s)) \\ &- \beta f(\psi(s + 1 + \tilde{s}_4 - s_2 + \bar{s})))\,.\end{aligned} \tag{42}$$

Note that $\eta_4(s_1 - \bar{s}) = \phi(s_1 - \bar{s})$ and $\eta_5(s_2 - \bar{s}) = \phi(s_2 - \bar{s})$. Since $\tilde{s}_3 \ge \tilde{s}_1 - \bar{s}$ and $\tilde{s}_4 \ge \tilde{s}_2 - \bar{s}$, by (30) and (31), there hold

$$-\frac{\beta}{c} f(\phi(s+1)) \ge -\frac{\beta}{c} f(\psi(s + 1 + \tilde{s}_3 - s_1 + \bar{s})) \quad \text{for} \quad s \ge s_1 - \bar{s} \tag{43}$$

and

$$-\frac{\beta}{c} f(\phi(s+1)) \le -\frac{\beta}{c} f(\phi(s + 1 + \tilde{s}_4 - s_2 + \bar{s})) \quad \text{for} \quad s \ge s_2 - \bar{s}\,. \tag{44}$$

Following from the comparison arguments for scalar ODE's again, we have

$$\eta_1(s) \ge \eta_4(s) \quad \text{for} \quad s \ge s_1 - \bar{s}$$

and

$$\eta_1(s) \le \eta_5(s) \quad \text{for} \quad s \ge s_2 - \bar{s}\,,$$

that is, (37) and (38) hold. Now (39) and (40) follow from the similar arguments to (32) and (33).

Continuing the above process, we have

$$\phi(s) \le -\phi(s_1) \quad \text{for} \quad s \le s_1 \tag{45}$$

and

$$\phi(s) \ge \phi(s_1) \quad \text{for} \quad s \le s_2\,. \tag{46}$$

Equation (28) then follows from (29), (45) and (46), and the lemma follows from (27) and (28). ■

Corollary 2.5.

(1) *If $\phi(s_*;\, c) = x^-$ for some $s_* < 0$, then $\phi(s;\, c)$ is monotone.*
(2) *$\lim_{s\to-\infty} \phi(s;\, c) = x^-$ iff there is $s_* < 0$ such that $\phi(s;\, c) = x^-$ for $s \le s_*$.*
(3) *If $\phi(s;\, c)$ is not monotone for $s < 0$, then $\phi(s;\, c)$ is bounded and either $\phi(s;\, c)$ is not monotone for $s \in (-\infty,\, s_0)$ with any $s_0 < 0$, or $|\phi(s;\, c)| \le 1$ for $s \ll -1$.*

Proof. It directly follows from Lemmas 2.2–2.4. ■

Lemma 2.6. *If $|\phi(s;\, c)| \le 1$ for $s \ll -1$ and $-c\sigma = -1 + a + \beta e^{\sigma}$ has no positive roots σ, then $\phi(s;\, c)$ is not monotone for $s \in (-\infty,\, s_0)$ with any $s_0 < 0$ and $\#\{s|\phi(s;\, c) = 0,\, s < 0\} = \infty$.*

Proof. See [Gyori & Ladas, 1991]. ■

Lemma 2.7. *Suppose that $a < 1 + \beta$. If there is $c^* < 0$ such that $\lim_{s\to-\infty} \phi(s;\, c^*) = x^-$, then $x^- < \phi(s;\, c) < x^+$ for any $s < 0$ and $c < c^*$, and $\phi(s;\, c)$ is nondecreasing and unbounded for $c^* < c < 0$.*

Proof. First of all, by Corollary 2.5, $\phi(s;\, c^*)$ is monotone and there is $s_* < 0$ such that $\phi(s;\, c^*) = x^-$ for $s \le s_*$.

Case 1. Suppose that $c^* < c < 0$. We shall prove $\phi(s;\, c)$ is unbounded. Clearly, it suffices to prove

$\phi(s;\,c) < x^-$ for some $s < 0$. Let $s_0 = 0$. By Lemma 2.1,

$$\phi(s;\,c) > \phi(s;\,c^*) \quad \text{for} \quad s > s_0\,. \tag{47}$$

We claim that

$$\phi(s;\,c) < \phi(s;\,c^*) \quad \text{for} \quad s_0 - 1 \le s < s_0\,. \tag{48}$$

Note that $\phi(0;\,c) = \phi(0,\,c^*) = 1$ and

$$\phi'(0;\,c) = \frac{1}{c}(1-a-\beta) > \frac{1}{c^*}(1-a-\beta) = \phi'(0;\,c^*)\,.$$

