Chapter 4

Vector Calculus

4.1 The Line Integrals

4.1.1 Curves

Definition 4.1. A subset $C \subseteq \mathbb{R}^n$ is called a ***curve*** if C is the image of an interval $I \subseteq \mathbb{R}$ under the continuous map $\gamma : I \to \mathbb{R}^n$ (that is, $C = \gamma(I)$). The continuous map $\gamma : I \to \mathbb{R}^n$ is called a ***parametrization*** of the curve. A curve C is called ***simple*** if it has an injective parametrization; that is, there exists $\gamma : I \to \mathbb{R}^n$ such that $\gamma(I) = C$ and $\gamma(x) = \gamma(y)$ implies that $x = y$. A curve C with parametrization $\gamma : I \to \mathbb{R}^n$ is called ***closed*** if $I = [a, b]$ for some closed interval $[a, b] \subseteq \mathbb{R}$ and $\gamma(a) = \gamma(b)$. A ***simple closed*** curve C is a closed curve with parametrization $\gamma : [a, b] \to \mathbb{R}^n$ such that γ is one-to-one on (a, b).

Example 4.2. A line segment joining two points $P_0, P_1 \in \mathbb{R}^n$ is a curve. It can be parameterized by $\gamma : [0, 1] \to \mathbb{R}^n$ defined by $\gamma(t) = tP_1 + (1 - t)P_0$.

Example 4.3. A circle on the plane is a simple closed curve. In fact, a circle centered at the (x_0, y_0) with radius r has the following parametrization: $\gamma : [0, 2\pi] \to \mathbb{R}^2$ defined by $\gamma(\theta) = (x_0 + r\cos\theta, y_0 + r\sin\theta)$.

Example 4.4. Figure eight is the zero level set of $F(x, y) = x^4 - a^2(x^2 - y^2)$ for some $a \neq 0$. It can also be parameterized by $\gamma : [0, 4\pi] \to \mathbb{R}^2$ defined by $\gamma(\theta) = \left(a\cos\frac{\theta}{2}, \frac{a}{2}\sin\theta\right)$.

Definition 4.5 (Length of Curves)**.** The length of curve $C \subseteq \mathbb{R}^n$ parameterized by $\gamma : [a, b] \to \mathbb{R}^n$ is defined as the number

$$\ell(C) \equiv \sup\left\{ \sum_{i=1}^{k} \left\|\gamma(t_i) - \gamma(t_{i-1})\right\|_{\mathbb{R}^n} \,\middle|\, k \in \mathbb{N} \text{ and } a = t_0 < t_1 < \cdots < t_k = b \right\}.$$

Definition 4.6 (Rectifiable curves)**.** A curve $C \subseteq \mathbb{R}^n$ with parametrization $\gamma : I \to \mathbb{R}^n$ is called ***rectifiable*** if there is an homeomorphism $\varphi : \widetilde{I} \to I$, where $\widetilde{I}$ is again an interval, such that the map $\gamma \circ \varphi : \widetilde{I} \to \mathbb{R}^n$ is Lipschitz.

Remark 4.7. 1. By an homeomorphism it means a continuous bijection whose inverse is also continuous.

2. We can think of a curve as an equivalence class of continuous maps $\gamma : I \to \mathbb{R}^n$, where two parametrization $\gamma : I \to \mathbb{R}^n$ and $\widetilde{\gamma} : \widetilde{I} \to \mathbb{R}^n$ are equivalent if and only if there is an homeomorphism $\varphi : \widetilde{I} \to I$ such that $\widetilde{\gamma} = \gamma \circ \varphi$. Each element of the equivalence class is a parametrization of the curve and thus a rectifiable curve is a curve which has a Lipschitz continuous parametrization.

3. The length of a rectifiable curve parameterized by $\gamma : [a, b] \to \mathbb{R}^n$ is finite since by choosing a Lipschitz parametrization $\widetilde{\gamma} : [c, d] \to \mathbb{R}^n$, the number

$$\Big\{ \sum_{i=1}^{k} \|\widetilde{\gamma}(t_i) - \widetilde{\gamma}(t_{i-1})\|_{\mathbb{R}^n} \,\Big|\, k \in \mathbb{N} \text{ and } c = t_0 < t_1 < \cdots < t_k = d \Big\}$$

is bounded from above by $M(d - c)$, where M is the Lipschitz constant of $\widetilde{\gamma}$.

Example 4.8 (Non-rectifiable curves)**.** Let $C \subseteq \mathbb{R}^2$ be a curve parameterized by

$$\gamma(t) = \begin{cases} \big(t, t \sin \frac{\pi}{t}\big) & \text{if } \ t \in (0, 1]\,, \\ (0, 0) & \text{if } \ t = 0\,. \end{cases}$$

Since

$$\ell\big(\gamma([\tfrac{1}{n+1}, \tfrac{1}{n}])\big) \geqslant \|\gamma(\tfrac{1}{n+1}) - \gamma(\tfrac{1}{n+1/2})\|_{\mathbb{R}^2} + \|\gamma(\tfrac{1}{n+1/2}) - \gamma(\tfrac{1}{n})\|_{\mathbb{R}^2} \geqslant \frac{2}{n+1/2}$$

and $\sum\limits_{n=1}^{\infty} \dfrac{2}{n+1/2} = \infty$, by the remark above we conclude that $\gamma([0, 1])$ is not a rectifiable curve.

Definition 4.9. A curve $C \subseteq \mathbb{R}^n$ is said to be of class $\mathscr{C}^k$ or a $\mathscr{C}^k$-curve if there exists a parametrization $\gamma : I \to \mathbb{R}^n$ such that γ is k-times continuously differentiable. Such a parametrization is called a $\mathscr{C}^k$-parametrization of the curve. If there exists a parametrization $\gamma : I \to \mathbb{R}$ which is of class $\mathscr{C}^k$ for all $k \in \mathbb{N}$, then the curve is said to be ***smooth***. A curve $C \subseteq \mathbb{R}^n$ is said to be ***regular*** if there exists a $\mathscr{C}^1$-parametrization $\gamma : I \to \mathbb{R}^n$ such that $\gamma'(t) \neq \mathbf{0}$ for all $t \in I$.

Theorem 4.10. *Let $C \subseteq \mathbb{R}^n$ be a curve with $\mathscr{C}^1$-parametrization $\gamma : [a,b] \to \mathbb{R}^n$. Then*

$$\ell(C) = \int_a^b \|\gamma'(t)\|_{\mathbb{R}^n}\, dt\,.$$

Proof. Let $\varepsilon > 0$ be given. Since $\gamma : [a,b] \to \mathbb{R}^n$ is $\mathscr{C}^1$, there exists $\delta > 0$ such that

$$\|\gamma'(t) - \gamma'(s)\|_{\mathbb{R}^n} < \frac{\varepsilon}{4\sqrt{n}(b-a)} \quad \text{whenever} \quad s,t \in [a,b], |s-t| < \delta\,.$$

By the definition of the length of curves, there exists a partition $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_k = b\}$ of $[a,b]$ such that

$$\ell(C) - \frac{\varepsilon}{4} < \sum_{i=1}^{k} \|\gamma(t_i) - \gamma(t_{i-1})\|_{\mathbb{R}^n} \leqslant \ell(C)\,.$$

W.L.O.G., we can assume that $\|\mathcal{P}\| < \delta$. For each component γ_j of γ, the mean value theorem implies that for some $\xi_i \in [t_{i-1}, t_i]$,

$$\gamma_j(t_i) - \gamma_j(t_{i-1}) = \gamma_j'(\xi_i)(t_i - t_{i-1})\,;$$

thus for each $i \in \{1, \cdots, k\}$ and $s_i \in [t_{i-1}, t_i]$,

$$\big|\gamma_j(t_i) - \gamma_j(t_{i-1}) - \gamma_j'(s_i)(t_i - t_{i-1})\big| \leqslant \big|\gamma_j'(\xi_i) - \gamma_j'(s_i)\big||t_i - t_{i-1}| < \frac{\varepsilon}{4\sqrt{n}(b-a)}|t_i - t_{i-1}|\,.$$

As a consequence, for each $i \in \{1, \cdots, k\}$ and $s_i \in [t_{i-1}, t_i]$,

$$\begin{aligned} \Big|\|\gamma(t_i) - \gamma(t_{i-1})\|_{\mathbb{R}^n} - \|\gamma'(s_i)\|_{\mathbb{R}^n}|t_i - t_{i-1}|\Big| &< \Big|\|\gamma(t_i) - \gamma(t_{i-1})\|_{\mathbb{R}^n} - \|\gamma'(s_i)(t_i - t_{i-1})\|_{\mathbb{R}^n}\Big| \\ \leqslant \|\gamma(t_i) - \gamma(t_{i-1}) - \gamma'(s_i)(t_i - t_{i-1})\|_{\mathbb{R}^n} &\leqslant \Big[\sum_{j=1}^{n}\Big(\frac{\varepsilon}{4\sqrt{n}(b-a)}|t_i - t_{i-1}|\Big)^2\Big]^{\frac{1}{2}} \\ < \frac{\varepsilon}{4(b-a)}|t_i - t_{i-1}| \end{aligned}$$

which further implies that

$$\Big|\sum_{i=1}^{k} \|\gamma(t_i) - \gamma(t_{i-1})\|_{\mathbb{R}^n} - \sum_{i=1}^{k} \|\gamma'(s_i)\|_{\mathbb{R}^n}|t_i - t_{i-1}|\Big| < \frac{\varepsilon}{4}\,.$$

Therefore, for $a = t_0 \leqslant s_0 \leqslant t_1 \leqslant s_1 \cdots \leqslant s_k \leqslant t_k = b$,

$$\ell(C) - \frac{\varepsilon}{2} < \sum_{i=1}^{k} \|\gamma'(s_i)\|_{\mathbb{R}^n}|t_i - t_{i-1}| < \ell(C) + \frac{\varepsilon}{2}\,.$$

Since $\|\gamma'\|$ is Riemann integrable over $[a,b]$, we must have

$$\ell(C) - \varepsilon < L(\|\gamma'\|_{\mathbb{R}^n}, \mathcal{P}) \leqslant \int_a^b \|\gamma'(t)\|_{\mathbb{R}^n} dt \leqslant U(\|\gamma'\|_{\mathbb{R}^n}, \mathcal{P}) < \ell(C) + \varepsilon\,,$$

and the theorem is concluded because $\varepsilon > 0$ is given arbitrarily. □

Example 4.11. The length of the elliptic helix C parameterized by

$$\gamma(t) = (a\cos t, b\sin t, ct) \qquad t \in \Big[0, \frac{\pi}{2}\Big]$$

can be computed by

$$\ell(C) = \int_0^{\frac{\pi}{2}} \|\gamma'(t)\|_{\mathbb{R}^3} dt = \int_0^{\frac{\pi}{2}} \sqrt{a^2\sin^2 t + b^2\cos^2 t + c^2}\, dt\,.$$

1. When $a < b$, letting $k = \sqrt{\dfrac{b^2 - a^2}{b^2 + c^2}}$, then

$$\ell(C) = \sqrt{b^2 + c^2} \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 t}\, dt\,.$$

2. When $a > b$, letting $k = \sqrt{\dfrac{a^2 - b^2}{a^2 + c^2}}$, then

$$\ell(C) = \sqrt{a^2 + c^2} \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \cos^2 t}\, dt = \sqrt{a^2 + c^2} \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 t}\, dt\,.$$

The integral $E(k, \phi) \equiv \displaystyle\int_0^{\phi} \sqrt{1 - k^2\sin^2 t}\, dt$, where $0 < k^2 < 1$, is called the ***elliptic integral function of the second kind***, and $\mathrm{E}(k) \equiv E\big(k, \frac{\pi}{2}\big)$ is called the ***complete elliptic integral of the second kind***.

Definition 4.12. Let $C \subseteq \mathbb{R}^n$ be a curve with finite length. An ***arc-length parametrization*** of C is an injective parametrization $\gamma : [a, b] \to \mathbb{R}^n$ such that the length of the curve $\gamma([a, s])$ is exactly $s - a$; that is,

$$\ell\big(\gamma([a, s])\big) = s - a \qquad \forall\, s \in [a, b]\,.$$

Example 4.13. Let C be the circle centered at the origin with radius R. Then the parametrization

$$\gamma(s) = \Big(R\cos\frac{s}{R}, R\sin\frac{s}{R}\Big) \qquad s \in [0, 2\pi R]\,,$$

is an arc-length parametrization of C. To see this, we note that

$$\ell\big(\gamma([0, s])\big) = \int_0^s \big\|\gamma'(t)\big\|_{\mathbb{R}^2} dt = \int_0^s \Big\|\Big(-\sin\frac{s}{R}, \cos\frac{s}{R}\Big)\Big\|_{\mathbb{R}^2} dt = \int_0^s dt = s \qquad \forall\, s \in [0, 2\pi R]\,.$$

In general, the arc-length parametrization of a rectifiable curve exists, and we have the following

Theorem 4.14. *Let $C \subseteq \mathbb{R}^n$ be a rectifiable simple curve. Then there exists an arc-length parametrization of C.*

Proof. We only prove the case that C has a regular $\mathscr{C}^1$-parametrization $\gamma : [a,b] \to \mathbb{R}^n$.

Let $s(t) = \int_a^t \|\gamma'(t')\|_{\mathbb{R}^n}\, dt'$. Note that the $s : [a,b] \to \mathbb{R}$ is strictly increasing since the fundamental theorem of Calculus implies that $s'(t) = \|\gamma'(t)\|_{\mathbb{R}^n} > 0$ for all $t \in [a,b]$. The Inverse Function Theorem (Theorem A.10) then guarantees that s has a $\mathscr{C}^1$-inverse $u : [0, \ell(C)] \to [a,b]$ and we have $u'(t) = \dfrac{1}{s'(u(t))}$. Define $\widetilde{\gamma} = \gamma \circ u$. Then the chain rule implies that $\widetilde{\gamma} : [0, \ell(C)] \to \mathbb{R}^n$ is a $\mathscr{C}^1$-parametrization of C, and Theorem 4.10 implies that

$$\begin{aligned}\ell\big(\widetilde{\gamma}([0,s])\big) &= \int_0^s \|\widetilde{\gamma}'(t)\|_{\mathbb{R}^n}\, dt = \int_0^s \|\gamma'(u(t))u'(t)\|_{\mathbb{R}^n}\, dt = \int_0^s \|\gamma'(u(t))\|_{\mathbb{R}^n}\big|u'(t)\big|\, dt \\ &= \int_0^s s'(u(t))\frac{1}{\big|s'(u(t))\big|}\, dt = \int_0^s 1 dt = s\end{aligned}$$

which implies that $\widetilde{\gamma} : [0, \ell(C)]$ is an arc-length parametrization of C. □

Theorem 4.15. *Let $C \subseteq \mathbb{R}^n$ be a $\mathscr{C}^1$-curve with an arc-length parametrization $\gamma : I \to \mathbb{R}^n$. Then $\|\gamma'(s)\|_{\mathbb{R}^n} = 1$ for all $s \in I$.*

Proof. Suppose that $I = [a,b]$. Since $\gamma : I \to \mathbb{R}^n$ is an arc-length parametrization of C, we must have

$$s - a = \int_a^s \|\gamma'(t)\|_{\mathbb{R}^n}\, dt \qquad \forall\, s \in I\,.$$

Differentiating both sides of the equality above in t, the fundamental theorem of Calculus implies that $1 = \|\gamma'(s)\|_{\mathbb{R}^n}$ for all $s \in I$. □

4.1.2 The line element and line integrals

Line elements

Definition 4.16. A curve $C \subseteq \mathbb{R}^n$ is said to be piecewise $\mathscr{C}^k$ (smooth, regular) if there exists a parametrization $\gamma : [a,b] \to \mathbb{R}^n$ and a finite set of points $\{a = t_0 < t_1 < \cdots < t_N = b\}$ such that $\gamma : [t_i, t_{i+1}] \to \mathbb{R}^n$ is $\mathscr{C}^k$ (smooth, regular) for all $i \in \{0, 1, \cdots, N-1\}$.

Definition 4.17. Let $\mathscr{R}_{\mathcal{C}}$ be the collection of all piecewise regular curves. The line element is a set function $s : \mathscr{R}_{\mathcal{C}} \to \mathbb{R}$ that satisfies the following properties:

1. $s(C) > 0$ for all $C \in \mathscr{R}_{\mathcal{C}}$.

2. If $C \in \mathscr{R}_{\mathcal{C}}$ is the union of finitely many regular curves $C_1, \cdots, C_k$ that do not overlap except at their end-points, then
$$s(C) = s(C_1) + \cdots + s(C_k)\,.$$

3. The value of s agrees with the length on straight line segments; that is,
$$s(L) = \ell(L) \qquad \text{for all line segaments } L\,.$$

Line integrals of scalar functions

Definition 4.18. Let $C \subseteq \mathbb{R}^n$ be a simple rectifiable curve with an injective Lipschitz parametrization $\gamma : [a, b] \to \mathbb{R}^n$, and $f : C \to \mathbb{R}$ be a real-valued function. The ***line integral*** of f along C, denoted by $\displaystyle\int_C f\,ds$, is the number

$$\sup\Big\{\sum_{i=1}^{k}\big(\inf_{\xi\in\gamma([t_{i-1},t_i])} f(\xi)\big)\ell\big(\gamma([t_{i-1},t_i])\big)\,\Big|\,k\in\mathbb{N}, a=t_0<t_1<\cdots<t_k=b\Big\}$$

provided that it is identical to

$$\inf\Big\{\sum_{i=1}^{k}\big(\sup_{\xi\in\gamma([t_{i-1},t_i])} f(\xi)\big)\ell\big(\gamma([t_{i-1},t_i])\big)\,\Big|\,k\in\mathbb{N}, a=t_0<t_1<\cdots<t_k=b\Big\}.$$

When C is a closed curve, we also use $\displaystyle\oint_C f\,ds$ to denote the line integral of f along C to emphasize that the curve C is a closed loop.

Remark 4.19. Since the parametrization γ is required to be injective, the line integral of f along C is independent of the choice of the parametrization.

Remark 4.20. In particular, if $f \equiv 1$, then $\ell(C) = \displaystyle\int_C 1\,ds \equiv \int_C ds$.

Remark 4.21. If the curve C is a line segment $\{(x, 0)\,|\,a \leqslant x \leqslant b\}$, then the line integral of f along C is simply the Riemann integral of f over $[a, b]$ (by treating f as a function of x).

Remark 4.22 (The interpretation of the line integrals)**.** Let C be a piecewise smooth curve, and $f(x)$ denote the density of the curve C at position x. Suppose that f is continuous on C and $x = \gamma(t)$. Then $f(x)$ is computed by

$$f(x) = f(\gamma(t)) = \lim_{\Delta t \to 0} \frac{\mathrm{m}(\gamma([t, t+\Delta t]))}{\ell(\gamma([t, t+\Delta t]))},$$

where $\mathrm{m}(\cdot)$ denotes the mass. Let $\varepsilon > 0$ be given. Then by the continuity of $f \circ \gamma$ and the definition of limit, there exists $\delta > 0$ such that

$$\big|(f\circ\gamma)(t) - (f\circ\gamma)(s)\big| < \frac{\varepsilon}{4\ell(C)} \quad \text{if} \quad t, s \in [a,b],\ |t-s| < \delta$$

and

$$\big|f(\gamma(t))\ell(\gamma([t,t+\Delta t])) - \mathrm{m}(\gamma([t,t+\Delta t]))\big| \leqslant \ell(\gamma([t,t+\Delta t]))\frac{\varepsilon}{4\ell(C)} \quad \text{if} \quad |\Delta t| < \delta\,;$$

thus if $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_k = b\}$ is a partition of $[a,b]$ with $\|\mathcal{P}\| < \delta$, the total mass of the curve $\mathrm{m}(C)$, given by $\mathrm{m}(C) = \sum\limits_{i=1}^{k} \mathrm{m}(\gamma([t_{i-1}, t_i]))$, validates the following estimate:

$$\Big|\mathrm{m}(C) - \sum_{i=1}^{k} f(\gamma(s_{i-1}))\ell(\gamma([t_{i-1}, t_i]))\Big| \leqslant \frac{\varepsilon}{2}\,.$$

As a consequence,

$$\mathrm{m}(C) - \varepsilon < \sum_{i=1}^{k} \inf_{\xi \in \gamma([t_{i-1},t_i])} f(\xi)\ell(\gamma([t_{i-1},t_i])) \leqslant \sum_{i=1}^{k} \sup_{\xi \in \gamma([t_{i-1},t_i])} f(\xi)\ell(\gamma([t_{i-1},t_i])) < \mathrm{m}(C) + \varepsilon$$

which implies that the line integral of f along C is exactly the mass of the curve.