Hence, if (48) does not hold, then there is $s_0 - 1 \le \tilde{s} < s_0$ such that $\phi(s;\,c) < \phi(s;\,c^*)$ for $\tilde{s} < s < s_0$ and $\phi(\tilde{s};\,c) = \phi(\tilde{s};\,c^*)$. Therefore, $\phi'(\tilde{s};\,c) \le \phi'(\tilde{s};\,c^*)$. Since $f(\phi(\tilde{s}+1;\,c)) = f(\phi(\tilde{s}+1;\,c^*)) = 1$ and $a < 1+\beta$, we have

$$\begin{aligned}&\phi(\tilde{s};\,c) - af(\phi(\tilde{s};\,c)) - \beta f(\phi(\tilde{s}+1;\,c))\\ &\quad = \phi(\tilde{s};\,c^*) - af(\phi(\tilde{s};\,c^*)) - \beta f(\phi(\tilde{s}+1;\,c^*))\\ &\quad < 0\,.\end{aligned}$$

Hence,

$$\begin{aligned}\phi'(\tilde{s};\,c) &= \frac{1}{c}(\phi(\tilde{s};\,c) - af(\phi(\tilde{s};\,c)) - \beta f(\phi(\tilde{s}+1;\,c)))\\ &> \frac{1}{c^*}(\phi(\tilde{s};\,c^*) - af(\phi(\tilde{s};\,c^*))\\ &\quad - \beta f(\phi(\tilde{s}+1;\,c^*)))\\ &= \phi'(\tilde{s};\,c^*)\,,\end{aligned}$$

a contradiction. Therefore, (48) holds.

Now if $\phi(s_0-1;\,c^*) = x^-$, then $\phi(s_0-1;\,c) < x^-$ and the lemma follows. Otherwise, let $s_1 = s_0 - 1$ and $s_0 - 1 < \bar{s}_1 < s_0$ be such that $\phi(\bar{s}_1;\,c) = \phi(s_1;\,c^*)$. We claim that

$$\phi'(\bar{s}_1;\,c) > \phi'(\bar{s}_1;\,c^*)\,, \tag{49}$$

$$\phi(s - s_1 + \bar{s}_1;\,c) > \phi(s;\,c^*) \quad \text{for} \quad s > s_1\,, \tag{50}$$

and

$$\phi(s - s_1 + \bar{s}_1;\,c) < \phi(s;\,c^*) \quad \text{for} \quad s_1 - 1 \le s < s_1\,. \tag{51}$$

In fact, $-1 < f(\phi(\bar{s}_1 + 1;\,c^*)) \le 1$. If $f(\phi(\bar{s}_1+1;\,c^*)) = 1$, then by (47),

$$\begin{aligned}&\phi(\bar{s}_1;\,c) - af(\phi(\bar{s}_1;\,c)) - \beta f(\phi(\bar{s}_1+1;\,c))\\ &\quad \le \phi(\bar{s}_1;\,c^*) - af(\phi(\bar{s}_1;\,c^*)) - \beta f(\phi(\bar{s}_1+1;\,c^*))\\ &\quad < 0\,,\end{aligned}$$

which implies (49). If $-1 < f(\phi(\bar{s}_1+1;\,c^*)) < 1$, then by (47),

$$\begin{aligned}&\phi(\bar{s}_1;\,c) - af(\phi(\bar{s}_1;\,c)) - \beta f(\phi(\bar{s}_1+1;\,c))\\ &\quad < \phi(\bar{s}_1;\,c^*) - af(\phi(\bar{s}_1;\,c^*)) - \beta f(\phi(\bar{s}_1+1;\,c^*))\\ &\quad \le 0\,,\end{aligned}$$

which also implies (49). Hence, (49) holds.

If (50) does not hold, by (49), there is $\tilde{s} > s_1$ such that $\phi(s - s_1 + \bar{s}_1;\,c) > \phi(s;\,c^*)$ for $s_1 < s < \tilde{s}$ and $\phi(\tilde{s} - s_1 + \bar{s}_1;\,c) = \phi(\tilde{s};\,c^*)$. Then $\phi'(\tilde{s} - s_1 + \bar{s}_1;\,c) \le \phi'(\tilde{s};\,c^*)$. But by the similar arguments to (49), we have $\phi'(\tilde{s} - s_1 + \bar{s}_1;\,c) > \phi'(\tilde{s};\,c^*)$, a contradiction. Hence (50) holds.

Similarly, if (51) does not hold, then there is $s_1 - 1 \le \tilde{s} < s_1$ such that $\phi(s - s_1 + \bar{s}_1;\,c) < \phi(s;\,c^*)$ for $\tilde{s} < s < s_1$ and $\phi(\tilde{s} - s_1 + \bar{s}_1;\,c) = \phi(\tilde{s};\,c^*)$. Hence, $\phi'(\tilde{s} - s_1 + \bar{s}_1;\,c) \le \phi'(\tilde{s};\,c^*)$. But, by (50) and the similar arguments to (49), we have $\phi'(\tilde{s} - s_1 + \bar{s}_1;\,c) > \phi'(\tilde{s};\,c^*)$, a contradiction. Hence, (51) also holds.

Again, if $\phi(s_1 - 1;\,c^*) = x^-$, then $\phi(\bar{s}_1 - 1;\,c) < x^-$ and the lemma follows. Otherwise, let $s_2 = s_1 - 1$ and $s_1 - 1 < \bar{s}_2 < s_1$ be such that $\phi(s - s_2 + \bar{s}_2 - s_1 + \bar{s}_1;\,c) = \phi(s_2;\,c^*)$. Using similar arguments as above, we have

$$\phi(s - s_2 + \bar{s}_2 - s_1 + \bar{s}_1;\,c) > \phi(s;\,c^*) \quad \text{for} \quad s > s_2\,,$$

and

$$\begin{aligned}&\phi(s - s_2 + \bar{s}_2 - s_1 + \bar{s}_1;\,c) < \phi(s;\,c^*)\\ &\qquad\qquad \text{for} \quad s_2 - 1 \le s < s_2\,.\end{aligned}$$

Continuing the above process, there is $\tilde{s}$ such that $\phi(\tilde{s};\,c) < x^- = \phi(s_*;\,c^*)$ and then the lemma follows.

Case 2. Suppose that $c < c^*$. If there is $\tilde{s}$ such that $\phi(\tilde{s};\,c) = x^-$, by Corollary 2.5, $\phi(s;\,c)$ is monotone. Then by the similar arguments as in Case 1, there is $\tilde{s}_*$ such that $\phi(\tilde{s}_*;\,c^*) < x^-$, a contradiction. Therefore, $\phi(s;\,c) > x^-$ for $s < 0$. By the arguments of Lemma 2.4, $x^- < \phi(s;\,c) < x^+$ for $s < 0$. ■

Corollary 2.8. *Suppose that* $a < 1+\beta$. *Then there is at most one* $c^* < 0$ *such that* $\lim_{s\to-\infty}\phi(s;\,c^*) = x^-$.

Now we consider (14) and (16) with $f = f_0$, where f_0 is as in (13).

Definition 2.1. Suppose that $f = f_0$. Then $\phi(s;\, c)$ is called a solution of (16) if it is absolutely continuous and satisfies the differential inclusion,

$$-c\phi'(s;c) \in -\phi(s;c) + af(\phi(s;c)) + \beta f(\phi(s+1;c))$$

for almost all $s \in \mathbb{R}$.

We end up this section with the following classification of bounded solutions of (16).

Definition 2.2.

(i) A bounded solution $\phi(s;\, c)$ of (16) and (17) is called oscillating if $\phi(s;\, c)$ is not monotone for $s \in (-\infty,\, s_0)$ with any $s_0 < 0$. If $\phi(s;\, c)$ is oscillating, then $x_i(t) = \phi(i - ct;\, c)$ is said to be an oscillating traveling wave solution of (14).

(ii) An oscillating traveling wave $x_i(t) = \phi(i - ct;\, c)$ of (14) is called eventually periodic if there exists s_0 in $\mathbf{R}^1$ such that $\phi(s;\, c)$ is periodic for $s \le s_0$.

(iii) If a solution $\phi(s;\, c)$ of (16) is periodic, then $x_i(t) = \phi(i - ct;\, c)$ is called periodic wave train solution of (14).

3. Traveling Waves in CNN with Piecewise-Linear Output

In this section, we study (14) with piecewise-linear output and prove Theorem A and the existence of eventually periodic and periodic traveling wave solutions. We therefore first consider (16) with $f = f_1$, and assume that $a > 0$, $\beta > 0$, $a + \beta > 1$, and $\phi(s;\, c)$ is the solution of (16) given by (18) for $s \ge 0$, unless specified otherwise. Then we have

Theorem 3.1.

(1) *Suppose that $a \ge 1 + \beta$. There is $c_* < 0$ such that*

  (i) *if $c \le c_*$, then $\phi(s;\, c)$ is nondecreasing and $\lim_{s\to-\infty} \phi(s;\, c) = 0$,*

  (ii) *if $c_* < c < 0$, then $\phi(s;\, c)$ is oscillating and $|\phi(s;\, c)| < 1$ for $s < 0$.*

(2) *Suppose that $a < 1 + \beta$. There are c_*, c_p, and c^* with $c_* \le c_p \le c^* < 0$ such that*

  (i) *if $c \le c_*$, then $\phi(s;\, c)$ is nondecreasing and $\lim_{s\to-\infty} \phi(s;\, c) = 0$,*

  (ii) *if $c_* < c < c_p$, then $\phi(s;\, c)$ is oscillating,*

  (iii) *if $c_p \le c < c^*$, then $\phi(s;\, c)$ is eventually periodic and $\sup_{s<0} |\phi(s;\, c)| > 1$,*

  (iv) *if $c = c^* > c_*$, then $\phi(s;\, c)$ is nondecreasing and there is $s_* < 0$ such that $\phi(s;\, c) = x^-$ for $s \le s_*$,*

  (v) *if $c^* < c < 0$, then $\phi(s;\, c)$ is nondecreasing and unbounded.*

Note that Theorem A follows from Theorem 3.1. To prove Theorem 3.1, we show the following lemmas first.