Theorem 4.23. *Let $C \subseteq \mathbb{R}^n$ be a simple curve with $\mathscr{C}^1$-parametrization $\gamma : [a,b] \to \mathbb{R}^n$, and $f : C \to \mathbb{R}$ be a real-valued continuous function. Then*

$$\int_C f\,ds = \int_a^b f\big(\gamma(t)\big)\|\gamma'(t)\|_{\mathbb{R}^n}\,dt\,. \tag{4.1}$$

Proof. Let $\varepsilon > 0$ be given. Since $f\circ\gamma$ and γ' are continuous on $[a,b]$, $|f\circ\gamma| + \|\gamma'\|_{\mathbb{R}^n} \leqslant M$ on $[a,b]$ for some $M > 0$, and there exists $\delta > 0$ such that

$$\big|(f\circ\gamma)(s) - (f\circ\gamma)(t)\big| < \frac{\varepsilon}{8(M+1)(b-a)} \quad \text{whenever} \quad s, t \in [a,b],\ |s-t| < \delta$$

and

$$\left\|\gamma'(s) - \gamma'(t)\right\|_{\mathbb{R}^n} < \frac{\varepsilon}{8(M+1)(b-a)} \quad \text{whenever} \quad s,t \in [a,b],\ |s-t| < \delta\,.$$

Moreover, since $f \circ \gamma$ and γ' are both continuous on $[a,b]$, the integral $\int_a^b f\big(\gamma(t)\big)\|\gamma'(t)\|_{\mathbb{R}^n}\,dt$ exists; thus there exists a partition $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_k = b\}$ of $[a,b]$ such that

$$\sum_{i=1}^{k}\Big(\sup_{s\in[t_{i-1},t_i]}\big(f(\gamma(s))\|\gamma'(s)\|_{\mathbb{R}^n}\big) - \inf_{s\in[t_{i-1},t_i]}\big(f(\gamma(s))\|\gamma'(s)\|_{\mathbb{R}^n}\big)\Big)|t_i - t_{i-1}| < \frac{\varepsilon}{2}\,. \tag{4.2}$$

By choosing of a refinement of $\mathcal{P}$ if necessary, we can assume that $\|\mathcal{P}\| < \delta$. Let $s_i, r_i \in [t_{i-1}, t_i]$ be such that

$$\sup_{t\in[t_{i-1},t_i]}\big(f(\gamma(t))\|\gamma'(t)\|_{\mathbb{R}^n}\big) = f(\gamma(s_i))\left\|\gamma'(s_i)\right\|_{\mathbb{R}^n} \quad \text{and} \quad \sup_{\xi\in\gamma([t_{i-1},t_i])} f(\xi) = f(\gamma(r_i))\,.$$

Moreover, by Theorem 4.10 and the mean value theorem for integrals, there exists $q_i \in [t_{i-1}, t_i]$ such that

$$\ell(\gamma([t_{i-1},t_i])) = \int_{t_{i-1}}^{t_i}\left\|\gamma'(s)\right\|_{\mathbb{R}^n} ds = \left\|\gamma'(q_i)\right\|_{\mathbb{R}^n}|t_i - t_{i-1}|\,;$$

thus

$$\Big|\ell(\gamma([t_{i-1},t_i])) - \left\|\gamma'(s_i)\right\|_{\mathbb{R}^n}|t_i - t_{i-1}|\Big| \leqslant \frac{\varepsilon}{8(M+1)(b-a)}|t_i - t_{i-1}|\,.$$

Therefore, by the fact that $s_i, r_i, q_i \in [t_{i-1}, t_i]$ and $|t_i - t_{i-1}| < \delta$,

$$\begin{aligned}
&\Big|\sup_{s\in[t_{i-1},t_i]}\big(f(\gamma(s))\|\gamma'(s)\|_{\mathbb{R}^n}\big)|t_i - t_{i-1}| - \sup_{\xi\in\gamma([t_{i-1},t_i])} f(\xi)\ell(\gamma([t_{i-1},t_i]))\Big| \\
&\quad = \Big|f(\gamma(s_i))\left\|\gamma'(s_i)\right\|_{\mathbb{R}^n} - f(\gamma(r_i))\left\|\gamma'(q_i)\right\|_{\mathbb{R}^n}\Big||t_i - t_{i-1}| \\
&\quad \leqslant \big|f(\gamma(s_i)) - f(\gamma(r_i))\big|\left\|\gamma'(s_i)\right\|_{\mathbb{R}^n}|t_i - t_{i-1}| + \big|f(\gamma(r_i))\big|\left\|\gamma'(s_i) - \gamma'(q_i)\right\|_{\mathbb{R}^n}|t_i - t_{i-1}| \\
&\quad < \frac{\varepsilon}{4(b-a)}|t_i - t_{i-1}|\,,
\end{aligned}$$

and summing the inequality above over i we obtain that

$$\Big|\sum_{i=1}^{k}\sup_{s\in[t_{i-1},t_i]}\big(f(\gamma(s))\|\gamma'(s)\|_{\mathbb{R}^n}\big)|t_i - t_{i-1}| - \sum_{i=1}^{k}\sup_{\xi\in\gamma([t_{i-1},t_i])} f(\xi)\ell(\gamma([t_{i-1},t_i]))\Big| < \frac{\varepsilon}{4}\,.$$

Similarly,

$$\Big|\sum_{i=1}^{k}\inf_{s\in[t_{i-1},t_i]}\big(f(\gamma(s))\|\gamma'(s)\|_{\mathbb{R}^n}\big)|t_i - t_{i-1}| - \sum_{i=1}^{k}\inf_{\xi\in\gamma([t_{i-1},t_i])} f(\xi)\ell(\gamma([t_{i-1},t_i]))\Big| < \frac{\varepsilon}{4}\,;$$

thus using (4.2) we find that

$$\begin{aligned}\int_a^b (f\circ\gamma)(t)\|\gamma'(t)\|_{\mathbb{R}^n}\,dt-\varepsilon &< L\big((f\circ\gamma)\|\gamma'\|_{\mathbb{R}^n},\mathcal{P}\big)-\frac{\varepsilon}{4}\\ &\leqslant \sum_{i=1}^k \inf_{\xi\in\gamma([t_{i-1},t_i])} f(\xi)\ell(\gamma([t_{i-1},t_i]))\leqslant \sum_{i=1}^k \sup_{\xi\in\gamma([t_{i-1},t_i])} f(\xi)\ell(\gamma([t_{i-1},t_i]))\\ &\leqslant U\big((f\circ\gamma)\|\gamma'\|_{\mathbb{R}^n},\mathcal{P}\big)+\frac{\varepsilon}{4}<\int_a^b (f\circ\gamma)(t)\|\gamma'(t)\|_{\mathbb{R}^n}\,dt+\varepsilon\,.\end{aligned}$$

Since $\varepsilon>0$ is chosen arbitrary, we conclude (4.1). □

Example 4.24. Let C be the upper half part of the circle centered at the origin with radius $R>0$ in the xy-plane. Evaluate the line integral $\displaystyle\int_C y\,ds$.

First, we parameterize C by

$$\gamma(t)=(R\cos t,R\sin t)\qquad t\in[0,\pi]\,.$$

Then

$$\int_C y\,ds=\int_0^\pi R\sin t\big\|(-R\sin t,R\cos t)\big\|_{\mathbb{R}^2}dt=\int_0^\pi R^2\sin t\,dt=2R^2\,.$$

Example 4.25. Find the mass of a wire lying along the first octant part of the curve of intersection of the elliptic paraboloid $z=2-x^2-2y^2$ and the parabolic cylinder $z=x^2$ between $(0,1,0)$ and $(1,0,1)$ if the density of the wire at position (x,y,z) is $\varrho(x,y,z)=xy$.

Note that we can parameterize the curve C by

$$\gamma(t)=(t,\sqrt{1-t^2},t^2)\qquad t\in[0,1]\,.$$

Therefore, the mass of the curve can be computed by

$$\begin{aligned}\int_C \varrho\,ds&=\int_0^1 t\sqrt{1-t^2}\,\big\|(1,\frac{-t}{\sqrt{1-t^2}},2t)\big\|_{\mathbb{R}^3}dt=\int_0^1 t\sqrt{1-t^2}\,\frac{\sqrt{1-t^2+t^2+4t^2(1-t^2)}}{\sqrt{1-t^2}}\,dt\\ &=\int_0^1 t\sqrt{2-(1-2t^2)^2}\,dt=\frac14\int_{-1}^1\sqrt{2-u^2}du=\frac14\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}2\cos^2\theta\,d\theta\\ &=\frac14\Big[\theta+\frac{\sin(2\theta)}{2}\Big]\Big|_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}}=\frac{\pi}{8}+\frac14\,.\end{aligned}$$

Line integrals of vector fields

We recall that a vector field is a vector-valued function whose domain and co-domain are subsets of identical Euclidean space $\mathbb{R}^n$.

Let C be a simple regular curve parameterized by $\gamma : I \to \mathbb{R}^n$, and $\boldsymbol{F} : C \to \mathbb{R}^n$ be a vector field. The ***line integral of F along*** C ***in the direction of*** γ (or the ***oriented line integral of F along*** C) is defined as the line integral of the scalar function $\boldsymbol{F} \cdot \mathbf{T}$ along C, where $\mathbf{T}$ is the unit tangent of C given by

$$\mathbf{T} = \frac{\gamma'}{\|\gamma'\|_{\mathbb{R}^n}} \circ \gamma^{-1} \quad \text{on} \quad C\,. \tag{4.3}$$

Given another parametrization $\phi : \widetilde{I} \to \mathbb{R}^n$ of C such that $(\phi' \circ \phi^{-1}) \cdot (\gamma' \circ \gamma^{-1}) > 0$ (that is, the orientation of C given by ϕ and γ are the same), using the chain rule we obtain that

$$\gamma' = \frac{d}{dt}(\phi \circ \phi^{-1} \circ \gamma)(t) = (\phi' \circ \phi^{-1} \circ \gamma)(t)(\phi^{-1} \circ \gamma)'(t)\,. \tag{4.4}$$

Since $\phi^{-1} \circ \gamma : I \to \widetilde{I}$, $(\phi^{-1} \circ \gamma)'$ is a scalar function; thus (4.4) and the fact that $(\phi' \circ \phi^{-1}) \cdot (\gamma' \circ \gamma^{-1}) > 0$ imply that $\gamma' \circ \gamma^{-1} = c(\phi' \circ \phi^{-1})$ for some positive scalar function $c : C \to \mathbb{R}$. Therefore,

$$\frac{\phi'}{\|\phi'\|_{\mathbb{R}^n}} \circ \phi^{-1} = \frac{\gamma'}{\|\gamma'\|_{\mathbb{R}^n}} \circ \gamma^{-1} \quad \text{on} \quad C\,. \tag{4.5}$$

In other words, the tangent vector $\mathbf{T}$ is well-defined on C; thus the line integral of $\boldsymbol{F}$ along C in the direction of the parametrization γ is a well-defined quantity.

Suppose that $I = [a, b]$. Using (4.1), we find that

$$\int_C \boldsymbol{F} \cdot \mathbf{T}\, ds = \int_a^b (\boldsymbol{F} \circ \gamma)(t) \cdot \frac{\gamma'(t)}{\|\gamma'(t)\|_{\mathbb{R}^n}} \|\gamma'(t)\|_{\mathbb{R}^n}\, dt = \int_a^b (\boldsymbol{F} \circ \gamma)(t) \cdot \gamma'(t)\, dt\,.$$

Let $\boldsymbol{r} : \widetilde{I} \to \mathbb{R}^n$ be an arc-length parametrization of C such that $(\boldsymbol{r}' \circ \boldsymbol{r}^{-1}) \cdot (\gamma' \circ \gamma^{-1}) > 0$ on C. Then (4.5) implies that $\mathbf{T} = \dfrac{d\boldsymbol{r}}{ds}$. In terms of notation, we also write $\mathbf{T}\, ds$ as $d\boldsymbol{r}$; thus

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \boldsymbol{F} \cdot \mathbf{T}\, ds = \int_a^b (\boldsymbol{F} \circ \gamma)(t) \cdot \gamma'(t)\, dt\,.$$

Remark 4.26 (The interpretation of line integrals of vector fields)**.** Consider the work done by moving an object along a smooth curve C parameterized by $\gamma : I \to \mathbb{R}^n$ with a continuous variable force $\boldsymbol{F} : C \to \mathbb{R}^n$ from $\gamma(a)$ to $\gamma(b)$ (that is, in the direction of the parametrization of γ). Since the work done by a constant force is the inner product of the displacement and the force, we find the the work done by the force $\boldsymbol{F}$ along the small portion $\gamma([t_i, t_{i+1}])$, from $\gamma(t_i)$ to $\gamma(t_{i+1})$, of the curve, where $|t_i - t_{i+1}| \ll 1$, is approximately

$$(\boldsymbol{F} \cdot \mathbf{T})(\gamma(t_i))\ell\big(\gamma([t_i, t_{i+1}])\big) \equiv \boldsymbol{F}(\gamma(t_i)) \cdot \mathbf{T}(\gamma(t_i))\ell\big(\gamma([t_i, t_{i+1}])\big)\,.$$

Summing over all the portions, we conclude that the work done by the force $\boldsymbol{F}$ along the curve C, in the direction of the parametrization γ, is approximately

$$\sum_{i=0}^{k-1}(\boldsymbol{F} \cdot \mathbf{T})(\gamma(t_i))\ell\big(\gamma([t_i, t_{i+1}])\big)$$

which converges to the line integral $\int_C (\boldsymbol{F} \cdot \mathbf{T})\, ds$. Therefore, the line integral of vector fields $\boldsymbol{F}$ along C in the direction of the parametrization γ is simply the work done by the force $\boldsymbol{F}$ in moving an object along the curve C from the starting point to the end point.

Example 4.27. Let $\boldsymbol{F}(x, y) = (y^2, 2xy)$. Evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ from $(0, 0)$ to $(1, 1)$ along

1. the straight line $y = x$,
2. the curve $y = x^2$, and
3. the piecewise smooth path consisting of the straight line segments from $(0, 0)$ to $(0, 1)$ and from $(0, 1)$ to $(1, 1)$.

For the straight line case, we parameterize the path by $\gamma(t) = (t, t)$ for $t \in [0, 1]$. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 (t^2, 2t^2) \cdot (1, 1)dt = \int_0^1 3t^2 dt = 1\,.$$

For the case of parabola, we parameterize the path by $\gamma(t) = (t, t^2)$ for $t \in [0, 1]$. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 (t^4, 2t^3) \cdot (1, 2t)dt = \int_0^1 5t^4 dt = 1\,.$$

For the piecewise linear case, we let C_1 denote the line segment joining $(0, 0)$ and $(0, 1)$, and let C_2 denote the line segment joining $(0, 1)$ and $(1, 1)$. Note that we can parameterize C_1 and C_2 by

$$\gamma_1(t) = (0, t) \quad t \in [0, 1] \quad \text{and} \quad \gamma_2(t) = (t, 1) \quad t \in [0, 1]\,,$$

respectively. Therefore,

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{F}\cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^1 (t^2,0)\cdot(0,1)\,dt + \int_0^1 (1,2t)\cdot(1,0)\,dt = 1\,.$$

We note that in this example the line integrals of $\boldsymbol{F}$ over three different paths joining $(0,0)$ and $(1,1)$ are identical.

Example 4.28. Let $\boldsymbol{F}(x,y) = (y,-x)$. Evaluate the line integral $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ from $(1,0)$ to $(0,-1)$ along

1. the straight line segment joining these points, and
2. three-quarters of the circle of unit radius centered at the origin and traversed counter-clockwise.

For the first case, we parameterize the path by $\gamma(t) = (1-t,-t)$ for $t\in[0,1]$. Then

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^1 (-t,t-1)\cdot(-1,-1)\,dt = \int_0^1 1\,dt = 1\,.$$

For the second case, we parameterize the path by $\gamma(t) = (\cos t,\sin t)$ for $t\in\left[0,\frac{3\pi}{2}\right]$. Then

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{\frac{3\pi}{2}} (\sin t,-\cos t)\cdot(-\sin t,\cos t)\,dt = \int_0^{\frac{3\pi}{2}} (-1)\,dt = -\frac{3\pi}{2}\,.$$

We note that in this example the line integrals of $\boldsymbol{F}$ over different paths joining $(1,0)$ and $(0,-1)$ might be different.

4.2 Conservative Vector Fields

In the previous section, we define the line integral of a force along a curve in a given orientation. In Example 4.27, we see that the line integrals along three different paths connecting two given points are the same, while in Example 4.28 the line integrals along two different paths (connecting two given points) are different. In this section, we are interested in the rule of judging whether the line integral is path independent or not.

Definition 4.29 (Conservative Fields)**.** A vector field $\mathbf{F}:\mathcal{D}\subseteq\mathbb{R}^{\rm n}\to\mathbb{R}^{\rm n}$ is said to be ***conservative*** if $\mathbf{F}=\nabla\phi$ for some scalar function $\varphi:\mathcal{D}\to\mathbb{R}$. Such a ϕ is called a (scalar) potential for $\mathbf{F}$ on $\mathcal{D}$.

Theorem 4.30. *Let $\mathcal{D}$ be an open, connected domain in $\mathbb{R}^n$, and let* $\mathbf{F}$ *be a smooth vector field defined on $\mathcal{D}$. Then the following three statements are equivalent:*

(1) $\mathbf{F}$ *is conservative in $\mathcal{D}$.*

(2) $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ *for every piecewise smooth, closed curve C in $\mathcal{D}$.*

(3) *Given any two point $P_0, P_1 \in \mathcal{D}$, $\int_C \mathbf{F}\cdot d\mathbf{r}$ has the same value for all piecewise smooth curves in $\mathcal{D}$ starting at P_0 and ending at P_1.*

Proof. (1) $\Rightarrow$ (2): Suppose that $\boldsymbol{F} = \nabla\phi$ in $\mathcal{D}$ for some scalar function $\phi : \mathcal{D} \to \mathbb{R}$. Let $C \subseteq \mathbb{R}^n$ be a piecewise smooth closed curve parameterized by $\gamma : [a,b] \to \mathbb{R}^n$ such that $\gamma : [t_{i-1}, t_i] \to \mathbb{R}^n$ is smooth for all $1 \leqslant i \leqslant N$, where $a = t_0 < t_1 < \cdots < t_N = b$. Let $C_i = \gamma([t_{i-1}, t_i])$. Then the chain rule implies that

$$\begin{aligned}\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} &= \sum_{i=1}^N \int_{C_i} \nabla\phi\cdot d\boldsymbol{r} = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (\nabla\phi\circ\gamma)(t)\cdot\gamma'(t)\,dt \\ &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \frac{d}{dt}(\phi\circ\gamma)(t)\,dt = \sum_{i=1}^N (\phi\circ\gamma)(t)\Big|_{t=t_{i-1}}^{t=t_i} = \phi(\gamma(b)) - \phi(\gamma(a)) = 0\,.\end{aligned}$$

(2) $\Rightarrow$ (3): Let C_1 and C_2 be two piecewise smooth curves in $\mathcal{D}$ starting at P_0 and ending at P_1 parameterized by $\gamma_1 : [a,b] \to \mathbb{R}^n$ and $\gamma_2 : [c,d] \to \mathbb{R}^n$, respectively. Define $\gamma : [a, b+d-c] \to \mathbb{R}^n$ by

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [a,b]\,, \\ \gamma_2(b+d-t) & \text{if } t \in [b, b+d-c]\,. \end{cases}$$

Then $C = \gamma([a, b+d-c])$ is a piecewise smooth closed curve; thus

$$\begin{aligned} 0 = \oint_C \boldsymbol{F}\cdot d\boldsymbol{r} &= \int_a^b (\boldsymbol{F}\circ\gamma_1)(t)\cdot\gamma_1'(t)\,dt - \int_b^{b+d-c} (\boldsymbol{F}\circ\gamma_2)(b+d-t)\gamma_2'(b+d-t)\,dt \\ &= \int_{C_1} \boldsymbol{F}\cdot d\boldsymbol{r} - \int_c^d (\boldsymbol{F}\circ\gamma_2)(t)\gamma_2'(t)dt = \int_{C_1} \boldsymbol{F}\cdot d\boldsymbol{r} - \int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{r}\,. \end{aligned}$$

(3) $\Rightarrow$ (1): Let $P_0 \in \mathcal{D}$. For $x \in \mathcal{D}$, define $\phi(x) = \int_C \boldsymbol{F}\cdot d\boldsymbol{r}$, where C is any piecewise smooth curve starting at P_0 and ending at x. Note that by assumption, $\phi : \mathcal{D} \to \mathbb{R}$ is well-defined.