Lemma 3.2. *There is $c_* < 0$ such that for any $c \le c_*$, $\phi(s;\, c)$ is nondecreasing and $\lim_{s\to-\infty} \phi(s;\, c) = 0$, and for any $c > c_*$, $-c\sigma = -1 + a + \beta e^{\sigma}$ has no positive roots σ.*

Proof. See [Hsu & Lin, 1998]. ■

Lemma 3.3. *Suppose that $a < 1 + \beta$. Then there is $c_0 < 0$ such that $\phi(s;\, c)$ satisfies*

$$\begin{aligned} &\phi(s^*;\, c) = -1\,, \\ &\phi(s;\, c) > -1 \text{ is monotone for } s > s^* \end{aligned} \tag{52}$$

with some $s^* \ge -1$ iff $c \ge c_0$. Moreover, if $a \ne 1$, then

$$c_0 = (a-1)\cdot\left\{\ln\frac{1-a+\beta}{a+\beta-1}\right\}^{-1}, \tag{53}$$

and if $a = 1$, then

$$c_0 = \frac{-\beta}{2}\,. \tag{54}$$

Proof. We prove the case that $a \ne 1$. First, for any $s^* \ge -1$ satisfying (52), we have

$$\phi'(s;\, c) = \frac{1}{c}(\phi(s;\, c) - a\phi(s;\, c) - \beta) \quad \text{for } s \in [s^*,\, 0]$$

and then

$$\phi(s;\, c) = \left(\frac{(1-a-\beta)}{1-a}\right)e^{\frac{1}{c}(1-a)s} + \frac{\beta}{1-a}$$

for $s \in [s^*,\, 0]$. If $\phi(s^*;\, c) = -1$, we must have

$$s^* = -\frac{c}{c_0}\,,$$

where c_0 is as in (53). The lemma then follows. ■

Lemma 3.4. *Suppose that $a < 1 + \beta$. Let*

$$\tilde{c} = \left\{\frac{1}{c_0} - \ln\frac{2\beta}{1-a+\beta}\right\}^{-1}. \tag{55}$$

Then $\phi(-1;\, \tilde{c}) = x^-$.

Proof. By Lemma 3.3, for any $c_0 \le c < 0$, there is $s^* \ge -1$ satisfying (52). Therefore, if $c_0 \le c < 0$,

$$\phi'(s;\, c) = \frac{1}{c}\{\phi(s;\, c) + a - \beta\} \quad \text{for } s \in [-1,\, s^*]\,.$$

It then follows that

$$\phi(s;\, c) = (a-1-\beta)e^{\frac{1}{c}(s-s^*)} - a + \beta \quad \text{for } s \in [-1,\, s^*]\,.$$

Let $\tilde{c}$ be as in (57). Clearly, $\phi(-1;\, \tilde{c}) = x^-$. ■

Lemma 3.5. *Suppose that $a \ge 1 + \beta$. Then $-1 < \phi(s;\, c) < 1$ for any $c < 0$ and $s < 0$.*

Proof. First, consider

$$\dot{\psi}(s;\, c) = \frac{1}{c}(\psi(s;\, c) - a\psi(s;\, c) - \beta) \quad \text{for } s \le 0$$

with $\psi(0;\, c) = 1$. We have

$$\psi(s;\, c) = \frac{1-a-\beta}{1-a}e^{\frac{1-a}{c}s} + \frac{\beta}{1-a}\,.$$

Hence, $\psi'(s;\, c) > 0$ for any $s \le 0$, and then

$$-1 \le \frac{\beta}{1-a} < \psi(s;\, c) < 1 \quad \text{for } s < 0\,.$$

This implies that

$$-1 < \psi(s;\, c) = \phi(s;\, c) < 1 \quad \text{for } -1 \le s < 0\,.$$

Next, if there is some $s_2 \in [-2,\, -1)$ such that $\phi(s_2;\, c) = -1$ and $\phi(s;\, c) > -1$ for $s \in (s_2,\, -1]$, then

$$0 \le \phi'(s_2;\, c) = \frac{1}{c}(a-1) - \frac{\beta}{c}f(\phi(s_2+1;\, c))$$
$$< \frac{1}{c}(a-1) - \frac{\beta}{c} \le 0\,,$$

a contradiction. If there is some $s_2 \in [-2,\, -1)$ such that $\phi(s_2;\, c) = 1$ and $\phi(s;\, c) < 1$ for $s \in (s_2,\, -1]$, then

$$0 \ge \phi'(s_2) = \frac{1}{c}(1-a) - \frac{\beta}{c}f(\phi(s_2+1;\, c))$$
$$> \frac{1}{c}(1-a) + \frac{\beta}{c} \ge 0\,,$$

a contradiction again. Therefore, we must have $\phi(s;\, c) \in (-1,\, 1)$ for $s \in [-2,\, -1]$.

Continuing the above process, we have that $\phi(s;\, c) \in (-1,\, 1)$ for $s < 0$. ■

Proof of Theorem 3.1. Let $c_* \le 0$ be as in Lemma 3.2.