Choose $\delta > 0$ such that $B(x,\delta) \subseteq \mathcal{D}$. Let C be a piecewise smooth curve joining P_0, and L be the line segment joining x and $x + h\mathrm{e}_j$, where $0 < h < \delta$ and $\mathrm{e}_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ is the unit vector whose j-th component is 1. Then with the parametrization of L: $\gamma(t) = x + t\mathrm{e}_j$ for $t \in [0,h]$, we have

$$\frac{\phi(x+h\mathrm{e}_j) - \phi(x)}{h} = \frac{1}{h}\int_L \boldsymbol{F}\cdot d\boldsymbol{r} = \frac{1}{h}\int_0^h \boldsymbol{F}(x+t\mathrm{e}_j)\cdot \mathrm{e}_j\,dt\,;$$

thus passing to the limit as $h \to 0$, we find that

$$\frac{\partial \phi}{\partial x_j}(x) = \boldsymbol{F}(x)\cdot \mathrm{e}_j\,.$$

As a consequence, $\boldsymbol{F}(x) = (\nabla\phi)(x)$ which implies that $\boldsymbol{F}$ is conservative. □

Let $\mathcal{D} \subseteq \mathbb{R}^2$, and $\boldsymbol{F} = (M,N) : \mathcal{D} \to \mathbb{R}^2$. If $\boldsymbol{F}$ is conservative, then $M = \phi_x$ and $N = \phi_y$ for some scalar function $\phi : \mathcal{D} \to \mathbb{R}$; thus if ϕ is of class $\mathscr{C}^2$, we must have $M_y = N_x$. In other words, if $\boldsymbol{F} : \mathcal{D} \to \mathbb{R}^2$ is a smooth vector field, then it is necessary that $M_y = N_x$. The converse statement is not true in general, and we have the following counter-example.

Example 4.31. Let $\mathcal{D} \subseteq \mathbb{R}^2$ be the annular region $\mathcal{D} = \big\{(x,y)\,\big|\,1 < x^2+y^2 < 4\big\}$, and consider the vector field $\boldsymbol{F}(x,y) = \big(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\big)$. Then

$$\frac{\partial}{\partial y}\frac{y}{x^2+y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\frac{-x}{x^2+y^2}\,;$$

however, if $\boldsymbol{F} = \nabla\phi$ for some differentiable scalar function $\phi : \mathcal{D} \to \mathbb{R}$, we must have

$$\phi_x(x,y) = \frac{y}{x^2+y^2}$$

which further implies that

$$\phi(x,y) = \arctan\frac{x}{y} + f(y)\,.$$

Using that $\phi_y(x,y) = \frac{y}{x^2+y^2}$, we conclude that f is a constant function; thus

$$\phi(x,y) = \arctan\frac{x}{y} + C\,.$$

Since ϕ is not differentiable on the positive x-axis, $\boldsymbol{F} \neq \nabla\phi$.

Definition 4.32. A connected domain $\mathcal{D}$ is said to be ***simply connected*** if every simple closed curve can be continuously shrunk to a point in $\mathcal{D}$ without any part ever passing out of $\mathcal{D}$.

Theorem 4.33. *Let $\mathcal{D} \subseteq \mathbb{R}^2$ be simply connected, and $\boldsymbol{F} = (M, N) : \mathcal{D} \to \mathbb{R}^2$ be of class $\mathscr{C}^1$. If $M_y = N_x$, then $\boldsymbol{F}$ is conservative.*

The theorem above can be proved using Theorem 4.30 and Green's theorem (Theorem 4.90), and is left till Section 4.8 (where Green's theorem is introduced).

4.3 The Surface Integrals

4.3.1 Surfaces

Definition 4.34. A subset $\Sigma \subseteq \mathbb{R}^3$ is called a surface if for each $p \in \Sigma$, there exist an open neighborhood $\mathcal{U} \subseteq \Sigma$ of p, an open set $\mathcal{V} \subseteq \mathbb{R}^2$, and a continuous map $\varphi : \mathcal{U} \to \mathcal{V}$ such that $\varphi : \mathcal{U} \to \mathcal{V}$ is one-to-one, onto, and its inverse $\psi = \varphi^{-1}$ is also continuous. Such a pair $\{\mathcal{U}, \varphi\}$ is called a coordinate chart (or simply chart) at p, and $\{\mathcal{V}, \psi\}$ is called a (local) parametrization at p.

Remark 4.35. In some literatures the surface is defined in the following equivalent but reversed way: A subset $\Sigma \subseteq \mathbb{R}^3$ is a surface if for each $p \in \Sigma$, there exists a neighborhood $\mathcal{U} \subseteq \mathbb{R}^3$ of p and a map $\psi : \mathcal{V} \to \mathcal{U} \cap \Sigma$ of an open set $\mathcal{V} \subseteq \mathbb{R}^2$ onto $\mathcal{U} \cap \Sigma \subseteq \mathbb{R}^3$ such that ψ is a homeomorphism; that is, ψ has an inverse $\varphi = \psi^{-1} : \mathcal{U} \cap \Sigma \to \mathcal{V}$ which is continuous. The mapping ψ is called a parametrization or a system of (local) coordinates in (a neighborhood of) p.

Definition 4.36 (Regular surfaces)**.** A surface $\Sigma \subseteq \mathbb{R}^3$ is said to be regular if for each $p \in \Sigma$, there exists a differentiable local parametrization $\{\mathcal{V}, \psi\}$ of Σ at p such that $D\psi(q)$, the derivative of ψ at q, has full rank for all $q \in \mathcal{V}$; that is, $D\psi(q) : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one for all $q \in \mathcal{V}$. The range of the map $D\psi\big(\psi^{-1}(p)\big)$ is called the ***tangent plane*** of Σ at p, and is denoted by $\mathbf{T}_p\Sigma$.

In the following, we always assume that $D\psi(q)$ has full rank for all $q \in \mathcal{V}$ if $\{\mathcal{V}, \psi\}$ is a local parametrization of a regular surface $\Sigma \subseteq \mathbb{R}^3$.

Remark 4.37. Write $\psi : \mathcal{V} \to \Sigma$ as

$$\psi(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)\,.$$

Then if $q = (u_0, v_0)$,

$$\big[(D\psi)(q)\big] = \begin{bmatrix} x_u(u_0,v_0) & x_v(u_0,v_0) \\ y_u(u_0,v_0) & y_v(u_0,v_0) \\ z_u(u_0,v_0) & z_v(u_0,v_0) \end{bmatrix} = \big[[\psi,_1(u_0,v_0)] \vdots [\psi,_2(u_0,v_0)]\big]\,.$$

The injectivity of $D\psi(q)$ is then translated to that the two vectors

$$\psi,_1(u_0,v_0) \equiv \psi_u(u_0,v_0) = \big(x_u(u_0,v_0), y_u(u_0,v_0), z_u(u_0,v_0)\big)$$
$$\psi,_2(u_0,v_0) \equiv \psi_v(u_0,v_0) = \big(x_v(u_0,v_0), y_v(u_0,v_0), z_v(u_0,v_0)\big)$$

are linearly independent. Therefore, the range of $D\psi(q)$ is the span of the two vectors $\psi,_1(q)$ and $\psi,_2(q)$ and is indeed a plane for all $q \in \mathcal{V}$.

Let $p \in \Sigma$ and $q = \psi^{-1}(p)$. Since $D\psi(q)$ is injective, each $\boldsymbol{v} \in \mathbf{T}_p\Sigma$ corresponds a unique vector $(a,b) \in \mathbb{R}^2$ such that $\boldsymbol{v} = a\psi,_1(q) + b\psi,_2(q)$. This vector $(a,b) \in \mathbb{R}^2$ satisfies $[\boldsymbol{v}] = \big[D\psi(q)\big][a,b]^{\mathrm{T}}$, and can be computed by

$$\begin{bmatrix} a \\ b \end{bmatrix} = \Big(\big[D\psi(q)\big]^{\mathrm{T}}\big[D\psi(q)\big]\Big)^{-1}\big[D\psi(q)\big]^{\mathrm{T}}[\boldsymbol{v}]\,.$$

Example 4.38. Let $\mathbb{S}^2 = \big\{(x,y,z) \in \mathbb{R}^3 \,\big|\, x^2 + y^2 + z^2 = 1\big\}$ be the unit sphere in $\mathbb{R}^3$. If $p = (x_0, y_0, z_0) \in \mathbb{S}^2$, then either x_0, y_0 or z_0 is non-zero. Suppose that $z_0 \neq 0$. Let $r = 1 - \sqrt{x_0^2 + y_0^2} > 0$. Define

$$\psi(x,y) = \begin{cases} (x, y, \sqrt{1 - x^2 - y^2}) & \text{if } z_0 > 0\,, \\ (x, y, -\sqrt{1 - x^2 - y^2}) & \text{if } z_0 < 0\,, \end{cases}$$

$\mathcal{V} = B\big((x_0,y_0), r\big)$, and $\mathcal{U} = \psi(\mathcal{V})$. Then $\psi : \mathcal{V} \to \mathcal{U}$ is a bijection. Let $\varphi = \psi^{-1}$. Then $\{\mathcal{U}, \varphi\}$ is a coordinate chart at p; thus $\mathbb{S}^2$ is a surface.

There exists another coordinate chart. Let $\mathcal{U}_1 = \mathbb{S}^2\backslash(0,0,-1)$ and $\mathcal{U}_2 = \mathbb{S}^2\backslash(0,0,1)$. Define the map $\varphi_1 : \mathcal{U}_1 \to \mathbb{R}^2$ by that $\varphi_1(p)$ is the unique point on $\mathbb{R}^2$ such that $(0,0,-1)$, $\varphi_1(p)$ and $(x,y,0)$ are on the same straight line. Similarly, define $\varphi_2 : \mathcal{U}_2 \to \mathbb{R}^2$ by that $\varphi_2(p)$ is the unique point on $\mathbb{R}^2$ such that $(0,0,1)$, $\varphi_2(p)$ and $(x,y,0)$ are on the same straight line. It is easy to check that if $p \in \mathbb{S}^2$, then either $\{\mathcal{U}_1, \varphi_1\}$ or $\{\mathcal{U}_2, \varphi_2\}$ is a coordinate chart at p.

A third kind of coordinate chart is given as follows. Let $\mathcal{U} = (0, 2\pi) \times (0, \pi)$, and define

$$\psi(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$

Then $\psi : \mathcal{U} \to \mathbb{S}^2 \backslash \{(x, 0, z) \,|\, 0 \leqslant x \leqslant 1, x^2 + z^2 = 1\}$ is a continuous bijection with a continuous inverse. We note that for any $\mathcal{U} = (\theta_0, \theta_0 + 2\pi) \times (\phi_0, \phi_0 + \pi)$, ψ is a homeomorphism between $\mathcal{U}$ and an open subset of $\mathbb{S}^2$.

Next, we would like to define the derivative of f when $f : \Sigma \to \mathbb{R}^3$ is a vector-valued function. We first talk about what the directional derivative is. Let $\Sigma \subseteq \mathbb{R}^3$ be a regular surface, $p \in \Sigma$, and $\boldsymbol{v} \in \mathbf{T}_p\Sigma$. It is intuitive to define the directional derivative of f at p in the direction $\boldsymbol{v}$ by

$$\frac{d}{dt}\Big|_{t=0} (f \circ \boldsymbol{x})(t)\,, \tag{4.6}$$

if the derivative exists, where $\boldsymbol{x} : (-\delta, \delta) \to \Sigma$ is a $\mathscr{C}^1$-parametrization of a curve on Σ such that $\boldsymbol{x}(0) = p$ and $\boldsymbol{x}'(0) = \boldsymbol{v}$. The first question arising naturally is that if the derivative in (4.6) depends on the choices of $\boldsymbol{x}$. Suppose that $\boldsymbol{y} : (-\delta, \delta) \to \Sigma$ is a $\mathscr{C}^1$-parametrization of another curve on Σ such that $\boldsymbol{y}(0) = p$ and $\boldsymbol{y}'(0) = \boldsymbol{v}$ (note that the curve $\boldsymbol{x}((-\delta, \delta))$ and $\boldsymbol{y}((-\delta, \delta))$ in general are different). Let $\{\mathcal{V}, \psi\}$ be a parametrization of Σ at p, and $q = \psi^{-1}(p)$. Then the chain rule (Theorem 2.49) implies that

$$\boldsymbol{v} = \boldsymbol{x}'(0) = \frac{d}{dt}\Big|_{t=0} (\psi \circ \psi^{-1} \circ \boldsymbol{x})(t) = (D\psi)(q)\Big(\frac{d}{dt}\Big|_{t=0} (\psi^{-1} \circ \boldsymbol{x})(t)\Big)$$

and similarly, $\boldsymbol{v} = (D\psi)(q)\Big(\frac{d}{dt}\Big|_{t=0} (\psi^{-1} \circ \boldsymbol{y})(t)\Big)$. Therefore,

$$(D\psi)(q)\Big(\frac{d}{dt}\Big|_{t=0} (\psi^{-1} \circ \boldsymbol{x})(t)\Big) = (D\psi)(q)\Big(\frac{d}{dt}\Big|_{t=0} (\psi^{-1} \circ \boldsymbol{y})(t)\Big)\,.$$

The injectivity of $(D\psi)(\psi^{-1}(p))$ then shows that

$$\frac{d}{dt}\Big|_{t=0} (\psi^{-1} \circ \boldsymbol{x})(t) = \frac{d}{dt}\Big|_{t=0} (\psi^{-1} \circ \boldsymbol{y})(t)\,.$$

Using the chain rule again,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} (f \circ \boldsymbol{x})(t) &= \frac{d}{dt}\Big|_{t=0} (f \circ \psi \circ \psi^{-1} \circ \boldsymbol{x})(t) = D(f \circ \psi)(\psi^{-1}(p))\Big(\frac{d}{dt}\Big|_{t=0} (\psi^{-1} \circ \boldsymbol{x})(t)\Big) \\ &= D(f \circ \psi)(\psi^{-1}(p))\Big(\frac{d}{dt}\Big|_{t=0} (\psi^{-1} \circ \boldsymbol{y})(t)\Big) = \frac{d}{dt}\Big|_{t=0} (f \circ \boldsymbol{y})(t)\,.\end{aligned}$$

In other words, the derivative in (4.6) is independent of the choice of $\boldsymbol{x}$ as long as $\boldsymbol{x}(0) = p$ and $\boldsymbol{x}'(0) = \boldsymbol{v}$. This observation implies the following

Theorem 4.39. *Let $\Sigma \subseteq \mathbb{R}^3$ be a regular surface, $\{\mathcal{V}_1, \psi_1\}$ and $\{\mathcal{V}_2, \psi_2\}$ be two local $\mathscr{C}^1$-parameterizations of Σ at a point $p \in \Sigma$, and $\mathcal{U} = \psi_1(\mathcal{V}_1) \cap \psi_2(\mathcal{V}_2) \subseteq \Sigma$. Then for $(i,j) = (1,2)$ and $(2,1)$, the transition function $\psi_j^{-1} \circ \psi_i : \psi_i^{-1}(\mathcal{U}) \to \psi_j^{-1}(\mathcal{U})$ is of class $\mathscr{C}^1$.*

Proof. We first note that $\psi_j^{-1} \circ \psi_i$ is continuous on $\psi_i^{-1}(\mathcal{U})$. Moreover, by the chain rule we find that $\dfrac{\partial(\psi_j^{-1} \circ \psi_i)}{\partial u}$ is the unique 2-vector satisfying

$$\Big[\frac{\partial \psi_i}{\partial u}(u,v)\Big] = \Big[\frac{\partial}{\partial u}(\psi_j \circ \psi_j^{-1} \circ \psi_i)(u,v)\Big] = \Big[(D\psi_j)(\psi_j^{-1} \circ \psi_i)(u,v)\Big]\Big[\frac{\partial(\psi_j^{-1} \circ \psi_i)}{\partial u}(u,v)\Big].$$

Similarly, $\dfrac{\partial(\psi_j^{-1} \circ \psi_i)}{\partial v}$ is the unique 2-vector satisfying

$$\Big[\frac{\partial \psi_i}{\partial v}(u,v)\Big] = \Big[\frac{\partial}{\partial v}(\psi_j \circ \psi_j^{-1} \circ \psi_i)(u,v)\Big] = \Big[(D\psi_j)(\psi_j^{-1} \circ \psi_i)(u,v)\Big]\Big[\frac{\partial(\psi_j^{-1} \circ \psi_i)}{\partial v}(u,v)\Big].$$

Therefore, we obtain that

$$\big[D\psi_i\big] = \big[(D\psi_j) \circ (\psi_j^{-1} \circ \psi_i)\big]\Big[\big[\frac{\partial(\psi_j^{-1} \circ \psi_i)}{\partial u}\big] \vdots \big[\frac{\partial(\psi_j^{-1} \circ \psi_i)}{\partial v}\big]\Big]. \tag{4.7}$$

Since $\big[D\psi_j\big]$ has full rank, $\big[D\psi_j\big]^{\mathrm{T}}\big[D\psi_j\big]$ is an invertible 2×2 matrix (for if $A^{\mathrm{T}}Ax = 0$ then $\|Ax\|^2_{\mathbb{R}^n} = x^{\mathrm{T}}A^{\mathrm{T}}Ax = 0$ which implies $x = 0$ since A has full rank); thus (4.7) implies that

$$\Big[\big[\frac{\partial(\psi_j^{-1} \circ \psi_i)}{\partial u}\big] \vdots \big[\frac{\partial(\psi_j^{-1} \circ \psi_i)}{\partial v}\big]\Big] = \Big(\big(\big[D\psi_j\big]^{\mathrm{T}}\big[D\psi_j\big]\big) \circ (\psi_j^{-1} \circ \psi_i)\Big)^{-1}\big[(D\psi_j) \circ (\psi_j^{-1} \circ \psi_i)\big]^{\mathrm{T}}\big[D\psi_i\big];$$

thus the partial derivatives of $\psi_j^{-1} \circ \psi_i$ exist and are continuous. Theorem 2.30 then implies that $\psi_j^{-1} \circ \psi_i$ is of class $\mathscr{C}^1$. □

Similar to how the directional derivative is defined, we intend to define the differentiability of f through the differentiability of the function $f \circ \psi : \mathcal{V} \to \mathbb{R}^n$, where $\{\mathcal{V}, \psi\}$ is a local parametrization of Σ (at some point). Again, we need to talk about if this definition depends on the choice of local parameterizations. Nevertheless, if $\{\mathcal{V}_1, \psi_1\}$ and $\{\mathcal{V}_2, \psi_2\}$ are two $\mathscr{C}^1$-local parametrization of Σ at p, and $f \circ \psi_1$ is differentiable at $\psi_1^{-1}(p)$, then the chain rule and Theorem 4.39 imply that $f \circ \psi_2$ is also differentiable at $\psi_2^{-1}(p)$ since $f \circ \psi_2 = (f \circ \psi_1) \circ (\psi_1^{-1} \circ \psi_2)$. This induces the following

Definition 4.40. Let $\Sigma \subseteq \mathbb{R}^3$ be a $\mathscr{C}^1$-regular surface. A scalar function $f : \Sigma \to \mathbb{R}$ is said to be differentiable at $p \in \Sigma$ if for every parametrization $\{\mathcal{V}, \psi\}$ of Σ at p, the function

$f \circ \psi : \mathcal{V} \to \mathbb{R}^n$ is differentiable at $\psi^{-1}(p)$. The derivative of f at p, denoted by df_p, is a linear map on $T_p\,\Sigma$ satisfying

$$(df_p)(\boldsymbol{v}) = \frac{d}{dt}\Big|_{t=0}(f \circ \boldsymbol{x})(t)\,,$$

where $\boldsymbol{x} : (-\delta, \delta) \to \Sigma$ is a $\mathscr{C}^1$-parametrization of a curve on Σ such that $\boldsymbol{x}(0) = p$ and $\boldsymbol{x}'(0) = \boldsymbol{v}$. A scalar function $f : \Sigma \to \mathbb{R}$ is said to be of class $\mathscr{C}^1$ if $f \circ \psi$ is of class $\mathscr{C}^1$ for all local parametrization $\{\mathcal{V}, \psi\}$.

4.3.2 The metric tensor and the first fundamental form

Definition 4.41 (Metric)**.** Let $\Sigma \subseteq \mathbb{R}^3$ be a regular surface. The metric tensor associated with the local parametrization $\{\mathcal{V}, \psi\}$ (at $p \in \Sigma$) is the matrix $g = [g_{\alpha\beta}]_{2\times 2}$ given by

$$g_{\alpha\beta} = \psi_{,\alpha} \cdot \psi_{,\beta} = \sum_{i=1}^{3} \frac{\partial \psi^i}{\partial y_\alpha}\frac{\partial \psi^i}{\partial y_\beta} \qquad \text{in} \quad \mathcal{V}$$

or equivalently, $g = [D\psi]^{\mathrm{T}}[D\psi]$.

Proposition 4.42. *Let $\Sigma \subseteq \mathbb{R}^3$ be a regular surface, and $g = [g_{\alpha\beta}]_{2\times 2}$ be the metric tensor associated with the local parametrization $\{\mathcal{V}, \psi\}$ (at $p \in \Sigma$). Then the metric tensor g is positive definite; that is,*

$$\sum_{\alpha,\beta=1}^{2} g_{\alpha\beta} v^\alpha v^\beta > 0 \qquad \forall\, \boldsymbol{v} = \sum_{\gamma=1}^{2} v^\gamma \frac{\partial \psi}{\partial y^\gamma} \neq \boldsymbol{0}\,.$$

Proof. Since $D\psi$ has full rank on $\mathcal{V}$, every tangent vector $\boldsymbol{v}$ can be expressed as the linear combination of $\Big\{\dfrac{\partial \psi}{\partial y_1}, \dfrac{\partial \psi}{\partial y_2}\Big\}$. Write $\boldsymbol{v} = \sum\limits_{\gamma=1}^{2} v^\gamma \dfrac{\partial \psi}{\partial y^\gamma}$. Then if $\boldsymbol{v} \neq \boldsymbol{0}$,

$$0 < \|\boldsymbol{v}\|_{\mathbb{R}^3}^2 = \sum_{i=1}^{3}\sum_{\alpha,\beta=1}^{2} v^\alpha \frac{\partial \psi^i}{\partial y_\alpha} v^\beta \frac{\partial \psi^i}{\partial \psi_\beta} = \sum_{\alpha,\beta=1}^{2} g_{\alpha\beta} v^\alpha v^\beta\,. \qquad \square$$

Definition 4.43 (The first fundamental form)**.** Let $\Sigma \subseteq \mathbb{R}^3$ be a regular surface, and $g = [g_{\alpha\beta}]_{2\times 2}$ be the metric tensor associated with the local parametrization $\{\mathcal{V}, \psi\}$ (at $p \in \Sigma$). The first fundamental form associated with the local parametrization $\{\mathcal{V}, \psi\}$ (at $p \in \Sigma$) is the scalar function $\mathrm{g} = \det(g)$.

Theorem 4.44. *Let $\Sigma \subseteq \mathbb{R}^3$ be a regular surface, and $\{\mathcal{V}, \psi\}$ be a local parametrization at $p \in \Sigma$. Then*

$$\sqrt{\mathrm{g}} = \|\psi_{,1} \times \psi_{,2}\|_{\mathbb{R}^3}\,. \tag{4.8}$$

Proof. Using the permutation symbol and Kronecker's delta, we have

$$\begin{aligned}\|\psi,_1 \times \psi,_2\|^2_{\mathbb{R}^3} &= \sum_{i=1}^{3}\Big(\sum_{j,k=1}^{3}\varepsilon_{ijk}\psi^j,_1\psi^k,_2\Big)\Big(\sum_{r,s=1}^{3}\varepsilon_{irs}\psi^r,_1\psi^s,_2\Big)\\ &= \sum_{j,k,r,s=1}^{3}\Big[\Big(\sum_{i=1}^{3}\varepsilon_{ijk}\varepsilon_{irs}\Big)\psi^j,_1\psi^k,_2\psi^r,_1\psi^s,_2\Big]\\ &= \sum_{j,k,r,s=1}^{3}\big(\delta_{jr}\delta_{ks}-\delta_{js}\delta_{kr}\big)\psi^j,_1\psi^k,_2\psi^r,_1\psi^s,_2\,,\end{aligned}$$

where we use the identity

$$\sum_{i=1}^{3}\varepsilon_{ijk}\varepsilon_{irs} = \delta_{jr}\delta_{ks}-\delta_{js}\delta_{kr} \tag{4.9}$$

to conclude the last equality. Therefore,

$$\begin{aligned}\|\psi,_1 \times \psi,_2\|^2_{\mathbb{R}^3} &= \sum_{j,k=1}^{3}\big(\psi^j,_1\psi^k,_2\psi^j,_1\psi^k,_2 - \psi^j,_1\psi^k,_2\psi^j,_2\psi^k,_1\big)\\ &= g_{11}g_{22}-g_{12}g_{21} = \det(g) = \mathrm{g}\,.\end{aligned}$$

Finally, (4.8) is concluded from the fact that g is positive definite. □

Remark 4.45. Let $L \in \mathscr{B}(\mathbb{R}^2; \mathbf{T}_p\Sigma)$ be given by

$$L(a\mathrm{e}_1 + b\mathrm{e}_2) = a\psi,_1 + b\psi,_2\,,$$

where $\mathcal{B}_2 = \{\mathrm{e}_1, \mathrm{e}_2\}$ is the standard basis of $\mathbb{R}^2$. Let $\mathcal{B}' = \{e_1, e_2\}$ be an orthonormal basis of $\mathbf{T}_p\Sigma$, and $\mathcal{B}_3 = \{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$ be the standard basis of $\mathbb{R}^3$. Then

$$[L]_{\mathcal{B}_2,\mathcal{B}'} = \begin{bmatrix}\psi,_1\cdot e_1 & \psi,_2\cdot e_1\\ \psi,_1\cdot e_2 & \psi,_2\cdot e_2\end{bmatrix} = \begin{bmatrix}[e_1]^{\mathrm{T}}_{\mathcal{B}_3}\\ [e_2]^{\mathrm{T}}_{\mathcal{B}_3}\end{bmatrix}\Big[[\psi,_1]_{\mathcal{B}_3}\vdots[\psi,_2]_{\mathcal{B}_3}\Big]\,.$$

By the fact that $\{e_1, e_2\}$ is an orthonormal basis,

$$\begin{aligned}[L]^{\mathrm{T}}_{\mathcal{B}_2,\mathcal{B}'}[L]_{\mathcal{B}_2,\mathcal{B}'} &= \begin{bmatrix}[\psi,_1]^{\mathrm{T}}_{\mathcal{B}_3}\\ [\psi,_2]^{\mathrm{T}}_{\mathcal{B}_3}\end{bmatrix}\Big[[e_1]_{\mathcal{B}_3}\vdots[e_2]_{\mathcal{B}_3}\Big]\begin{bmatrix}[e_1]^{\mathrm{T}}_{\mathcal{B}_3}\\ [e,_2]^{\mathrm{T}}_{\mathcal{B}_3}\end{bmatrix}\Big[[\psi,_1]_{\mathcal{B}_3}\vdots[\psi,_2]_{\mathcal{B}_3}\Big]\\ &= \begin{bmatrix}[\psi,_1]^{\mathrm{T}}_{\mathcal{B}_3}\\ [\psi,_2]^{\mathrm{T}}_{\mathcal{B}_3}\end{bmatrix}\Big[[\psi,_1]_{\mathcal{B}_3}\vdots[\psi,_2]_{\mathcal{B}_3}\Big] = \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\,,\end{aligned}$$

where $[g_{\alpha\beta}]_{2\times 2}$ is the metric tensor associated with the parametrization $\{\mathcal{V}, \psi\}$. Therefore, $\det([L]_{\mathcal{B}_2,\mathcal{B}'}) = \sqrt{\mathrm{g}}$ as long as $\mathcal{B}'$ is an orthonormal basis of $\mathbf{T}_p\Sigma$.

Since a natural way to write $L\boldsymbol{v}$, where $\boldsymbol{v} = a\mathrm{e}_1 + b\mathrm{e}_2 \in \mathbb{R}^2$, is

$$L\boldsymbol{v} = \Big[[\psi,_1] \vdots [\psi,_2]\Big] \begin{bmatrix} a \\ b \end{bmatrix} = [\nabla\psi] \begin{bmatrix} a \\ b \end{bmatrix},$$

sometimes we also use $\nabla\psi$ to denote L, and then write $\sqrt{\mathrm{g}}$ as $\det(\nabla\psi)$ (even though $[\nabla\psi]$ is a 3×2 matrix) and call $\sqrt{\mathrm{g}}$ the Jacobian of the map ψ.

Example 4.46. Let Σ be the sphere centered at the origin with radius R. Consider the local parametrization $\psi(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$ with $(\theta, \phi) \in \mathcal{V} \equiv (0, 2\pi) \times (0, \pi)$. Then

$$\psi,_1(\theta, \phi) = \psi_\theta(\theta, \phi) = (-R\sin\theta\sin\phi, R\cos\theta\sin\phi, 0),$$
$$\psi,_2(\theta, \phi) = \psi_\phi(\theta, \phi) = (R\cos\theta\cos\phi, R\sin\theta\cos\phi, -R\sin\phi);$$

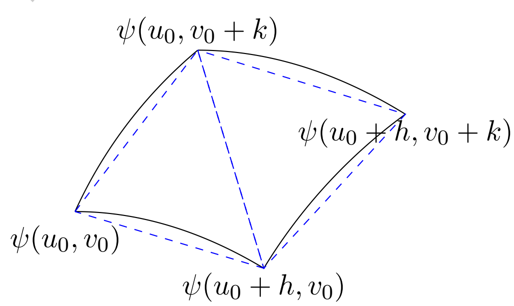
thus the metric tensor and the first fundamental form associated with the parametrization $\{\mathcal{V}, \psi\}$ are

$$g(\theta, \phi) = \big[D\psi\big]^{\mathrm{T}}\big[D\psi\big](\theta, \phi) = \begin{bmatrix} R^2\sin^2\phi & 0 \\ 0 & R^2 \end{bmatrix}$$

and $\mathrm{g} = \det(g) = R^4\sin^2\phi$.

What does the first fundamental form do for us?

Let $p = \psi(u_0, v_0)$ be a point in Σ. Then the surface area of the region $\psi\big([u_0, u_0 + h] \times [v_0, v_0 + k]\big)$, where h, k are very small, can be approximated by the sum of the area of two triangles, one with vertices $\psi(u_0, v_0)$, $\psi(u_0 + h, v_0)$, $\psi(u_0, v_0 + k)$ and the other with vertices $\psi(u_0 + h, v_0)$, $\psi(u_0, v_0 + k)$, $\psi(u_0 + h, v_0 + k)$.



Here we remark that the approximation of the surface area of a regular $\mathscr{C}^1$-surface obeys

$$\lim_{(h,k)\to(0,0)} \frac{\text{the surface area of } \psi\big([u_0, u_0 + h] \times [v_0, v_0 + k]\big)}{\text{the sum of area of the two triangles given in the context}} = 1. \tag{4.10}$$

The area of the triangle with vertices $\psi(u_0,v_0)$, $\psi(u_0+h,v_0)$, $\psi(u_0,v_0+k)$ is

$$A_1 = \frac{1}{2}\big\|\big(\psi(u_0+h,v_0)-\psi(u_0,v_0)\big)\times\big(\psi(u_0,v_0+k)-\psi(u_0,v_0)\big)\big\|_{\mathbb{R}^3}.$$

By the mean value theorem, for each component $j\in\{1,2,3\}$, we have

$$\begin{aligned}\psi^j(u_0+h,v_0)-\psi^j(u_0,v_0) &= \psi,_1(u_0+\theta_1^j h,v_0)h\,,\\ \psi^j(u_0,v_0+k)-\psi^j(u_0,v_0) &= \psi,_2(u_0,v_0+\theta_2^j k)k\end{aligned}$$

for some $\theta_i^j\in(0,1)$; thus if ψ is of class $\mathscr{C}^1$,

$$\begin{aligned}\psi(u_0+h,v_0)-\psi(u_0,v_0) &= \psi,_1(u_0,v_0)h+\boldsymbol{E}_1(u_0,v_0;h)h\,,\\ \psi(u_0,v_0+k)-\psi(u_0,v_0) &= \psi,_2(u_0,v_0)k+\boldsymbol{E}_2(u_0,v_0;k)k\,,\end{aligned}$$

where $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are bounded vector-valued functions satisfying that $\lim\limits_{h\to0}\boldsymbol{E}_1(u_0,v_0;h)=0$ and $\lim\limits_{k\to0}\boldsymbol{E}_2(u_0,v_0;k)=0$. Therefore,

$$\lim_{(h,k)\to(0,0)}\frac{\big(\psi(u_0+h,v_0)-\psi(u_0,v_0)\big)\times\big(\psi(u_0,v_0+k)-\psi(u_0,v_0)\big)}{hk}-\psi,_1(u_0,v_0)\times\psi,_2(u_0,v_0)=\mathbf{0}\,.$$

Since $\sqrt{\mathrm{g}}=\|\psi,_1\times\psi,_2\|_{\mathbb{R}^3}$, we have

$$A_1=\frac{1}{2}\sqrt{\mathrm{g}(u_0,v_0)}hk+f_1(u_0,v_0;h,k)hk$$

for some function f_1 which converges to 0 as $(h,k)\to(0,0)$ and is bounded since $\nabla\psi$ is bounded. Similarly, the area of the triangle with vertices $\psi(u_0+h,v_0)$, $\psi(u_0,v_0+k)$, $\psi(u_0+h,v_0+k)$ is

$$A_2=\frac{1}{2}\sqrt{\mathrm{g}(u_0,v_0)}hk+f_2(u_0,v_0;h,k)hk\,.$$

Taking (4.10) into account, we find that

$$\text{the surface area of }\psi\big([u_0,u_0+h]\times[v_0,v_0+k]\big)=\sqrt{\mathrm{g}(u_0,v_0)}hk+f(u_0,v_0;h,k)hk \tag{4.11}$$

for some bounded function $f(\cdot,\cdot;\cdot,\cdot)$ which converges to 0 as the last two variables h,k approach 0.

Now consider the surface area of $\psi([a,a+L]\times[b,b+W])$. Let $\varepsilon>0$ be given. Choose $N>0$ such that

$$\big|f(u,v;h,k)\big|<\frac{\varepsilon}{2LW}\quad\forall\,0<h<\frac{L}{N},0<k<\frac{W}{N}\text{ and }(u,v)\in[a,a+L]\times[b,b+W]\,,$$

and

$$\Big|\sum_{j=1}^{m}\sum_{i=1}^{n}\sqrt{\mathrm{g}\big(a+\frac{i-1}{n}L, b+\frac{j-1}{m}M\big)}\frac{L}{n}\frac{W}{m}-\int_{[a,a+L]\times[b,b+W]}\sqrt{\mathrm{g}}\,d\mathbb{A}\Big|<\frac{\varepsilon}{2}\quad \text{if } n,m\geqslant N\,.$$

Then for $n, m \geqslant N$, with (h,k) denoting $\big(\frac{L}{n},\frac{W}{m}\big)$ (4.11) implies that

$$\begin{aligned}
&\Big|\text{the surface area of } \psi([a,a+L]\times[b,b+W])-\int_{[a,a+L]\times[b,b+W]}\sqrt{\mathrm{g}}\,d\mathbb{A}\Big|\\
&\quad=\Big|\sum_{j=1}^{m}\sum_{i=1}^{n}\text{the surface area of } \psi([a+(i-1)h,a+ih]\times[b+(j-1)k,b+jk])\\
&\qquad-\int_{[a,a+L]\times[b,b+W]}\sqrt{\mathrm{g}}\,d\mathbb{A}\Big|\\
&\quad\leqslant\Big|\sum_{j=1}^{m}\sum_{i=1}^{n}\sqrt{\mathrm{g}\big(a+(i-1)h,b+(j-1)k\big)}hk-\int_{[a,a+L]\times[b,b+W]}\sqrt{\mathrm{g}}\,d\mathbb{A}\Big|\\
&\qquad+\Big|\sum_{j=1}^{m}\sum_{i=1}^{n}f(a+(i-1)h,b+(j-1)k;h,k)hk\Big|\\
&\quad<\frac{\varepsilon}{2}+\frac{\varepsilon}{2LW}\sum_{j=1}^{m}\sum_{i=1}^{n}hk=\varepsilon\,.
\end{aligned}$$

The discussion above verifies the following

Theorem 4.47. *Let $\Sigma\subseteq\mathbb{R}^3$ be a regular $\mathscr{C}^1$-surface, $\{\mathcal{V},\psi\}$ be a local $\mathscr{C}^1$-parametrization of Σ at p, and g be the first fundamental form associated with $\{\mathcal{V},\psi\}$. Then*

$$\textit{the surface area of } \psi(\mathcal{V})=\int_{\mathcal{V}}\sqrt{\mathrm{g}}\,d\mathbb{A}\,.$$

Example 4.48. Recall from Example 4.46 that the first fundamental form g of the parametrization $\{\mathcal{V},\psi\}$ of the 2-sphere centered at the origin with radius R, where

$$\psi(\theta,\phi)=(R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$$

and $\mathcal{V}=(0,2\pi)\times(0,\pi)$, is given by $\mathrm{g}(\theta,\phi)=R^4\sin^2\phi$. Therefore,

$$\begin{aligned}
&\text{the surface area of } \psi\big((0,2\pi)\times(0,\pi)\big)=\int_{(0,2\pi)\times(0,\pi)}R^2\sin\phi\,d(\theta,\phi)\\
&\qquad=R^2\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi d\theta=4\pi R^2\,.
\end{aligned}$$

Since the difference of the 2-sphere and $\psi\big((0,2\pi)\times(0,\pi)\big)$ has zero area, we find that the surface area of the 2-sphere with radius R is $4\pi R^2$.

Example 4.49. Let $\Sigma \subseteq \mathbb{R}^3$ be the upper half sphere; that is, $\Sigma = \big\{(x,y,z) \in \mathbb{R}^3 \,\big|\, x^2+y^2+z^2 = R^2, z > 0\big\}$, and $\{\mathcal{V}, \psi\}$ be a global parametrization of Σ given by

$$\psi(u,v) = (u, v, \sqrt{R^2-u^2-v^2}), \quad (u,v) \in \mathcal{V} = \big\{(u,v) \in \mathbb{R}^2 \,\big|\, u^2+v^2 \leqslant R^2\big\}.$$

To find the surface area using this parametrization, we first compute $\{\psi_{,1}, \psi_{,2}\}$ as follows:

$$\psi_{,1}(u,v) = \Big(1, 0, \frac{-u}{\sqrt{R^2-u^2-v^2}}\Big) \quad \text{and} \quad \psi_{,2}(u,v) = \Big(0, 1, \frac{-v}{\sqrt{R^2-u^2-v^2}}\Big),$$

thus the first fundamental form associated with the parametrization $\{\mathcal{V}, \psi\}$ is

$$\begin{aligned} \mathrm{g}(u,v) &= \|\psi_{,1}(u,v) \times \psi_{,2}(u,v)\|^2_{\mathbb{R}^3} = \Big\| \Big(\frac{u}{\sqrt{R^2-u^2-v^2}}, \frac{v}{\sqrt{R^2-u^2-v^2}}, 1\Big)\Big\|^2_{\mathbb{R}^3} \\ &= \frac{R^2}{R^2-u^2-v^2}. \end{aligned}$$

Therefore, the surface area of Σ is

$$\begin{aligned} \int_\Sigma dS &= \int_{\mathcal{V}} \frac{R}{\sqrt{R^2-u^2-v^2}}\, d\mathbb{A} = \int_{-R}^{R} \int_{-\sqrt{R^2-u^2}}^{\sqrt{R^2-u^2}} \frac{R}{\sqrt{R^2-u^2-v^2}}\, dv du \\ &= R \int_{-R}^{R} \arcsin \frac{v}{\sqrt{R^2-u^2}} \Big|_{v=-\sqrt{R^2-u^2}}^{v=\sqrt{R^2-u^2}} du = R \int_{-R}^{R} \pi\, du = 2\pi R^2. \end{aligned}$$

Note the the computation above also shows that the surface area of the sphere in $\mathbb{R}^3$ with radius R is $4\pi R^2$ which is the same as what we have conclude in Example 4.48.