(1) (i) follows from Lemma 3.2, and (ii) follows from Lemmas 2.6, 3.2 and 3.5.

(2) Let

$$C = \{\overline{c}|\text{ for any } \overline{c} < c < 0,\ \phi(s;\, c) \text{ is unbounded}\}\,.$$

By Lemmas 3.3 and 3.4, $C \ne \emptyset$. Define c^* by

$$c^* = \inf\{c|c \in C\}\,. \tag{56}$$

Clearly, $c_* \le c^* < 0$, and $\phi(s;\, c^*)$ is bounded and monotone.

Case 1. $c^* = c_*$. Let $c_p = c^*$. Then (i) follows from Lemma 3.2, and (ii)–(v) follow from the definitions of c^* and c_p.

Case 2. $c^* > c_*$. By Lemmas 2.2, 2.3, 2.6 and 3.2, we must have $\phi(s;\, c^*) = x^-$ for $s \le s_*$ and some $s_* < 0$, and (iv) then follows. Again, (i) follows from Lemma 3.2 and (v) follows from the definition of c^*. Let

$$C_p = \{\overline{c}|x^- < \phi(s;\, c) \le -1 \quad \text{for} \quad \overline{c} \le c < c^* \text{ and } \quad s^* - 1 \le s \le s^*\}\,,$$

where $s^* = s^*(c) < 0$ is such that $\phi(s^*;\, c) = -1$, and $\phi(s;\, c) > -1$ for $s > s^*$. By Lemma 2.7, $C_p \ne \emptyset$. Define c_p by

$$c_p = \inf\{c|c \in C_p\}\,. \tag{57}$$

Then (ii) follows from Corollary 2.5 and Lemmas 2.6, 2.7, and 3.2. (iii) follows from Lemma 2.2. ■

Remark 3.1. If $c^* > c_*$, then $c_* \le c_p < c^*$.

Proposition 3.6. *If $1 \le a < 1 + \beta$, then $c_* < c^*$, where c_*, c^* are as in Theorem 3.1(2). Moreover, if $(5/4) \le a < 1 + \beta$, then $c_p \le c_0$, where c_0 is as in Lemma 3.3, and c_p is as in (57).*

Proof. First, we assume that $1 \le a < 1 + \beta$ and prove $c_* < c^*$. Clearly, it is suffice to prove that $\phi(s;\, c)$ is oscillating for $c > c_*$ with $c - c_* \ll 1$. Note that for any $c_* < c < 0$, $\{s < 0|\phi(s;\, c) = 0\}$ is not empty. For otherwise, we have $\phi'(s;\, c) = (1/c)((1-a)\phi(s;\, c) - \beta f(\phi(s+1;\, c)) > 0$ for $s < 0$ and then $0 < \phi(s;\, c) < 1$ for $s < 0$. By Lemmas 2.6 and 3.2, $\#\{s < 0|\phi(s;\, c) = 0\} = \infty$, a contradiction. For given $c_* < c < 0$, let $s_0 < 0$ be such that $\phi(s_0;\, c) = 0$ and $\phi(s;\, c) > 0$ for $s > s_0$. Then

$$\phi'(s_0;\, c) > 0,\ \phi(s;\, c) < 0 \text{ for } s < s_0,\ s_0 - s \ll 1\,. \tag{58}$$

Since $\lim_{s\to-\infty}\phi(s;\,c_*)=0$, we have $\phi(s_0-1;\,c)>-1$ provided that $c_*<c$ and $c-c_*\ll 1$. Now if $\phi(s_0-1;\,c)\geq 0$, by (58), $\phi(s;\,c)$ is not monotone for $s<0$. If $\phi(s_0-1;\,c)<0$, then $\phi'(s_0-1;\,c)=(1/c)(1-a)\phi(s_0-1;\,c)<0$. By (58) again, $\phi(s;\,c)$ is also not monotone for $s<0$. Therefore, following from Corollary 2.5 and Lemmas 2.6, 3.2, $\phi(s;\,c)$ is oscillating for $c>c_*$ with $c-c_*\ll 1$, and then $c_*<c^*$.

Next, we assume that $(5/4)\leq a<1+\beta$ and prove $c_p\leq c_0$. By Lemma 3.3, for any $c_0\leq c<c^*$, there is $s^*\geq -1$ such that $\phi(s^*;\,c)=-1$ and $\phi(s;\,c)>-1$ for $s>s^*$. Then by Lemma 2.2, it is suffice to prove that $x^-<\phi(s;\,c)\leq -1$ for $s\in[s^*-1,\,s^*]$. By direct computation, for any $c_0\leq c<0$,

$$\phi(s;\,c)=\left(\frac{1-a-\beta}{1-a}\right)e^{\frac{1-a}{c}s}+\frac{\beta}{1-a}\,,\quad s^*\leq s\leq 0\,,$$

and

$$\phi(s;\,c)=(a-1-\beta)e^{\frac{1}{c}(s-s^*)}-a+\beta\,,\quad -1\leq s\leq s^*\,.$$

Consider

$$\psi'(s;\,c)=\frac{1}{c}\left(\psi(s;\,c)+a-\beta\left(\frac{1-a-\beta}{1-a}\right)e^{\frac{1-a}{c}(s+1)}-\frac{\beta^2}{1-a}\right),\quad s^*-1\leq s\leq -1$$

with $\psi(-1;\,c)=\phi(-1;\,c)$. Then

$$\psi(s;\,c)=\left(\phi(-1;\,c)+a-\frac{\beta^2}{1-a}\right)e^{\frac{1}{c}(s+1)}+\frac{\beta(1-a-\beta)}{a(1-a)}e^{\frac{1}{c}(s+1)}(e^{-\frac{a}{c}(s+1)}-1)-a+\frac{\beta^2}{1-a}\,.$$

Hence,

$$\psi(s^*-1;\,c)=\left(\phi(-1;\,c)+a-\frac{\beta^2}{1-a}\right)e^{\frac{1}{c}s^*}+\frac{\beta(1-a-\beta)}{a(1-a)}e^{\frac{1}{c}s^*}(e^{-\frac{a}{c}s^*}-1)-a+\frac{\beta^2}{1-a}\,.$$

Note that $e^{\frac{1-a}{c}s^*}=((a-1-\beta)/(1-a-\beta))$ and $s^*/c=(1/(1-a))\ln((a-1-\beta)/(1-a-\beta))$. Then it is not difficult to see that $\phi(-1;\,c)$ and $\psi(s^*-1;\,c)$ are decreasing as c increases for $c_0\leq c<0$. Clearly, $\phi(s;\,c)\leq -1$ for $s\in[s^*-1,\,s^*]$ iff $\psi(s;\,c)\leq -1$ for $s\in[s^*-1,\,-1]$, and $\psi(s;\,c)\leq -1$ for $s\in[s^*-1,\,-1]$ iff $\psi(s^*-1;\,c)\leq -1$.

Observe that $\psi(s^*-1;\,c_0)\leq -1$ iff

$$-\frac{1}{c_0}\geq \ln\frac{-1+a+\dfrac{\beta(\beta+1)}{a}-\dfrac{\beta}{a}(a+\beta-1)e^{-\frac{1}{c_0}}}{a-1-\beta}\,,$$

which is equivalent to $H(a,\,\beta)\geq e^{\frac{1}{c_0}}$, where

$$H(a,\,\beta)=\frac{a-a^2+\beta-\beta^2}{a-a^2-\beta-\beta^2}\,.$$

Let $\mathcal{H}(a,\,\beta)\equiv H(a,\,\beta)-e^{\frac{1}{c_0}}$. Then for any fixed $a>1$, we have

$$\lim_{\beta\to\infty}\mathcal{H}(a,\,\beta)=0\,,\ \text{and}\ \lim_{\beta\to a-1}\mathcal{H}(a,\,\beta)=\frac{a-1}{a}\,. \tag{59}$$

By an elementary computation, if $\mathcal{H}(a,\,\beta)$ has a critical point at some β_0, then

$$\mathcal{H}(a,\,\beta_0)=a(a-1)^2+\beta_0^2a(4a-5)\,. \tag{60}$$

Then by (59) and (60), when $(5/4)\leq a<1+\beta$, $\mathcal{H}(a,\,\beta)\geq 0$ (i.e. $H(a,\,\beta)\geq e^{\frac{1}{c_0}}$), and hence $c_p\leq c_0$. ∎

Corollary 3.7. *Suppose that $a<1+\beta$. Let c_p and c^* be as in Theorem 3.1(2). Then for any $c_p\leq c<c^*$, $\tilde\phi(s;\,c)$ is a periodic solution of (16) with period $\omega(c)$, where $\tilde\phi(s;\,c)=\phi(s;\,c)$ for $s\leq 0$, $\tilde\phi(s;\,c)=\phi(s-(k+1)\omega(c);\,c)$ for $s\in[k\omega(c),\,(k+1)\omega(c)]$, $k=0,\,1,\,2,\ldots$, and $\omega(c)$ is the period of $\phi(s;\,c)$ for $s<0$.*

We point out that the eventually periodic and periodic solutions of (16) in Theorem 3.1(2)(iii) and Corollary 3.7 are mainly due to the piecewise linearity and symmetry of the output function f. Equation (16) has also periodic solutions with arbitrary small magnitude resulting from the linear part of the output function. In fact, we have

Theorem 3.8. *Suppose that $a<1+\beta$. Let $\nu_0\in[0,\,\pi]$ be such that*

$$\cos\nu_0=\frac{1-a}{\beta}\,,$$

and ν_k and c_k be defined by

$$\nu_k = \nu_0 + 2k\pi \quad \text{and} \quad c_k = \frac{-\beta \sin \nu_k}{\nu_k}\,.$$

Then the functions $\phi_{k,l}(s)$ defined by

$$\phi_{k,l}(s) = l \cdot \cos \nu_k s \quad \text{with } |l| \le 1$$

for $k \in \mathbf{Z}$ are periodic solutions of (16).