Remark 4.50. The example above provides one specific way of evaluating the surface integrals: if the surface Σ is in fact a subset of the graph of a function $f : \mathcal{D} \subseteq \mathbb{R}^2 \to \mathbb{R}$; that is, $\Sigma \subseteq \big\{x, y, f(x,y)) \,\big|\, (x,y) \in \mathcal{D}\big\}$, then Σ has a global parametrization

$$\psi(x,y) = \big(x, y, f(x,y)\big), \qquad (x,y) \in \mathcal{V},$$

where $\mathcal{V}$ is the projection of Σ onto the xy-plane along the z-direction. Then the first fundamental form associated to this parametrization is

$$\mathrm{g}(x,y) = \|\psi_{,1}(x,y) \times \psi_{,2}(x,y)\|^2_{\mathbb{R}^3} = 1 + \Big|\frac{\partial f}{\partial x}(x,y)\Big|^2 + \Big|\frac{\partial f}{\partial y}(x,y)\Big|^2;$$

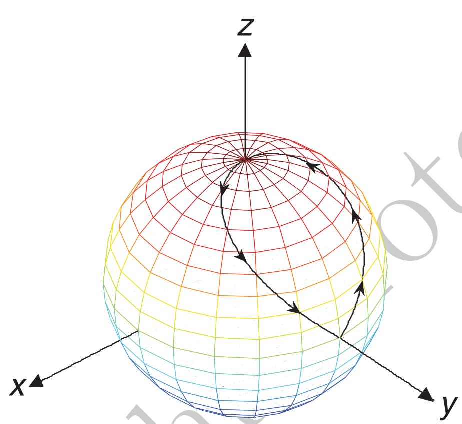
thus the surface area of Σ is

$$\int_\Sigma dS = \int_{\mathcal{V}} \sqrt{1 + \Big|\frac{\partial f}{\partial x}(x,y)\Big|^2 + \Big|\frac{\partial f}{\partial y}(x,y)\Big|^2}\, d(x,y).$$

Example 4.51. Let C be a smooth curve parameterized by

$$\boldsymbol{r}(t) = (\cos t \sin t, \sin t \sin t, \cos t), \qquad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

The clearly C is on the unit sphere $\mathbb{S}^2$ since $\|\boldsymbol{r}(t)\|_{\mathbb{R}^3} = 1$ for all $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Since C is a closed curve, C divides $\mathbb{S}^2$ into two parts. Let Σ denote the part with smaller area (see the following figure), and we are interested in finding the surface area of Σ.



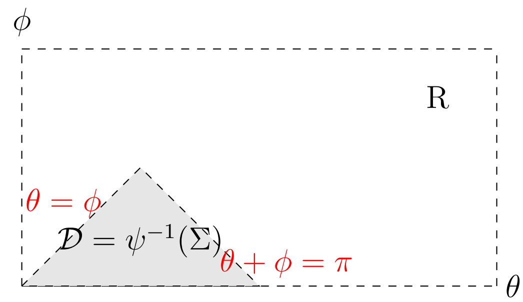
To compute the surface area of Σ, we need to find a way to parameterize Σ. Naturally we try to parameterize Σ using the spherical coordinate. In other words, let $\mathrm{R} = (0, 2\pi) \times (0, \pi)$ and $\psi : \mathrm{R} \to \mathbb{R}^3$ be defined by

$$\psi(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi),$$

and we would like to find a region $\mathcal{D} \subseteq \mathrm{R}$ such that $\psi(\mathcal{D}) = \Sigma$.

Suppose that $\gamma(t) = \big(\theta(t), \varphi(t)\big)$, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, is a curve in R such that $(\psi \circ \gamma)(t) = \boldsymbol{r}(t)$. Then for $t \in \left[0, \frac{\pi}{2}\right]$, the identity $\cos t = \cos\phi(t)$ implies that $\phi(t) = t$; thus the identities $\cos t \sin t = \cos\theta(t) \sin\phi(t)$ and $\sin t \sin t = \sin\theta(t) \sin\phi(t)$ further imply that $\theta(t) = t$.

On the other hand, for $t \in \left[-\frac{\pi}{2}, 0\right]$, the identity $\cos t = \cos\phi(t)$, where $\phi(t) \in (0, \pi)$, implies that $\phi(t) = -t$; thus the identities $\cos t \sin t = \cos\theta(t) \sin\phi(t)$ and $\sin t \sin t = \sin\theta(t) \sin\phi(t)$ further imply that $\theta(t) = \pi + t$.



Since the first fundamental form associate with $\{\mathrm{R}, \psi\}$ is the first fundamental form associated with $\{\mathrm{R}, \psi\}$ is

$$\begin{aligned}
\mathrm{g}(u, v) &= \big\|(\psi_\theta \times \psi_\phi)(u, v)\big\|_{\mathbb{R}^3}^2 \\
&= \big\|(-\sin\theta\sin\phi, \cos\theta\sin\phi, 0) \times (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)\big\|_{\mathbb{R}^3}^2 \\
&= \big\|(-\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -(\sin^2\theta + \cos^2\theta)\sin\phi\cos\phi)\big\|_{\mathbb{R}^3}^2 \\
&= (\cos^2\theta + \sin^2\theta)\sin^4\phi + \sin^2\phi\cos^2\phi = \sin^2\phi\,,
\end{aligned}$$

the area of the desired surface can be computed by

$$\begin{aligned}
\int_\Sigma dS &= \int_{\psi^{-1}(\Sigma)} \sqrt{\mathrm{g}}\, d\mathbb{A} = \int_0^{\frac{\pi}{2}} \int_\phi^{\pi-\phi} \sin\phi\, d\theta d\phi = \int_0^{\frac{\pi}{2}} (\pi - 2\phi)\sin\phi\, d\phi \\
&= \big(-\pi\cos\phi + 2\phi\cos\phi - 2\sin\phi\big)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} = \pi - 2\,.
\end{aligned}$$

Another way to parameterize Σ is to view Σ as the graph of function $z = \sqrt{1 - x^2 - y^2}$ over $\mathcal{D}$, where $\mathcal{D}$ is the projection of Σ along z-axis onto xy-plane. We note that the boundary of $\mathcal{D}$ can be parameterized by

$$\widetilde{\boldsymbol{r}}(t) = (\cos t\sin t, \sin t\sin t)\,, \qquad t \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big]\,.$$

Let $(x, y) \in \partial\mathcal{D}$. Then $x^2 + y^2 = y$; thus Σ can also be parameterized by $\psi : \mathcal{D} \to \mathbb{R}^3$, where

$$\psi(x, y) = \big(x, y, \sqrt{1 - x^2 - y^2}\big) \quad \text{and} \quad \mathcal{D} = \big\{(x, y)\,\big|\, x^2 + y^2 \leqslant y\big\}\,.$$

Therefore, with f denoting the function $f(x, y) = \sqrt{1 - x^2 - y^2}$, Remark 4.50 implies that the surface area of Σ can be computed by

$$\begin{aligned}
\int_{\mathcal{D}} \sqrt{1 + f_x^2 + f_y^2}\, d\mathbb{A} &= \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} \frac{1}{\sqrt{1 - x^2 - y^2}}\, dxdy \\
&= \int_0^1 \arcsin \frac{x}{\sqrt{1 - y^2}}\Big|_{x=-\sqrt{y-y^2}}^{x=\sqrt{y-y^2}} dy = 2\int_0^1 \arcsin\frac{\sqrt{y}}{\sqrt{1 + y}}\, dy\,;
\end{aligned}$$

thus making a change of variable $y = \tan^2\theta$ we conclude that

$$\begin{aligned}\text{the surface area of } \Sigma &= 2\int_0^{\frac{\pi}{4}} \arcsin\frac{\tan\theta}{\sec\theta}\,d(\tan^2\theta) = 2\int_0^{\frac{\pi}{4}} \theta\, d\big(\tan^2\theta\big)\\ &= 2\Big[\theta\tan^2\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^2\theta d\theta\Big]\\ &= 2\Big[\frac{\pi}{4} - \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1)\,d\theta\Big] = 2\Big[\frac{\pi}{4} - (\tan\theta - \theta)\Big|_{\theta=0}^{\theta=\frac{\pi}{4}}\Big]\\ &= 2\Big[\frac{\pi}{4} - \Big(1 - \frac{\pi}{4}\Big)\Big] = \pi - 2\,.\end{aligned}$$

4.3.3 The surface element and the surface integral

Let $\Sigma \subseteq \mathbb{R}^3$ be a regular surface, and $\{\mathcal{V}, \psi\}$ be a parametrization of Σ such that $\psi(\mathcal{V}) = \Sigma$. If $f : \Sigma \to \mathbb{R}$ is a bounded continuous function, the surface integral of f over Σ, denoted by $\int_\Sigma f\,dS$, is defined by

$$\int_\Sigma f\,dS = \int_{\mathcal{V}} (f\circ\psi)\sqrt{\mathrm{g}}\,d\mathbb{A}\,. \tag{4.12}$$

In particular, if $f \equiv 1$, the number $\int_\Sigma dS \equiv \int_\Sigma 1\,dS$ is the surface area of Σ.

Since the surface integrals defined by (4.12) seems to depend on a given parametrization, before proceeding we show that the surface integral is indeed independent of the choice of the parameterizations. Suppose that $\{\mathcal{V}_1, \psi_1\}$ and $\{\mathcal{V}_2, \psi_2\}$ are two local $\mathscr{C}^1$-parameterizations of a regular surface Σ at p, g_1, g_2 denote the metric tensors associated with the parameterizations $\{\mathcal{V}_1, \psi_1\}$, $\{\mathcal{V}_2, \psi_2\}$, respectively, and $\mathrm{g}_1 = \det(g_1)$, $\mathrm{g}_2 = \det(g_2)$ are corresponding first fundamental forms. Let $\Psi = \psi_2^{-1}\circ\psi_1$. Then the change of variables formula (Theorem 3.31) implies that

$$\int_{\mathcal{V}_2} (f\circ\psi_2)\sqrt{\mathrm{g}_2}\,d\mathbb{A} = \int_{\mathcal{V}_1} (f\circ\psi_2\circ\Psi)\big(\sqrt{\mathrm{g}_2}\circ\Psi\big)|J_\Psi|\,d\mathbb{A} = \int_{\mathcal{V}_1} (f\circ\psi_1)\big(\sqrt{\mathrm{g}_2}\circ\Psi\big)|J_\Psi|\,d\mathbb{A}\,,$$

where J_Ψ is the Jacobian of the map Ψ. Using (4.7), we find that

$$\big[D\Psi\big]^{\mathrm{T}}\big[(D\psi_2)\circ\Psi\big]^{\mathrm{T}}\big[(D\psi_2)\circ\Psi\big]\big[D\Psi\big] = \big[D\psi_1\big]^{\mathrm{T}}\big[D\psi_1\big]\,;$$

thus by the fact that $\mathrm{g}_1 = \det\big([D\psi_1]^{\mathrm{T}}[D\psi_1]\big)$ and $\mathrm{g}_2 = \det\big([D\psi_2]^{\mathrm{T}}[D\psi_2]\big)$, we obtain that

$$\det\big([D\Psi]\big)^2(\mathrm{g}_2\circ\Psi) = \mathrm{g}_1\,.$$

Since $J_\Psi = \det\big([D\Psi]\big)$, the identity above implies that $|J_\Psi|(\sqrt{\mathrm{g}_2}\circ\Psi) = \sqrt{\mathrm{g}_1}$, so we conclude that

$$\int_{\mathcal{V}_1} (f\circ\psi_1)\sqrt{\mathrm{g}_1}\, d\mathbb{A} = \int_{\mathcal{V}_2} (f\circ\psi_2)\sqrt{\mathrm{g}_2}\, d\mathbb{A}\,. \tag{4.13}$$

Therefore, the surface integral of f over Σ is independent of the choice of parameterizations of Σ. In particular, the surface area of a regular $\mathscr{C}^1$-surface which can be parameterized by a global parametrization is also independent of the choice of parameterizations.

As noticed in Remark 4.45, the first fundamental form $\sqrt{\mathrm{g}}$ associated with the parametrization $\{\mathcal{V},\psi\}$ can be viewed as the Jacobian of the map ψ. Therefore, we arrive at the conclusion that dS "$=$" $\sqrt{\mathrm{g}}\, d\mathbb{A}$. dS is called the ***surface element***. Moreover, similar to the reason provided in Remark 4.22, the surface integral of a positive continuous function f over Σ, where f is considered as the mass density of the surface given by

$$f(x) = \lim_{\substack{\mathrm{diam}(\Delta)\to 0\\ \psi^{-1}(x)\in\Delta}} \frac{\text{the mass of } \psi(\Delta)}{\text{the surface area of } \psi(\Delta)}$$

is the total mass of the surface.

Next, we study the surface area of general regular surfaces that cannot be parameterized using a single pair $\{\mathcal{V},\psi\}$. Let $\Sigma\subseteq\mathbb{R}^3$ be a regular surface, and $\{\mathcal{V}_i,\psi_i\}_{i\in\mathcal{I}}$ be a collection of local parameterizations satisfying that for each $p\in\Sigma$ there exists $i\in\mathcal{I}$ such that $\{\mathcal{V}_i,\psi_i\}$ is a local parametrization of Σ at p. If there exists a countable collection of non-negative functions $\{\zeta_j\}_{j\in\mathcal{J}}$ defined on Σ such that

1. For each $j\in\mathcal{J}$, $\mathrm{spt}(\zeta_j)\equiv$ the closure of $\big\{x\in\Sigma\,\big|\,\zeta_j(x)\neq 0\big\}\subseteq\mathcal{V}_i$ for some $i\in\mathcal{I}$;
2. $\sum_{j\in\mathcal{J}}\zeta_j(x) = 1$ for all $x\in\Sigma$,

then intuitively we can compute the surface area by

$$\int_\Sigma dS = \sum_{j\in\mathcal{J}}\int_\Sigma \zeta_j\, dS\,, \tag{4.14}$$

where the surface integral of ζ_j over Σ is defined by (4.12) since $\mathrm{spt}(\zeta_j)\subseteq\psi(\mathcal{V}_i)$ and $\zeta_j = 0$ outside $\mathrm{spt}(\zeta_j)$. In other words, each term on the right-hand side of (4.14) can be evaluated by

$$\int_\Sigma \zeta_j\, dS = \int_{\mathcal{V}_i} (\zeta_j\circ\psi_i)\sqrt{\mathrm{g}_i}\, dS\,.$$

if $\operatorname{spt}(\zeta_j) \subseteq \psi_i(\mathcal{V}_i)$. Similarly, for a bounded continuous function f defined on Σ, the surface integral of f over Σ can be defined by

$$\int_\Sigma f\,dS = \sum_{j\in\mathcal{J}} \int_\Sigma (\zeta_j f)\,dS = \sum_{j\in\mathcal{J}} \sum_{\substack{\text{choose one } i \text{ such that} \\ \operatorname{spt}(\zeta_j)\subseteq \psi_i(\mathcal{V}_i)}} \int_{\mathcal{V}_i} (\zeta_j f)\circ\psi_i \sqrt{\mathrm{g}_i}\,dS\,. \tag{4.15}$$

Remark 4.52. Defining the surface integrals of a function as above, a question arises naturally: is the surface integral given by (4.15) independent of the choice of the parametrization and the partition-of-unity? In other words, if a regular $\mathscr{C}^k$-surface Σ admits two collections of local parametrization $\{\mathcal{U}_i, \varphi_i\}_{i\in\mathcal{I}}$ and $\{\mathcal{V}_j, \psi_j\}_{j\in\mathcal{J}}$, and $\{\zeta_i\}_{i\in\mathcal{I}}$ and $\{\lambda_j\}_{j\in\mathcal{J}}$ are $\mathscr{C}^k$-partition-of-unity subordinate to $\{\mathcal{U}_i\}_{i\in\mathcal{I}}$ and $\{\mathcal{V}_j\}_{j\in\mathcal{J}}$, respectively. Is it true that

$$\sum_{i\in\mathcal{I}} \sum_{\substack{\text{choose one } i \text{ such that} \\ \operatorname{spt}(\zeta_j)\subseteq \varphi_i(\mathcal{U}_i)}} \int_{\mathcal{U}_i} (\zeta_i f)\circ\varphi_i \sqrt{\mathrm{g}_i}\,dS = \sum_{j\in\mathcal{J}} \sum_{\substack{\text{choose one } j \text{ such that} \\ \operatorname{spt}(\lambda_k)\subseteq \psi_j(\mathcal{V}_j)}} \int_{\mathcal{V}_j} (\lambda_j f)\circ\psi_j \sqrt{g_j}\,dS\,,$$

where g_i and g_j are the first fundamental form associated with the parametrization $\{\mathcal{U}_i, \varphi_i\}$ and $\{\mathcal{V}_j, \psi_j\}$, respectively.

The answer to the question above is affirmative, and the surface integral given by (4.15) is indeed independent of the choice of parametrization of the surface and the partition-of-unity; however, we will not prove this and only treat this as a known fact.

Now we focus on the existence of a collection of functions $\{\zeta_j\}_{j\in\mathcal{J}}$ discussed above.

Definition 4.53. A collection of subsets of $\mathbb{R}^n$ is said to be ***locally finite*** if for every point $x \in \mathbb{R}^n$ there exists $r > 0$ such that $B(x,r)$, the ball centered at x with radius r, intersects at most finitely many sets in this collection.

Definition 4.54 (Partition of Unity)**.** Let $A \subseteq \mathbb{R}^n$ be a subset. A collection of functions $\{\zeta_j\}_{j\in\mathcal{J}}$ is said to be a ***partition-of-unity*** of A if

1. $0 \leqslant \zeta_j \leqslant 1$ for all $j \in \mathcal{J}$.
2. The collection of sets $\{\operatorname{spt}(\zeta_j)\}_{j\in\mathcal{J}}$ is locally finite.
3. $\sum\limits_{j\in\mathcal{J}} \zeta_j(x) = 1$ for all $x \in A$.

Let $\{\mathcal{U}_j\}_{j\in\mathcal{J}}$ be an open cover of A; that is, $\mathcal{U}_j$ is open for all $j \in \mathcal{J}$ and $A \subseteq \bigcup_{j\in\mathcal{J}} \mathcal{U}_j$. A partition-of-unity $\{\zeta_j\}_{j\in\mathcal{J}}$ of A is said to be ***subordinate*** to $\{\mathcal{U}_j\}_{j\in\mathcal{J}}$ (or $\{\mathcal{U}_j\}_{j\in\mathcal{J}}$ has a subordinate partition-of-unity of A) if $\operatorname{spt}(\zeta_j) \subseteq \mathcal{U}_j$ for all $j \in \mathcal{J}$.

We note the if $\{\zeta_j\}_{j\in\mathcal{J}}$ is a partition-of-unity of A, then the property of local finiteness of $\{\text{spt}(\zeta_j)\}_{j\in\mathcal{J}}$ ensures that for each point $x \in A$ has a neighborhood on which all but finitely many λ_j's are zero.

Lemma 4.55. *Let $A \subseteq \mathbb{R}^n$ be a subset, $\{\mathcal{U}_i\}_{i\in\mathcal{I}}$ be an open cover of A, and $\{\mathcal{V}_j\}_{j\in\mathcal{J}}$ be a collection of open sets such that each $\mathcal{V}_j$ is a subset of some $\mathcal{U}_i$; that is, for each $j \in \mathcal{J}$, $\mathcal{V}_j \subseteq \mathcal{U}_i$ for some $i \in \mathcal{I}$. If $\{\mathcal{V}_j\}_{j\in\mathcal{J}}$ has a subordinate $\mathscr{C}^k$-partition-of-unity of A, so has $\{\mathcal{U}_i\}_{i\in\mathcal{I}}$.*

Proof. Let $\{\zeta_j\}_{j\in\mathcal{J}}$ be a partition-of-unity of A subordinate to $\{\mathcal{V}_j\}_{j\in\mathcal{J}}$, and $f : \mathcal{J} \to \mathcal{I}$ be a map such that $\mathcal{V}_j \subseteq \mathcal{U}_{f(j)}$ (we note that such f in general is not unique). Define $\chi_i : \mathbb{R}^n \to [0,1]$ by

$$\chi_i(x) = \sum_{j\in f^{-1}(i)} \zeta_j(x)\,. \tag{4.16}$$

Then clearly $\text{spt}(\chi_i) \subseteq \mathcal{U}_i$ and $\sum\limits_{i\in\mathcal{I}} \chi_i(x) = 1$ for all $x \in A$. Moreover, since the sum (4.16) is a finite sum, χ_i is of class $\mathscr{C}^k$ for all $i \in \mathcal{I}$ since ζ_j if of class $\mathscr{C}^k$ for all $j \in \mathcal{J}$. Now we show that $\{\text{spt}(\chi_i)\}_{i\in\mathcal{I}}$ is locally finite. Let $x \in \mathbb{R}^n$ be given. By the local finiteness of $\{\text{spt}(\zeta_j)\}_{j\in\mathcal{J}}$ there exists $r > 0$ such that $\#\{j \in \mathcal{J} \,|\, B(x,r) \cap \text{spt}(\zeta_j) \neq \varnothing\} < \infty$. By the fact that $f^{-1}(i_1) \cap f^{-1}(i_2) = \varnothing$ if $i_1 \neq i_2$ (that is, each $j \in \mathcal{J}$ belongs to $f^{-1}(i)$ for exactly one $i \in \mathcal{I}$) and that

$$y \in B(x,r) \cap \text{spt}(\chi_i) \quad \Leftrightarrow \quad y \in B(x,r) \cap \text{spt}(\zeta_j) \text{ for some } j \in f^{-1}(i)\,,$$

we must have

$$\#\{i \in \mathcal{I} \,|\, B(x,r) \cap \text{spt}(\chi_i) \neq \varnothing\} \leqslant \#\{j \in \mathcal{J} \,|\, B(x,r) \cap \text{spt}(\zeta_j) \neq \varnothing\} < \infty\,. \qquad \square$$

Theorem 4.56. *Let $\Sigma \subseteq \mathbb{R}^3$ be a regular $\mathscr{C}^k$-surface. Then every open cover of Σ has a subordinate $\mathscr{C}^k$-partition-of-unity of Σ.*

Proof. Let $\{\mathcal{O}_i\}_{i\in\mathcal{I}}$ be a given open cover of Σ. Let $\{\mathcal{U}_j, \varphi_j\}_{j\in\mathcal{J}}$ be a collection of $\mathscr{C}^k$-charts of Σ such that $\{\mathcal{U}_j\}_{j\in\mathcal{J}}$ is a locally finite open cover of Σ and for each $j \in \mathcal{J}$, $\overline{\mathcal{U}}_j \subseteq \mathcal{O}_i$ for some $i \in \mathcal{I}$. By Lemma 4.55, it suffices to find a $\mathscr{C}^k$-partition-of-unity of Σ subordinate to $\{\mathcal{U}_j\}_{j\in\mathcal{J}}$.