Proof. Suppose that $i\nu_0$ with $\nu_0 \in [0, \pi]$ is the root of the characteristic equation of (16) about $\phi = x^0$, that is, $\Delta(i\nu_0, c, x^0) = 0$, where

$$\Delta(\sigma, c, x^0) = -c\sigma + 1 - a - \beta e^{\sigma}\,.$$

Then

$$\cos \nu_0 = \frac{1-a}{\beta} \quad \text{and} \quad c = \frac{-\beta \sin \nu_0}{\nu_0}$$

Hence, $\Delta(i\nu_k, c_k, x^0) = 0$ for all $k \in \mathbf{Z}$ and the result follows. ■

As mentioned, (16) and (17) with $f = f_\varepsilon$ ($\varepsilon > 0$) and $f = f_1$ have same dynamics. Moreover, denote $c_*(a, \beta)$ $c_p(a, \beta)$, and $c^*(a, \beta)$ as c_*, c_p, and c^* in Theorem 3.1, respectively. By Theorem 3.1 and Proposition 3.6, we have

Theorem 3.9. *Let $a > 0$ and $\beta > 0$ be fixed, and $c_*(\varepsilon) = c_*(a/\varepsilon, \beta/\varepsilon)$, $c_p(\varepsilon) = c_p(a/\varepsilon, \beta/\varepsilon)$, $c^*(\varepsilon) = c^*(a/\varepsilon, \beta/\varepsilon)$. Suppose that $\phi_\varepsilon(s; c)$ is the solution of (16) and (17) with $f = f_\varepsilon$. There is $\varepsilon_0 = \varepsilon_0(a, \beta) > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, the following holds.*

(1) *Suppose $a > \beta$. Then*
   (i) *if $c \le c_*$, then $\phi_\varepsilon(s; c)$ is nondecreasing and satisfying (8),*
   (ii) *if $c_* < c < 0$, then $\phi_\varepsilon(s; c)$ is oscillating and $|\phi_\varepsilon(s; c)| < \varepsilon$ for $s < 0$.*

(2) *Suppose $a \le \beta$. Then*
   (i) *if $c \le c_*$, then $\phi_\varepsilon(s; c)$ is nondecreasing and satisfies (8),*
   (ii) *if $c_* < c < c_p$, then $\phi_\varepsilon(s; c)$ is oscillating,*
   (iii) *if $c_p \le c < c_*$, then $\phi_\varepsilon(s; c)$ is eventually periodic and $\sup_{s<0} |\phi_\varepsilon(s; c)| > \varepsilon$,*
   (iv) *if $c = c^*$, then $\phi_\varepsilon(s; c^*)$ is nondecreasing and there exists s_* such that $\phi_\varepsilon(s; c^*) = x^-$ for all $s \le s_*$,*
   (v) *if $c^* < c < 0$, then $\phi(s; c)$ is nondecreasing and unbounded.*

Remark 3.2. By Proposition 3.6, $c_*(a/\varepsilon, \beta/\varepsilon) \le c_p(a/\varepsilon, \beta/\varepsilon) \to -\infty$ as $\varepsilon \to 0$.

4. Traveling Waves in CNN with Idealized Output

In this section, we shall prove Theorem B stated in the introduction. We therefore consider (16) and (17) with $f = f_0$. Assume that $a > 0$ and $\beta > 0$. Denote $\phi_0(s; c)$ as the solution of (16) and (17) with $f = f_0$.

Theorem 4.1.

(1) *Suppose $a \ge \beta$. Then for any $c < 0$, $\phi_0(s; c)$ is nondecreasing and there is $s_0 < 0$ such that $\phi_0(s; c) = 0$ for $s \le s_0$.*
(2) *Suppose $a < \beta$. Then there is $c^* < 0$ such that*
   (i) *if $c < c^*$, then $\phi_0(s; c)$ is eventually periodic,*
   (ii) *if $c = c^*$, then $\phi_0(s; c)$ is nondecreasing and there is $s_* < 0$ such that $\phi(s; c) = x^-$ for $s \le s_*$,*
   (iii) *if $c^* < c < 0$, then $\phi_0(s; c)$ is nondecreasing and unbounded.*

Remark 4.1. Theorem 4.1(1) and (2) can be viewed as the limit analogue of Theorem 3.9(1) and (2) as $\varepsilon \to 0$, respectively. Note that when $a > \beta$, oscillating solutions no longer exist due to the fact that $|\phi_\varepsilon(s; c)| < \varepsilon$ for $s < 0$. When $a < \beta$, nondecreasing solutions satisfying (8) no longer exist and oscillating solutions are eventually periodic due to the fact that $c_p(a/\varepsilon, \beta/\varepsilon) \to -\infty$ as $\varepsilon \to 0$. It should be pointed out that when $a = \beta$, the dynamics is similar to the case that $a > \beta$.

Proof of Theorem 4.1. First of all, we may assume that $a+\beta > 1$. For otherwise, let $k > 1/(a+\beta)$ and $\psi(s; c) = k\phi_0(s; c)$. Then ψ satisfies (16) and (17) with $f = f_0$ and a and β being replaced by ka and $k\beta$, respectively. Hence we may also assume that $\phi_0(0; c) = 1$.

(1) Suppose that $a \ge \beta$. For any $c < 0$, consider

$$\psi'(s) = \frac{1}{c}(\psi(s) - a - \beta)$$

with $\psi(0) = 1$. Then

$$\psi(s) = (1 - a - \beta)e^{\frac{1}{c}s} + a + \beta\,. \qquad (61)$$

Let $s_1^0 < 0$ be such that $\psi(s_1^0) = 0$, and

$$\phi_0(s;\, c) = \begin{cases} \psi(s), & s \ge s_1^0\,, \\ 0, & s < s_1^0\,. \end{cases}$$

Since $a \ge \beta$, $\phi_0(s;\, c)$ is the solution of (16) and (17) (see Definition 2.1).

(2) Suppose that $a < \beta$. Let $\psi(s)$ be as in (61) and $s_1^0 < 0$ be such that $\psi(s_1^0) = 0$. Then $s_1^0 = c(\ln(a+\beta)/(-1+a+\beta))$, and

$$\phi_0(s;\, c) = \begin{cases} (1 - a - \beta)e^{\frac{1}{c}s} + a + \beta \\ \qquad \text{for } s \ge s_1^0\,, \\ (a-\beta)e^{\frac{1}{c}(s - s_1^0)} - a + \beta \\ \qquad \text{for } s_1^0 - 1 \le s \le s_1^0\,. \end{cases} \qquad (62)$$

Let

$$c^* = \left(\ln \frac{\beta - a}{2\beta}\right)^{-1}\,. \qquad (63)$$

By (62), we have

$$\phi_0(s_1^0 - 1;\, c^*) = x^-,$$

and

$$\phi_0(s_1^0 - 1;\, c) < x^- \quad \text{for } c^* < c < 0\,.$$

It then follows that

$$\phi_0(s;\, c^*) = x^- \quad \text{for } s \le s_1^0 - 1\,,$$

and

$$\begin{aligned}\phi_0(s;\, c) &= (\phi(s_1^0 - 1;\, c) \\ &\quad + a + \beta)e^{\frac{1}{c}(s - s_1^0 + 1)} - a - \beta \\ &\qquad \text{for} \quad s \le s_1^0 - 1\,, \quad c^* < c < 0\,.\end{aligned}$$

Therefore, (ii) and (iii) hold.

If $c < c^*$, we may assume that $\phi_0(s_2^0;\, c) = 0$ for some $s_2^0 < s_1^0 - 1$ and $\phi_0(s;\, c)$ satisfies

$$-c\phi_0'(s;\, c) = -\phi_0(s;\, c) + a - \beta \qquad \text{for } s \in [s_2^0 - 1,\, s_2^0]\,. \qquad (64)$$

By (62) and (64), we have

$$\phi_0(s;\, c) = -\phi_0(s - s_2^0 + s_1^0;\, c) \qquad \text{for } s \in [s_2^0 - 1,\, s_2^0]\,. \qquad (65)$$

Then by (65), we have

$$\phi_0(s;\, c) = -\phi_0(s - s_2^0 + s_1^0;\, c) \qquad \text{for } s \in [s_2^0 - 2,\, s_2^0 - 1]\,. \qquad (66)$$

Continuing this process, we have that

$$\begin{aligned}\phi_0(s;\, c) &= -\phi_0(s - s_2^0 + s_1^0;\, c) \\ &= \phi_0(s - 2(s_1^0 - s_2^0);\, c) \quad \text{for } s \le 2s_2^0 - s_1^0\,.\end{aligned} \qquad (67)$$

Hence, $\phi_0(s;\, c)$ is a periodic function for $s \le s_1^0$ with period $2(s_1^0 - s_2^0)$. ■

Remark 4.2. Suppose that $a < \beta$. Let c^* be as in Theorem 4.1(2) and $\omega(c)$ be the period of the eventually periodic solution $\phi(s;\, c)$ in Theorem 4.1(2)(ii). Then by an elementary computation, we have

$$\begin{aligned}\omega_0(c) &= 2(s_1^0 - s_2^0) \\ &= \omega_0(c) = 2\left(1 - c \cdot \ln \frac{a+\beta}{2\beta + (a-\beta)e^{\frac{-1}{c}}}\right)\,,\end{aligned}$$

and

$$\lim_{c \to c^*} \omega_0(c) = \infty \quad \text{and} \quad \lim_{c \to -\infty} \omega_0(c) = \frac{4\beta}{a+\beta}\,.$$

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