W.L.O.G., we can assume that $\mathcal{U}_j$ and $\mathcal{V}_j \equiv \varphi(\mathcal{U}_j)$ is bounded for all $j \in \mathcal{J}$. Define $\psi_j = \varphi_j^{-1}$. Then $\{\mathcal{V}_j, \psi_j\}_{j\in\mathcal{J}}$ is a collection of local parametrization of Σ. Choose a collection of open sets $\{\mathcal{W}_j\}_{j\in\mathcal{J}}$ such that $\overline{\mathcal{W}}_j \subseteq \mathcal{V}_j$ for all $j \in \mathcal{J}$ and $\{\psi_j(\mathcal{W}_j)\}_{j\in\mathcal{J}}$ is still an open cover of Σ. For each $j \in \mathcal{J}$, let $\{B_k^{(j)}\}_{k=1}^{N_j}$ be a collection of open balls satisfying $\overline{\mathcal{W}}_j \subseteq \bigcup\limits_{k=1}^{N_j} B_k^{(j)}$ and $\mathrm{cl}(B_k^{(j)}) \subseteq \mathcal{V}_j$ for all $k \in \{1, \cdots, N_j\}$. For $j \in \mathcal{J}$ and $k \in \{1, \cdots, N_j\}$, with $c_{j,k}$ and $r_{j,k}$ denoting the center and the radius of $B_k^{(j)}$, respectively, let

$$\mu_{(j,k)}(x) = \begin{cases} \exp\Big(\dfrac{1}{\|x - c_{j,k}\|_{\mathbb{R}^2}^2 - r_{j,k}^2}\Big) & \text{if } x \in B_k^{(j)}\,, \\ 0 & \text{if } x \notin B_k^{(j)}\,, \end{cases}$$

and then define $\chi_j : \mathbb{R}^2 \to \mathbb{R}$ by $\chi_j(x) = \sum\limits_{k=1}^{N_j} \mu_{(j,k)}(x)$. Then $\chi_j > 0$ in $\overline{\mathcal{W}}_j$, and $\chi_j = 0$ outside $\bigcup\limits_{k=1}^{N_j} B_k^{(j)}$. Further define

$$\lambda_j(x) = \begin{cases} (\chi_j \circ \varphi_j)(x) & \text{if } x \in \mathcal{U}_j\,, \\ 0 & \text{if } x \in \mathcal{U}_j^\complement\,. \end{cases}$$

Then $\lambda_j > 0$ on $\psi_j(W_j)$ which implies that $\sum\limits_{j \in \mathcal{J}} \lambda_j > 0$. Finally, we define $\zeta_j = \dfrac{\lambda_j}{\sum_{j \in \mathcal{J}} \lambda_j}$. Then $\{\zeta_j\}_{j \in \mathcal{J}}$ is a $\mathscr{C}^k$-partition-of-unity subordinate to the open cover $\{\mathcal{U}_j\}_{j \in \mathcal{J}}$. □

Definition 4.57 (Piecewise Regular Surface)**.** A surface $\Sigma \subseteq \mathbb{R}^3$ is said to be piecewise regular if there are finite many curves $C_1, \cdots, C_k$ such that $\Sigma \backslash \bigcup\limits_{i=1}^{k} C_i$ is a disjoint union of regular surfaces.

Definition 4.58. Let $\Sigma \subseteq \mathbb{R}^3$ be a piecewise regular surface such that Σ is the disjoint union of regular surfaces Σ_i, where $i \in \mathcal{I}$ for some finite index set $\mathcal{I}$. For a continuous function $f : \Sigma \to \mathbb{R}$, the surface integral of f over Σ, still denoted by $\int_\Sigma f\,dS$, is defined by

$$\int_\Sigma f\,dS = \sum_{i \in \mathcal{I}} \int_{\Sigma_i} f\,dS\,.$$

Definition 4.59. Let $\mathscr{R}_\Sigma$ be the collection of piecewise regular surfaces in $\mathbb{R}^3$. The surface element is a set function $\mathscr{S} : \mathscr{R}_\Sigma \to \mathbb{R}$ that satisfies the following properties:

1. $\mathscr{S}(\Sigma) > 0$ for all $\Sigma \in \mathscr{R}_\Sigma$.

2. If Σ is the union of finitely many regular surfaces $\Sigma_1, \cdots, \Sigma_k$ that do not overlap except at their boundaries, then
$$\mathscr{S}(\Sigma) = \mathscr{S}(\Sigma_1) + \cdots + \mathscr{S}(\Sigma_k)\,.$$

3. The value of $\mathscr{S}$ agrees with the area on planar surfaces; that is,
$$\mathscr{S}(\mathcal{P}) = \mathbb{A}(\mathcal{P}) \qquad \text{for all planar surfaces } \mathcal{P}.$$

4.4 Oriented Surfaces

In the study of surfaces, orientability is a property that measures whether it is possible to make a consistent choice of surface normal vector at every point. A choice of surface normal allows one to use the right-hand rule to define a "counter-clockwise" direction of loops in the surface that is required in the presentation of the Stokes theorem (Theorem 4.86), a main result in vector calculus which will be introduced later in Section 4.7.2.

Definition 4.60. A regular surface $\Sigma \subseteq \mathbb{R}^3$ is said to be ***oriented*** if there exists a continuous vector-valued function $\mathbf{N} : \Sigma \to \mathbb{R}^3$ such that $\|\mathbf{N}\|_{\mathbb{R}^3} = 1$ and for all $p \in \Sigma$, $\mathbf{N}(p) \cdot \boldsymbol{v} = 0$ for all $\boldsymbol{v} \in \mathbf{T}_p\Sigma$. Such a vector-field $\mathbf{N}$ is called a ***unit normal*** of Σ.

Suppose that $\Sigma \subseteq \mathbb{R}^3$ is a connected regular surface. Since at each $p \in \Sigma$ the tangent plane $\mathbf{T}_p\Sigma$ of Σ at p has two normal directions, Σ has at most two continuous unit normal vector fields. If in addition that Σ is oriented, there are exactly two continuous unit normal vector fields of Σ, and one is the opposite of the other. The two unit normal vector fields define two ***sides*** of the surface.

Suppose further that this oriented surface Σ is the boundary of an open set $\Omega \subseteq \mathbb{R}^3$ (for example, a sphere is the boundary of a ball), then one of the unit normal vector fields $\mathbf{N} : \partial\Omega \to \mathbb{R}^3$ has the property that $p + t\mathbf{N}(p) \notin \Omega$ for all small but positive t. Such a normal is called the ***outward-pointing unit normal*** of $\partial\Omega$, and the opposite of the outward-pointing unit normal of $\partial\Omega$ is called the ***inward-pointing unit normal*** of $\partial\Omega$.

Example 4.61. Consider the unit sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$. Then $\mathbf{N} : \mathbb{S}^2 \to \mathbb{R}^3$ defined by $\mathbf{N}(p) = p$, where the right-hand side is treated as the vector $p - 0$, is a continuous unit normal vector field on Σ; thus $\mathbb{S}^2$ is an oriented surface. Let $B(0, 1) = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 < 1\}$ be the unit ball in $\mathbb{R}^3$. Then $\mathbf{N}$ is the outward-pointing unit normal of $\partial B(0, 1)$.

Let $\Sigma \subseteq \mathbb{R}^3$ be a regular surface, $p \in \Sigma$, and $\{\mathcal{V}, \psi\}$ be a local parametrization of Σ at p. Since $\psi_{,1}$ and $\psi_{,2}$ are linearly independent, $\psi_{,1} \times \psi_{,2} \neq \mathbf{0}$; thus the vector $\boldsymbol{n}$ given by

$$\boldsymbol{n} = \frac{\psi_{,1} \times \psi_{,2}}{\|\psi_{,1} \times \psi_{,2}\|_{\mathbb{R}^3}} \circ \psi^{-1}$$

is a unit normal vector field on $\psi(\mathcal{V})$. As a consequence, a regular $\mathscr{C}^1$-surface that can be parameterized by one single parametrization $\{\mathcal{V}, \psi\}$; that is, $\Sigma = \psi(\mathcal{V})$, is always oriented.

Such a normal vector fields is said to be compatible with the parametrization $\{\mathcal{V}, \psi\}$. To be more precise, we have the following

Definition 4.62. Let $\Sigma \subseteq \mathbb{R}^3$ be an oriented $\mathscr{C}^1$-surface, and $\mathbf{N} : \Sigma \to \mathbb{R}^3$ be a continuous unit normal vector field of Σ. For each $p \in \mathcal{V}$, $\mathbf{N}$ is said to be compatible with a local parametrization $\{\mathcal{V}, \psi\}$ of Σ at p if $\det\big([\psi_{,1} \,\vdots\, \psi_{,2} \,\vdots\, \mathbf{N} \circ \psi]\big) > 0$.

The following example provides a famous regular surface which is not oriented.

Example 4.63. A Möbius strip/band is a surface obtained, conceptually, by half-twisting a paper strip and then joining the ends of the strip together to form a loop (see the following figure for the idea).

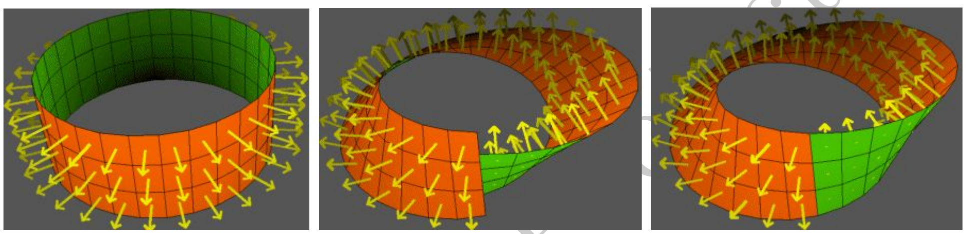


Figure 4.1: Normal vector fields on a Möbius strip

As one can see from Figure 4.1, a Möbius strip is not oriented. To see this mathematically, consider the following Möbius strip

$$\mathcal{M} = \Big\{\big(-(2 + v\cos\frac{u}{2})\sin u, (2 + v\cos\frac{u}{2})\cos u, v\sin\frac{u}{2}\big) \,\Big|\, (u, v) \in [0, 2\pi] \times (-1, 1)\Big\}$$

and choose a local parametrization $\psi : \mathcal{V} \to \mathbb{R}^3$ given by

$$\psi(u, v) = \big(-(2 + v\cos\frac{u}{2})\sin u, (2 + v\cos\frac{u}{2})\cos u, v\sin\frac{u}{2}\big)\,,$$

where $(u, v) \in \mathcal{V} \equiv (0, 2\pi) \times (-1, 1)$.

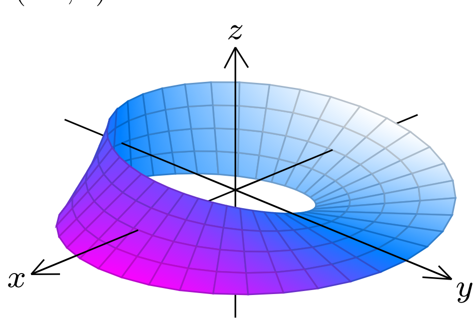


Figure 4.2: The Möbius strip/band $\psi([0, 2\pi] \times [-1, 1])$

Then the unit normal vector field on $\psi(\mathcal{V})$ compatible with the parametrization $\{\mathcal{V}, \psi\}$ is

$$\begin{aligned}(\mathbf{N}\circ\psi)(u,v) = \frac{\psi_{,1}\times\psi_{,2}}{\|\psi_{,1}\times\psi_{,2}\|_{\mathbb{R}^3}} &= \frac{2}{\sqrt{v^2+(4+2v\cos(u/2))^2}}\times \\ &\times\Big(\frac{v}{2}\cos u + (2+v\sin\frac{u}{2})\sin\frac{u}{2}\sin u, \\ &\quad -\frac{v}{2}\sin u + (2+v\cos\frac{u}{2})\sin\frac{u}{2}\cos u, -(2+v\cos\frac{u}{2})\cos\frac{u}{2}\Big),\end{aligned}$$

but $\mathbf{N}$ does not have a continuous extension on $\mathcal{M}$ since if $\widetilde{\mathbf{N}}$ is a continuous extension of $\mathbf{N}$; that is, $\widetilde{\mathbf{N}}$ is a unit normal vector field on $\mathcal{M}$ and $\mathbf{N} = \widetilde{\mathbf{N}}$ on $\psi(\mathcal{V})$, then

$$(0,0,-1) = \lim_{u\to 0^+}(\mathbf{N}\circ\psi)(u,0) = \widetilde{\mathbf{N}}(2,0,0) = \lim_{u\to 2\pi^-}(\mathbf{N}\circ\psi)(u,0) = (0,0,1)$$

which is a contradiction.

Another way of seeing that $\mathcal{M}$ is not oriented is the following. Let $r(t) = G(t,0) = (-2\sin t, 2\cos t, 0)$, and $C = r([0,2\pi]) \subseteq \mathcal{M}$ be a closed curve on $\mathcal{M}$. If there is a continuous unit normal vector field $\widetilde{\mathbf{N}}$ on $\mathcal{M}$, then $\widetilde{\mathbf{N}}$ is also continuous on C. However, $\widetilde{\mathbf{N}}$ is never continuous on C since by moving $\mathbf{N}$ continuously along C, starting from $r(0)$ and moving along C in the direction r' and back to $r(0) = r(2\pi)$, we obtain a different vector which implies that $\widetilde{\mathbf{N}}\circ r$ is not continuous at $r(0) = r(2\pi) = (2,0,0)$.

Definition 4.64. An open set $\Omega \subseteq \mathbb{R}^3$ is said to be of class $\mathscr{C}^k$ if the boundary $\partial\Omega$ is a regular $\mathscr{C}^k$-surface.

Theorem 4.65. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set of class $\mathscr{C}^1$. Then $\partial\Omega$ is oriented.*

4.5 Manifolds, Charts, Atlas and Differentiable Structure

In the following, we introduce a more abstract concept, the so-called manifolds, which is a generalization of regular surfaces.

Definition 4.66. A topological space $\mathcal{M}$ is called an n-dimensional ***manifold*** if it is locally homeomorphic to $\mathbb{R}^n$; that is, there is an open cover $\mathscr{U} = \{\mathcal{U}_i\}_{i\in\mathcal{I}}$ of $\mathcal{M}$ such that for each $i \in \mathcal{I}$ there is a map $\varphi_i : \mathcal{U}_i \to \mathbb{R}^n$ which maps $\mathcal{U}_i$ homeomorphically onto an open subset of $\mathbb{R}^n$. The pair $\{\mathcal{U}_i, \varphi_i\}$ is called a ***chart*** (or coordinate system) with domain $\mathcal{U}_i$, and

$\{\varphi_i(\mathcal{U}_i), \varphi_i^{-1}\}$ is called a local parametrization of $\mathcal{M}$. The collection of charts $\Phi = \{\mathcal{U}_i, \varphi_i\}_{i\in\mathcal{I}}$ is called an ***atlas***.

Two charts $\{\mathcal{U}_i, \varphi_i\}$ and $\{\mathcal{U}_j, \varphi_j\}$ are said to be ***$\mathscr{C}^r$-compatible*** or ***have $\mathscr{C}^r$-overlap*** if the coordinate change

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(\mathcal{U}_i \cap \mathcal{U}_j) \to \varphi_j(\mathcal{U}_i \cap \mathcal{U}_j)$$

is of class $\mathscr{C}^r$. An atlas Φ on $\mathcal{M}$ is called $\mathscr{C}^r$ if every pair of its charts is $\mathscr{C}^r$-compatible. A maximal $\mathscr{C}^r$-atlas α on $\mathcal{M}$ is called a ***differentiable structure***, and the pair $\{M, \alpha\}$ is called a manifold of class $\mathscr{C}^r$.

A function $f : \mathcal{M} \to \mathbb{R}$ is said to be of class $\mathscr{C}^r$ if $f \circ \varphi_i^{-1} : \mathcal{U}_i \to \mathbb{R}$ is of class $\mathscr{C}^r$ for all charts $\{\mathcal{U}_i, \varphi_i\}$.

In particular, a regular $\mathscr{C}^1$-curve $C \subseteq \mathbb{R}^3$ is a one-dimensional $\mathscr{C}^1$-manifold, and a regular $\mathscr{C}^1$-surface $\Sigma \subseteq \mathbb{R}^3$ is a two-dimensional $\mathscr{C}^1$-manifold.

Definition 4.67 (Metric)**.** Let $\Sigma \subseteq \mathbb{R}^{\mathrm{n}}$ be a $(n-1)$-dimensional manifold. The metric tensor associated with the local parametrization $\{\mathcal{V}, \psi\}$ (at $p \in \Sigma$) is the matrix $g = [g_{\alpha\beta}]_{(\mathrm{n}-1)\times(n-1)}$ given by

$$g_{\alpha\beta} = \psi_{,\alpha} \cdot \psi_{,\beta} = \sum_{i=1}^{\mathrm{n}} \frac{\partial \psi^i}{\partial y_\alpha} \frac{\partial \psi^i}{\partial y_\beta} \qquad \text{in} \quad \mathcal{V}\,.$$

Proposition 4.68. *Let $\Sigma \subseteq \mathbb{R}^{\mathrm{n}}$ be a $(n-1)$-dimensional manifold, and $g = [g_{\alpha\beta}]_{(\mathrm{n}-1)\times(\mathrm{n}-1)}$ be the metric tensor associated with the local parametrization $\{\mathcal{V}, \psi\}$ (at $p \in \Sigma$). Then the metric tensor g is positive definite; that is,*

$$\sum_{\alpha,\beta=1}^{\mathrm{n}-1} g_{\alpha\beta} v^\alpha v^\beta > 0 \qquad \forall\, \boldsymbol{v} = \sum_{\gamma=1}^{\mathrm{n}-1} v^\gamma \frac{\partial \psi}{\partial y^\gamma} \neq \mathbf{0}\,.$$

Definition 4.69 (The first fundamental form)**.** Let $\Sigma \subseteq \mathbb{R}^{\mathrm{n}}$ be a $(n-1)$-dimensional manifold, and $g = [g_{\alpha\beta}]_{(\mathrm{n}-1)\times(\mathrm{n}-1)}$ be the metric tensor associated with the local parametrization $\{\mathcal{V}, \psi\}$ (at $p \in \Sigma$). The first fundamental form associated with the local parametrization $\{\mathcal{V}, \psi\}$ (at $p \in \Sigma$) is the scalar function $\mathrm{g} = \det(g)$.

Definition 4.70 (Surface integrals)**.** Let $\mathcal{M}$ be an $(\mathrm{n}-1)$-dimensional $\mathscr{C}^1$-manifold, $\{\mathcal{U}_i\}_{i\in\mathcal{I}}$ be a collection of charts of $\mathcal{M}$ and $\{\zeta_i\}_{i\in\mathcal{I}}$ is a partition-of-unity of $\mathcal{M}$ subordinate to $\{\mathcal{U}_i\}_{i\in\mathcal{I}}$. The "surface integral" (or simply integral) of a scalar function $f : \mathcal{M} \to \mathbb{R}$ over $\mathcal{M}$, denoted by $\int_{\mathcal{M}} f\, dS$, is defined by

$$\int_{\mathcal{M}} f\, dS = \sum_{i\in I} \int_{\varphi_i(\mathcal{U}_i)} \big[(\zeta_i f) \circ \varphi^{-1}\big] \sqrt{\mathrm{g}_i}\, dx\,,$$

where g_i is the first fundamental form associated with the parametrization $\{\varphi_i(\mathcal{U}_i), \varphi^{-1}\}$.

Remark 4.71. Let $C \subseteq \mathbb{R}^3$ be a regular $\mathscr{C}^1$-curve. The line integral of a scalar function $f : C \to \mathbb{R}$ over C is the "surface integral" of f over C defined in (4.70). In other words, $dS = ds$ in the case that $\mathcal{M}$ is a one-dimensional manifold.

4.5.1 Some useful identities

Let $\Sigma \subseteq \mathbb{R}^n$ be the boundary of an open set Ω (thus an oriented surface), $\{\mathcal{V}, \psi\}$ be a local parametrization of Σ, and $\mathbf{N} : \Sigma \to \mathbb{R}^n$ be the normal vector on Σ which is compatible with the parametrization ψ; that is,

$$\det\left(\left[\,\psi_{,1} \,\vdots\, \psi_{,2} \,\vdots\, \cdots \,\vdots\, \psi_{,n-1} \,\vdots\, \mathbf{N} \circ \psi\,\right]\right) > 0\,.$$

Define $\Psi(y', y_n) = \psi(y') + y_n(\mathbf{N} \circ \psi)(y')$. Then $\Psi : \mathcal{V} \times (-\varepsilon, \varepsilon) \to \mathcal{T}$ for some tubular neighborhood $\mathcal{T}$ of Σ.

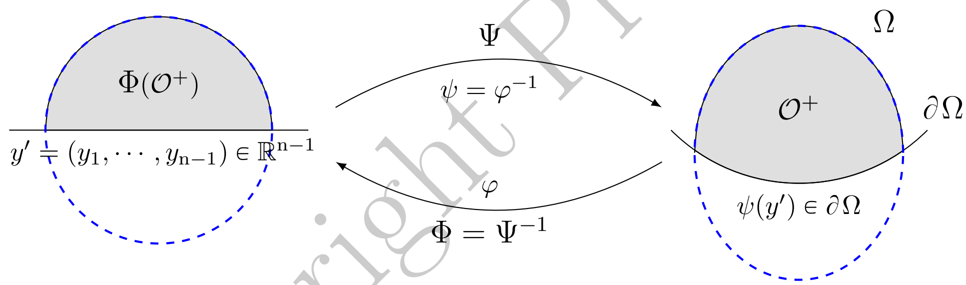


Figure 4.3: The map Ψ constructed from the local parametrization $\{\mathcal{V}, \psi\}$

Since $(\nabla\Psi)\big|_{\{y_n=0\}} = \left[\,\psi_{,1} \,\vdots\, \psi_{,2} \,\vdots\, \cdots \,\vdots\, \psi_{,n-1} \,\vdots\, \mathbf{N}\circ\psi\,\right]$, Corollary 1.65 and 1.66 implies that

$$\begin{aligned}\det(\nabla\Psi)^2\big|_{\{y_n=0\}} &= \left[\det\left((\nabla\Psi)^{\mathrm{T}}\right)\det(\nabla\Psi)\right]\Big|_{\{y_n=0\}} = \det\left((\nabla\Psi)^{\mathrm{T}}\nabla\Psi\right)\Big|_{\{y_n=0\}} \\ &= \det\left(\begin{bmatrix} g_{11} & g_{12} & \cdots & g_{(n-1)1} & 0 \\ g_{21} & g_{22} & \cdots & g_{(n-1)2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{(n-1)1} & g_{(n-1)2} & \cdots & g_{(n-1)(n-1)} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}\right) = g\,.\end{aligned}$$

Defining J as the Jacobian of the map Ψ; that is, $J = \det(\nabla\Psi)$, then the identity above implies that

$$J = \sqrt{g} \qquad \text{on} \quad \{y_n = 0\}\,.$$

Moreover, letting A denote the inverse of the Jacobian matrix of Ψ; that is, $\mathrm{A} = (\nabla\Psi)^{-1}$, and letting $\left[g^{\alpha\beta}\right]_{(\mathrm{n}-1)\times(\mathrm{n}-1)}$ be the inverse matrix of $\left[g_{\alpha\beta}\right]_{(\mathrm{n}-1)\times(\mathrm{n}-1)}$, we find that

$$\mathrm{A}\big|_{\{y_n=0\}} = \left[\sum_{\alpha=1}^{\mathrm{n}-1} g^{1\alpha}\psi_{,\alpha} \,\vdots\, \cdots\cdots\cdots\cdots \,\vdots\, \sum_{\alpha=1}^{\mathrm{n}-1} g^{(\mathrm{n}-1)\alpha}\psi_{,\alpha} \,\vdots\, \mathbf{N}\circ\psi \right]^{\mathrm{T}} .$$

As a consequence,

$$(\mathrm{J}\mathrm{A}^{\mathrm{T}}\mathrm{e}_{\mathrm{n}})\big|_{\{y_{\mathrm{n}}=0\}} = \sqrt{\mathrm{g}}\,(\mathbf{N}\circ\psi)\,. \tag{4.17}$$

4.6 The Divergence Theorem

Two differential operators play important roles in vector calculus. The first one is called the **divergence operator** which measures the flux of a vector field, and the second one is called the **curl operator** which measures the circulation (the speed of rotation) of a vector field. We will study this two operators in the following two sections.

4.6.1 Flux integrals

Let $\Sigma \subseteq \mathbb{R}^3$ be an oriented surface with a fixed unit normal vector field $\mathbf{N} : \Sigma \to \mathbb{R}^3$, and $\boldsymbol{u} : \Sigma \to \mathbb{R}^3$ be a vector-valued function. The flux integral of $\boldsymbol{u}$ over Σ with given orientation $\mathbf{N}$ is the surface integral of $\boldsymbol{u}\cdot\mathbf{N}$ over Σ.

Physical interpretation

Let $\Omega \subseteq \mathbb{R}^3$ be an open set which stands for a fluid container and fully contains some liquid such as water, and $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ be a vector-field which stands for the fluid velocity; that is, $\boldsymbol{u}(x)$ is the fluid velocity at point $x \in \Omega$. Furthermore, let $\Sigma \subseteq \Omega$ be a surface immersed in the fluid with given orientation $\mathbf{N}$, and $c : \Omega \to \mathbb{R}$ be the concentration of certain material dissolving in the liquid. Then the amount of the material carried across the surface in the direction $\mathbf{N}$ by the fluid in a time period of Δt is

$$\Delta t \cdot \int_\Sigma c\boldsymbol{u}\cdot\mathbf{N}\,dS\,.$$

Therefore, $\int_\Sigma c\boldsymbol{u}\cdot\mathbf{N}\,dS$ is the instantaneous amount of the material carried across the surface in the direction $\mathbf{N}$ by the fluid.

Example 4.72. Find the flux integral of the vector field $\boldsymbol{F}(x, y, z) = (x, y^2, z)$ upward through the first octant part Σ of the cylindrical surface $x^2 + z^2 = a^2$, $0 < y < b$.

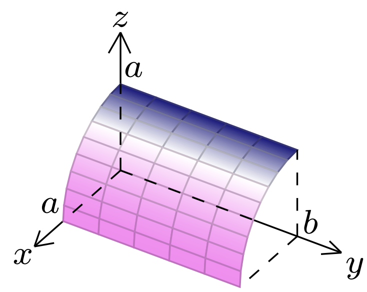


Figure 4.4: The surface Σ

Fist, we parameterize Σ by

$$\psi(u, v) = (u, v, \sqrt{a^2 - u^2}), \quad (u, v) \in \mathcal{V} = (0, a) \times (0, b)\,.$$

Since the first fundamental form g associated with $\{\mathcal{V}, \psi\}$ is $\mathrm{g} = \|\psi_{,1} \times \psi_{,2}\|^2_{\mathbb{R}^3} = \dfrac{a^2}{a^2 - u^2}$, and the upward-pointing unit normal is $\mathbf{N}(x, y, z) = (\dfrac{x}{a}, 0, \dfrac{z}{a})$, we have

$$\begin{aligned}\int_\Sigma \boldsymbol{F} \cdot \mathbf{N}\, dS &= \int_{\mathcal{V}} \frac{1}{a}(u^2 + a^2 - u^2)\frac{a}{\sqrt{a^2 - u^2}}\, d(u, v) = a^2 \int_{\mathcal{V}} \frac{1}{\sqrt{a^2 - u^2}}\, d(u, v)\\ &= a^2 \int_0^b \int_0^a \frac{1}{\sqrt{a^2 - u^2}}\, du dv = a^2 b \arcsin \frac{u}{a}\Big|_{u=0}^{u=a} = \frac{\pi a^2 b}{2}\,.\end{aligned}$$

4.6.2 Measurements of the flux - the divergence operator

Let $\Omega \subseteq \mathbb{R}^3$ be an open set, and $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ be a $\mathscr{C}^1$ vector field. Suppose that $\mathcal{O}$ is a bounded open set of class $\mathscr{C}^1$ such that $\overline{\mathcal{O}} \subseteq \Omega$ with outward-pointing unit normal vector field $\mathbf{N}$. Then the flux integral of $\boldsymbol{u}$ over $\partial\mathcal{O}$ in the direction $\mathbf{N}$ is

$$\int_{\partial\mathcal{O}} \boldsymbol{u} \cdot \mathbf{N}\, dS\,.$$

Consider a special case that $\mathcal{O} = B(a, r)$ for some ball in $\mathbb{R}^3$ centered at a with radius r. We first compute $\displaystyle\int_{\partial B(a,r)} \boldsymbol{u}^3 \mathbf{N}_3\, dS$. Consider

$$\begin{aligned}\psi_+(x_1, x_2) &= \big(x_1, x_2, a_3 + \sqrt{r^2 - (x_1 - a)^2 - (x_2 - a_2)^2}\big)\,, \qquad (x_1, x_2) \in D(a, r)\,,\\ \psi_-(x_2, x_2) &= \big(x_1, x_2, a_3 - \sqrt{r^2 - (x_1 - a)^2 - (x_2 - a_2)^2}\big)\,, \qquad (x_1, x_2) \in D(a, r)\,,\end{aligned}$$

where $D(a,r)$ is the disk in $\mathbb{R}^2$ given by $\{(x_1,x_2)\in\mathbb{R}^2 \,|\, (x_1-a_1)^2+(x_2-a_2)^2 \leqslant r^2\}$. Since $\partial B(a,r)\backslash(\psi_+(D(a,r))\cup\psi_-(D(a,r))$ is the equator of the sphere $\partial B(a,r)$ which has zero area, we must have

$$\int_{\partial B(a,r)} \boldsymbol{u}^3\mathbf{N}_3\,dS = \int_{\psi_+(D(a,r))} \boldsymbol{u}^3\mathbf{N}_3\,dS + \int_{\psi_-(D(a,r))} \boldsymbol{u}^3\mathbf{N}_3\,dS\,.$$

Note that $(\mathbf{N}\circ\psi_\pm)(x_1,x_2) = \dfrac{1}{r}\big(\psi_\pm(x_1,x_2)-a\big)$. In view of Example 4.49, we have

$$\begin{aligned}
&\int_{\psi_+(D(a,r))} \boldsymbol{u}^3\mathbf{N}_3\,dS \\
&\quad = \int_{D(a,r)} \boldsymbol{u}^3(\psi_+(x_1,x_2))\frac{\sqrt{r^2-(x_1-a_1)^2-(x_2-a_2)^2}}{r}\frac{r}{\sqrt{r^2-(x_1-a_1)^2-(x_2-a_2)^2}}\,d\mathbb{A} \\
&\quad = \int_{D(a,r)} \boldsymbol{u}^3(\psi_+(x_1,x_2))\,d\mathbb{A}\,.
\end{aligned}$$

and similarly,

$$\int_{\psi_+(D(a,r))} \boldsymbol{u}^3\mathbf{N}_3\,dS = -\int_{D(a,r)} \boldsymbol{u}^3(\psi_-(x_1,x_2))\,d\mathbb{A}\,.$$

Therefore,

$$\begin{aligned}
\int_{\partial B(a,r)} \boldsymbol{u}^3\mathbf{N}_3\,dS &= \int_{D(a,r)} \big[\boldsymbol{u}^3(\psi_+(x_1,x_2)) - \boldsymbol{u}^3(\psi_-(x_1,x_2))\big]\,d\mathbb{A} \\
&= \int_{D(a,r)} \Big(\int_{a_3-\sqrt{r^2-(x_1-a_1)^2-(x_2-a_2)^2}}^{a_3+\sqrt{r^2-(x_1-a_1)^2-(x_2-a_2)^2}} \frac{\partial \boldsymbol{u}^3}{\partial x_3}(x_1,x_2,x_3)\,dx_3\Big)d\,\mathbb{A} \\
&= \int_{B(a,r)} \frac{\partial \boldsymbol{u}^3}{\partial x_3}\,dx\,.
\end{aligned}$$

Similarly,

$$\int_{\partial B(a,r)} \boldsymbol{u}^1\mathbf{N}_1\,dS = \int_{B(a,r)} \frac{\partial u^1}{\partial x_1}\,dx \quad \text{and} \quad \int_{\partial B(a,r)} \boldsymbol{u}^2\mathbf{N}_2\,dS = \int_{B(a,r)} \frac{\partial u^2}{\partial x_2}\,dx\,;$$

thus we conclude that

$$\int_{\partial B(a,r)} \boldsymbol{u}\cdot\mathbf{N}\,dS = \int_{B(a,r)} \sum_{i=1}^{3} \frac{\partial \boldsymbol{u}^i}{\partial x_i}\,dx\,.$$

The computation above motivates the following

Definition 4.73 (The divergence operator)**.** Let $\boldsymbol{u}:\Omega\subseteq\mathbb{R}^{\mathrm{n}}\to\mathbb{R}^{\mathrm{n}}$ be a vector field. The divergence of $\boldsymbol{u}$ is a scalar function defined by

$$\mathrm{div}\boldsymbol{u} = \sum_{i=1}^{\mathrm{n}} \frac{\partial \boldsymbol{u}^i}{\partial x_i}\,.$$

Definition 4.74. A vector field $\boldsymbol{u} : \Omega \subseteq \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{n}}$ is called *solenoidal* or *divergence-free* if $\mathrm{div}\boldsymbol{u} = 0$ in Ω.

4.6.3 The divergence theorem

Theorem 4.75 (The divergence theorem)**.** *Let $\Omega \subseteq \mathbb{R}^{\mathrm{n}}$ be a bounded Lipschitz domain, and $\boldsymbol{v} \in \mathscr{C}^1(\Omega) \cap \mathscr{C}(\overline{\Omega})$. Then*

$$\int_\Omega \mathrm{div}\boldsymbol{v}\, dx = \int_{\partial\Omega} \boldsymbol{v} \cdot \mathbf{N}\, dS\,,$$

where $\mathbf{N}$ is the outward-pointing unit normal of Ω.

Proof. To embrace the beauty of geometry (and the context that we have introduced), we prove the case that Ω is a bounded open set of class $\mathscr{C}^3$.

Let $\{\mathcal{U}_m\}_{m=1}^K$ be an open cover of $\partial\Omega$ such that for each $m \in \{1, \cdots, K\}$ there exists a $\mathscr{C}^3$-parametrization $\psi_m : \mathcal{V}_m \subseteq \mathbb{R}^{\mathrm{n}-1} \to \mathcal{U}_m$ which is compatible with the orientation $\mathbf{N}$; that is,

$$\det\left([\psi_{m,1} \vdots \cdots \vdots \psi_{m,\mathrm{n}-1} \vdots \mathbf{N} \circ \psi_m]\right) > 0 \qquad \text{on} \quad \mathcal{V}_m\,.$$

Define $\vartheta_m(y', y_{\mathrm{n}}) = \psi_m(y') + y_{\mathrm{n}}(\mathbf{N} \circ \psi_m)(y')$ as in Section 4.5.1. Then there exists $\varepsilon_m > 0$ such that $\vartheta_m : \mathcal{V}_m \times (-\varepsilon_m, \varepsilon_m) \to \mathcal{W}_m$ is a $\mathscr{C}^2$-diffeomorphism for some open set in $\mathbb{R}^{\mathrm{n}}$ such that $\vartheta_m : \mathcal{V}_m \times (-\varepsilon_m, 0) \to \Omega \cap \mathcal{W}_m$ while $\vartheta_m : \mathcal{V}_m \times (0, \varepsilon_m) \to \mathrm{int}(\Omega^\complement) \cap \mathcal{W}_m$.

Choose an open set $\mathcal{W}_0 \subseteq \mathbb{R}^{\mathrm{n}}$ such that $\overline{\mathcal{W}_0} \subseteq \Omega$ and $\overline{\Omega} \subseteq \bigcup_{m=0}^{K} \mathcal{W}_m$, and define ϑ_0 as the identity map. Let $0 \leqslant \zeta_m \leqslant 1$ in $\mathscr{C}_c^\infty(\mathcal{U}_m)$ denote a partition-of-unity of $\overline{\Omega}$ subordinate to the open covering $\{\mathcal{W}_m\}_{m=0}^K$; that is,

$$\sum_{m=0}^{K} \zeta_m = 1 \quad \text{and} \quad \mathrm{spt}(\zeta_m) \subseteq \mathcal{U}_m \quad \forall\, m\,.$$

Let $\mathrm{J}_m = \det(\nabla\vartheta_m)$, $\mathrm{A}_m = (\nabla\vartheta_m)^{-1}$, and g_m denote the first fundamental form associated with $\{\mathcal{V}_m, \psi_m\}$. Using (4.17), $\sqrt{\mathrm{g}_m}(\mathbf{N} \circ \vartheta_m) = \mathrm{J}_m(\mathrm{A}_m)^{\mathrm{T}}\mathrm{e}_{\mathrm{n}}$ on $\mathcal{V}_m \times \{0\}$ for $m \in \{1, \cdots, K\}$.

Therefore, making change of variable $x = \vartheta_m(y)$ in each $\mathcal{W}_m$ we find that

$$\begin{aligned}
\int_{\partial\Omega} \boldsymbol{v}\cdot\mathbf{N}\,dS &= \sum_{m=1}^{K} \int_{\partial\Omega\cap\mathcal{W}_m} \zeta_m(\boldsymbol{v}\cdot\mathbf{N})\,dS \\
&= \sum_{m=1}^{K}\sum_{i=1}^{\mathrm{n}} \int_{\mathcal{V}_m\times\{y_{\mathrm{n}}=0\}} (\zeta_m\circ\vartheta_m)(\boldsymbol{v}^i\circ\vartheta_m)(\mathbf{N}^i\circ\vartheta_m)\sqrt{\mathrm{g}_m}\,dy' \\
&= \sum_{m=1}^{K}\sum_{i=1}^{\mathrm{n}} \int_{\mathcal{V}_m\times\{y_{\mathrm{n}}=0\}} (\zeta_m\circ\vartheta_m)(\boldsymbol{v}^i\circ\vartheta_m)\mathrm{J}_m(\mathrm{A}_m)^{\mathrm{n}}_i\,dy' \\
&= \sum_{m=1}^{K}\sum_{i=1}^{\mathrm{n}} \int_{\mathcal{V}_m\times(-\varepsilon_m,0)} \frac{\partial}{\partial y_{\mathrm{n}}}\big[(\zeta_m\circ\vartheta_m)\mathrm{J}_m(\mathrm{A}_m)^{\mathrm{n}}_i(\boldsymbol{v}^i\circ\vartheta_m)\big]\,dy\,.
\end{aligned}$$

On the other hand, for $\alpha\in\{1,\cdots,\mathrm{n}-1\}$ and $i\in\{1,\cdots,\mathrm{n}\}$,

$$\int_{\mathcal{V}_m\times(-\varepsilon_m,0)} \frac{\partial}{\partial y_\alpha}\big[(\zeta_m\circ\vartheta_m)\mathrm{J}_m(\mathrm{A}_m)^{\alpha}_i(\boldsymbol{v}^i\circ\vartheta_m)\big]\,dy = 0\,;$$

thus the Piola identity (2.6) implies that

$$\begin{aligned}
\int_{\partial\Omega} \boldsymbol{v}\cdot\mathbf{N}\,dS &= \sum_{m=1}^{K}\sum_{i,j=1}^{\mathrm{n}} \int_{\mathcal{V}_m\times(-\varepsilon_m,0)} \frac{\partial}{\partial y_j}\big[(\zeta_m\circ\vartheta_m)\mathrm{J}_m(\mathrm{A}_m)^{j}_i(\boldsymbol{v}^i\circ\vartheta_m)\big]\,dy \\
&= \sum_{m=1}^{K}\sum_{i,j=1}^{\mathrm{n}} \int_{\mathcal{V}_m\times(-\varepsilon_m,0)} \mathrm{J}_m(\mathrm{A}_m)^{j}_i(\zeta_m\circ\vartheta_m)_{,j}\,(\boldsymbol{v}^i\circ\vartheta_m)\,dy \\
&\quad + \sum_{m=1}^{K}\sum_{i,j=1}^{\mathrm{n}} \int_{\mathcal{V}_m\times(-\varepsilon_m,0)} (\zeta_m\circ\vartheta_m)\mathrm{J}_m(\mathrm{A}_m)^{j}_i(\boldsymbol{v}^i\circ\vartheta_m)_{,j}\;dy\,.
\end{aligned}$$

Making change of variable $y = \vartheta_m^{-1}(x)$ in each $\mathcal{V}_m\times(-\varepsilon_m,0)$ again, by the fact that

$$\sum_{i,j=1}^{\mathrm{n}} (\mathrm{A}_m)^j_i(\boldsymbol{v}^i\circ\theta_m)_{,j} = (\mathrm{div}\boldsymbol{v})\circ\theta_m \qquad\text{and}\qquad \int_{\mathcal{W}_0} \mathrm{div}\big(\zeta_0\boldsymbol{v}\big)\,dx = 0\,,$$

we conclude that

$$\begin{aligned}
\int_{\partial\Omega} \boldsymbol{v}\cdot\mathbf{N}\,dS &= \int_{\mathcal{W}_0} \mathrm{div}\big(\zeta_0\boldsymbol{v}\big)\,dx + \sum_{m=1}^{K}\int_{\mathcal{W}_m} (\boldsymbol{v}\cdot\nabla_x)\zeta_m\,dx + \sum_{m=1}^{K}\int_{\mathcal{W}_m} \zeta_m\mathrm{div}\boldsymbol{v}\,dx \\
&= \sum_{m=0}^{K}\int_{\mathcal{W}_m} (\boldsymbol{v}\cdot\nabla_x)\zeta_m\,dx + \sum_{m=0}^{K}\int_{\mathcal{W}_m} \zeta_m\mathrm{div}\boldsymbol{v}\,dx \\
&= \int_{\Omega} (\boldsymbol{v}\cdot\nabla_x)1\,dx + \int_{\Omega}\mathrm{div}\boldsymbol{v}\,dx = \int_{\Omega}\mathrm{div}\boldsymbol{v}\,dx\,.
\end{aligned}$$

□

Letting $\boldsymbol{v} = (0,\cdots,0,f,0,\cdots,0) = f\mathrm{e}_i$, we obtain the following

Corollary 4.76. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain, and $f \in \mathscr{C}^1(\Omega) \cap \mathscr{C}(\overline{\Omega})$. Then*

$$\int_\Omega \frac{\partial f}{\partial x_i}\, dx = \int_{\partial\Omega} f\, \mathbf{N}_i\, dS\,,$$

where $\mathbf{N}_i$ is the i-th component of the outward-pointing unit normal $\mathbf{N}$ of Ω.

Letting $\boldsymbol{v}$ be the product of a scalar function and a vector-valued function in Theorem 4.75, we conclude the following

Corollary 4.77. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain, and $\boldsymbol{v} \in \mathscr{C}^1(\Omega; \mathbb{R}^n) \cap \mathscr{C}(\overline{\Omega}; \mathbb{R}^n)$ be a vector-valued function and $\varphi \in \mathscr{C}^1(\Omega) \cap \mathscr{C}(\overline{\Omega})$ be a scalar function. Then*

$$\int_\Omega \varphi\, \mathrm{div}\boldsymbol{v}\, dx = \int_{\partial\Omega} (\boldsymbol{v}\cdot\mathbf{N})\varphi\, dS - \int_\Omega \boldsymbol{v}\cdot\nabla\varphi\, dx\,, \tag{4.18}$$

where $\mathbf{N}$ is the outward-pointing unit normal on $\partial\Omega$.

Example 4.78. Let Ω be the the first octant part bounded by the cylindrical surface $x^2 + z^2 = a^2$ and the plane $y = b$, and $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ be a vector-valued function defined by $\boldsymbol{F}(x, y, z) = (x, y^2, z)$.

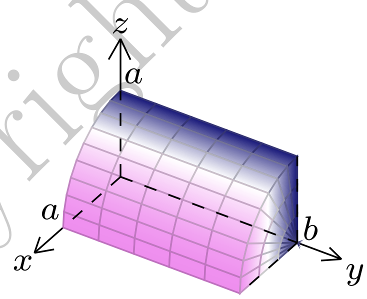


Figure 4.5: The domain Ω and its five pieces of boundaries

With $\mathbf{N}$ denoting the outward-pointing unit normal of $\partial\Omega$,

$$\begin{aligned}\int_\Omega \mathrm{div}\boldsymbol{F}\, d(x,y,z) &= \int_0^a \int_0^b \int_0^{\sqrt{a^2-x^2}} (2+2y)\, dzdydx = (b^2+2b)\int_0^a \int_0^{\sqrt{a^2-x^2}} dzdx \\ &= \frac{\pi a^2(b^2+2b)}{4}\,.\end{aligned}$$

On the other hand, we note that the boundary of Ω has five parts: Σ as given in Example 4.72, two rectangles $\mathrm{R}_1 = \{x = 0\} \times [0, b] \times [0, a]$, $\mathrm{R}_2 = [0, a] \times [0, b] \times \{z = 0\}$, and two quarter disc $\mathrm{D}_1 = \big\{(x,0,z) \in \mathbb{R}^3 \,\big|\, x^2+z^2 \leqslant a^2, x,z \geqslant 0\big\}$ and $\mathrm{D}_2 = \big\{(x,b,z) \in \mathbb{R}^3 \,\big|\, x^2+z^2 \leqslant a^2, x,z \geqslant 0\big\}$. Therefore,

$$\begin{aligned}
\int_{\mathrm{R}_1} \boldsymbol{F}\cdot\mathbf{N}\,dS &= \int_0^a\int_0^b (0,y^2,z)\cdot(-1,0,0)\,dydz = 0\,,\\
\int_{\mathrm{R}_2} \boldsymbol{F}\cdot\mathbf{N}\,dS &= \int_0^a\int_0^b (x,y^2,0)\cdot(0,0,-1)\,dydx = 0\,,\\
\int_{\mathrm{D}_1} \boldsymbol{F}\cdot\mathbf{N}\,dS &= \int_0^a\int_0^{\sqrt{a^2-x^2}} (x,0,z)\cdot(0,-1,0)\,dzdx = 0\,,
\end{aligned}$$

and

$$\int_{\mathrm{D}_1} \boldsymbol{F}\cdot\mathbf{N}\,dS = \int_0^a\int_0^{\sqrt{a^2-x^2}} (x,b^2,z)\cdot(0,1,0)\,dzdx = b^2\int_0^a\int_0^{\sqrt{a^2-x^2}} dzdx = \frac{\pi a^2 b^2}{4}\,.$$

Together with the result in Example 4.72, we find that

$$\begin{aligned}
\int_{\partial\Omega} \boldsymbol{F}\cdot\mathbf{N}\,dS &= \Big(\int_\Sigma + \int_{\mathrm{R}_1} + \int_{\mathrm{R}_2} + \int_{\mathrm{D}_1} + \int_{\mathrm{D}_2}\Big)\boldsymbol{F}\cdot\mathbf{N}\,dS = \frac{\pi a^2 b^2}{4} + \frac{\pi a^2 b}{2} = \frac{\pi a^2(b^2+2b)}{4}\\
&= \int_\Omega \mathrm{div}\boldsymbol{F}\,d(x,y,z)\,.
\end{aligned}$$

4.6.4 The divergence theorem on surfaces with boundary

This section is devoted to the divergence theorem on surfaces in $\mathbb{R}^3$ instead of domains of $\mathbb{R}^n$. To do so, we need to define what the divergence operator on a surface is, and this requires that we first define the vector fields on which the surface divergence operator acts.

Definition 4.79. Let $\Sigma \subseteq \mathbb{R}^3$ be an open $\mathscr{C}^1$-surface; that is, Σ is of class $\mathscr{C}^1$ and $\Sigma \cap \partial\Sigma = \varnothing$. A vector field $\boldsymbol{u}$ defined on Σ is called a tangent vector field on Σ, denoted by $\boldsymbol{u} \in \mathbf{T}\Sigma$, if $\boldsymbol{u}\cdot\mathbf{N} = 0$ on Σ, where $\mathbf{N} : \Sigma \to \mathbb{S}^2$ is a unit normal vector field on Σ.

Having established (4.18), we find that the divergence operator div is the formal adjoint of the operator $-\nabla$. The following definition is motivated by this observation.

Definition 4.80 (The surface gradient and the surface divergence)**.** Let $\Sigma \subseteq \mathbb{R}^n$ be a regular $\mathscr{C}^1$-surface. The surface gradient of a function $f : \Sigma \to \mathbb{R}$, denoted by $\nabla_\Sigma f$, is a vector-valued function from Σ to $\mathbf{T}_p\Sigma$ given, in a local parametrization $\{\mathcal{V}, \psi\}$, by

$$(\nabla_\Sigma f)\circ\psi = \sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta}\frac{\partial(f\circ\psi)}{\partial y_\alpha}\frac{\partial\psi}{\partial y_\beta}\,,$$

where $[g^{\alpha\beta}]$ is the inverse matrix of the metric tensor $[g_{\alpha\beta}]$ associated with $\{\mathcal{V},\psi\}$, and $\left\{\dfrac{\partial\psi}{\partial y_\beta}\right\}_{\beta=1}^{\rm n}$ are tangent vectors to Σ.

The surface divergence operator div_Σ is defined as the formal adjoint of $-\nabla_\Sigma$; that is, if $\boldsymbol{u}\in\mathbf{T}\Sigma$, then

$$-\int_\Sigma \boldsymbol{u}\cdot\nabla_\Sigma f\,dS = \int_\Sigma f\,\mathrm{div}_\Sigma\boldsymbol{u}\,dS \qquad \forall\, f\in\mathscr{C}^1_c(\Sigma;\mathbb{R})\,.$$

In a local parametrization $(\mathcal{V},\psi)$,

$$(\mathrm{div}_\Sigma\boldsymbol{u})\circ\psi = \frac{1}{\sqrt{\rm g}}\sum_{\alpha,\beta=1}^{\rm n-1}\frac{\partial}{\partial y_\alpha}\Big[\sqrt{\rm g}g^{\alpha\beta}\big((\boldsymbol{u}\circ\psi)\cdot\frac{\partial\psi}{\partial y_\beta}\big)\Big],$$

where ${\rm g}=\det(g)$ is the first fundamental form associated with $\{\mathcal{V},\psi\}$.

Remark 4.81. Suppose that $f:\mathcal{O}\subseteq\mathbb{R}^3\to\mathbb{R}$ for some open set containing Σ. Then the surface gradient of f at $p\in\Sigma$ is the projection of the gradient vector $(\nabla f)(p)$ onto the tangent plane $T_p\Sigma$. In other words, let $\mathbf{N}:\Sigma\to\mathbb{R}^3$ be a continuous unit normal vector field on Σ, then

$$(\nabla_\Sigma f)(p) = (\nabla f)(p) - \big[(\nabla f)(p)\cdot\mathbf{N}(p)\big]\mathbf{N}(p) \quad \text{(or simply } \nabla_\Sigma f = \nabla f - (\nabla f\cdot\mathbf{N})\mathbf{N}\text{)}\,.$$

Definition 4.82 (Surfaces with Boundary)**.** An oriented $\mathscr{C}^k$-surface $\Sigma\subseteq\mathbb{R}^3$ is said to have $\mathscr{C}^\ell$-boundary $\partial\Sigma$ if there exists a collection of pairs $\{\mathcal{V}_m,\psi_m\}_{m=1}^K$, called a collection of local parametrization of $\overline{\Sigma}$, if

1. $\mathcal{V}_m\subseteq\mathbb{R}^2$ is open and $\psi_m:\mathcal{V}_m\to\mathbb{R}^3$ is one-to-one map of class $\mathscr{C}^k$ for all $m\in\{1,\cdots,K\}$;

2. $\psi_m(\mathcal{V}_m)\cap\Sigma\neq\varnothing$ for all $m\in\{1,\cdots,K\}$ and $\overline{\Sigma}\subseteq\bigcup_{m=1}^K\psi_m(\mathcal{V}_m)$;

3. $\psi_m:\mathcal{V}_m\to\psi_m(\mathcal{V}_m)$ is a $\mathscr{C}^k$-diffeomorphism if $\psi_m(\mathcal{V}_m)\subseteq\Sigma$;

4. $\psi_m:\mathcal{V}_m^+\equiv\mathcal{V}_m\cap\{y_2>0\}\to\psi_m(\mathcal{V}_m)\cap\Sigma$ is a $\mathscr{C}^k$-diffeomorphism if $\mathcal{U}_m\cap\partial\Sigma\neq\varnothing$;

5. $\psi_m:\mathcal{V}_m\cap\{y_2=0\}\to\mathcal{U}_m\cap\partial\Sigma$ is of class $\mathscr{C}^\ell$ if $\mathcal{U}_m\cap\partial\Sigma\neq\varnothing$.

Now we are in the position of stating the divergence theorem on surfaces with boundary.

Theorem 4.83. *Let $\Sigma \subseteq \mathbb{R}^3$ be an oriented $\mathscr{C}^1$-surface with $\mathscr{C}^1$-boundary $\partial\Sigma$, $\mathbf{N} : \Sigma \to \mathbb{S}^2$ be a continuous unit normal vector field on Σ, and $\mathbf{T} : \partial\Sigma \to \mathbb{S}^2$ be tangent vector on $\partial\Sigma$ such that $\mathbf{T}$ is compatible with $\mathbf{N}$ (which means $\mathbf{T} \times \mathbf{N}$ points away from Σ). Then*

$$\int_{\partial\Sigma} \boldsymbol{u} \cdot (\mathbf{T} \times \mathbf{N})\, ds = \int_\Sigma \text{div}_\Sigma \boldsymbol{u}\, dS \qquad \forall\, \boldsymbol{u} \in \mathbf{T}\Sigma \cap \mathscr{C}^1(\Sigma; \mathbb{R}^3) \cap \mathscr{C}(\overline{\Sigma}; \mathbb{R}^3)\,,$$

where div_Σ *is the surface divergence operator.*

Proof. Let $\{\mathcal{V}_m, \psi_m\}_{m=1}^K$ denote a collection of local parametrization of $\overline{\Sigma}$ such that $\psi_m(\mathcal{V}_m) \cap \partial\Sigma = \varnothing$ for $1 \leqslant m \leqslant J$, and $\psi_m(\mathcal{V}_m) \cap \partial\Sigma$ is non-empty and connected for $J+1 \leqslant m \leqslant K$. W.L.O.G., we can assume that $\mathcal{V}_m = B_m \equiv B(0, r_m)$ for some $r_m > 0$. Write $\mathcal{U}_m = \psi_m(\mathcal{V}_m)$, and let $\{g_m\}_{m=1}^K$ be the associated metric tensor, as well as the associated first fundamental form $\mathrm{g}_m = \det(g_m)$. Let $\{\zeta_m\}_{m=1}^K$ be a partition-of-unity of $\overline{\Sigma}$ subordinate to $\{\mathcal{U}_m\}_{m=1}^K$. Then

$$\begin{aligned}\int_\Sigma \text{div}_\Sigma \boldsymbol{u}\, dS &= \sum_{m=1}^K \int_{\mathcal{U}_m \cap \Sigma} \zeta_m \text{div}_\Sigma \boldsymbol{u}\, dS \\ &= \sum_{m=1}^J \sum_{\alpha,\beta=1}^2 \int_{B_m} (\zeta_m \circ \psi_m) \frac{\partial}{\partial y_\alpha}\Big[\sqrt{\mathrm{g}_m} g_m^{\alpha\beta}\Big((\boldsymbol{u} \circ \psi_m) \cdot \frac{\partial \psi_m}{\partial y_\beta}\Big)\Big] dy \\ &\quad + \sum_{m=J+1}^K \sum_{\alpha,\beta=1}^2 \int_{B_m^+} (\zeta_m \circ \psi_m) \frac{\partial}{\partial y_\alpha}\Big[\sqrt{\mathrm{g}_m} g_m^{\alpha\beta}\Big((\boldsymbol{u} \circ \psi_m) \cdot \frac{\partial \psi_m}{\partial y_\beta}\Big)\Big] dy\,.\end{aligned}$$

Let $\boldsymbol{n}$ denote the outward-pointing unit normal on either ∂B_m for $1 \leqslant m \leqslant J$ or ∂B_m^+ for $J+1 \leqslant m \leqslant K$. Since $\zeta_m \circ \vartheta_m = 0$ on $\partial B(0, r_m)$ for $1 \leqslant m \leqslant J$, and $\zeta_m \circ \vartheta_m = 0$ on $\{y_2 > 0\} \cap \partial B(0, r_m)$ for $J+1 \leqslant m \leqslant K$, the divergence theorem (on $\mathbb{R}^2$) implies that

$$\begin{aligned}\int_\Sigma \text{div}_\Sigma \boldsymbol{u}\, dS &= -\sum_{m=1}^K \sum_{\alpha,\beta=1}^2 \int_{\psi_m^{-1}(\mathcal{U}_m \cap \Sigma)} \Big[\sqrt{\mathrm{g}_m} g_m^{\alpha\beta}\Big((\boldsymbol{u} \circ \psi_m) \cdot \frac{\partial \psi_m}{\partial y_\beta}\Big)\Big] \frac{\partial}{\partial y_\alpha}(\zeta_m \circ \psi_m)\, dy \\ &\quad + \sum_{m=J+1}^K \sum_{\alpha,\beta=1}^2 \int_{B_m \cap \{y_2 = 0\}} (\zeta_m \circ \psi_m) \boldsymbol{n}_\alpha \Big[\sqrt{\mathrm{g}_m} g_m^{\alpha\beta}\Big((\boldsymbol{u} \circ \psi_m) \cdot \frac{\partial \psi_m}{\partial y_\beta}\Big)\Big]\, dy_1 \\ &= -\sum_{m=1}^K \int_{\psi_m^{-1}(\mathcal{U}_m \cap \Sigma)} (\boldsymbol{u} \cdot \nabla_\Sigma \zeta_m) \circ \psi_m \sqrt{\mathrm{g}_m}\, dy \\ &\quad + \sum_{m=J+1}^K \int_{B_m \cap \{y_2 = 0\}} (\zeta_m \circ \psi_m)(\boldsymbol{u} \circ \psi_m) \cdot \Big[\sum_{\alpha,\beta=1}^2 \boldsymbol{n}_\alpha \sqrt{\mathrm{g}_m} g_m^{\alpha\beta} \frac{\partial \psi_m}{\partial y_\beta}\Big]\, dy_1\,.\end{aligned}$$

Since

$$\sum_{m=1}^{K}\int_{\psi_m^{-1}(\mathcal{U}_m\cap\Sigma)}(\boldsymbol{u}\cdot\nabla_\Sigma\zeta_m)\circ\psi_m\sqrt{\mathrm{g}_m}\,dy=\sum_{m=1}^{K}\int_{\mathcal{U}_m\cap\Sigma}(\boldsymbol{u}\cdot\nabla_\Sigma\zeta_m)\,dS=\int_\Sigma(\boldsymbol{u}\cdot\nabla\sum_{m=1}^{K}\zeta_m)\,dS=0\,,$$

we conclude that

$$\int_\Sigma \mathrm{div}_\Sigma\boldsymbol{u}\,dS=\sum_{m=J+1}^{K}\int_{B_m\cap\{y_2=0\}}(\zeta_m\circ\psi_m)(\boldsymbol{u}\circ\psi_m)\cdot\Big[\sum_{\alpha,\beta=1}^{2}\boldsymbol{n}_\alpha\sqrt{\mathrm{g}_m}g_m^{\alpha\beta}\frac{\partial\psi_m}{\partial y_\beta}\Big]\,dy_1\,.$$

On the other hand,

$$\begin{aligned}\int_{\partial\Sigma}\boldsymbol{u}\cdot(\mathbf{T}\times\mathbf{N})\,ds&=\sum_{m=J+1}^{K}\int_{\partial\Sigma\cap\mathcal{U}_m}\zeta_m\boldsymbol{u}\cdot(\mathbf{T}\times\mathbf{N})\,ds\\&=\sum_{m=J+1}^{K}\int_{B_m\cap\{y_2=0\}}(\zeta_m\circ\psi_m)(\boldsymbol{u}\circ\psi_m)\cdot\Big[(\mathbf{T}\times\mathbf{N})\circ\psi_m\Big|\frac{\partial\psi_m}{\partial y_1}\Big|\Big]\,dy_1\,.\end{aligned}$$

Therefore, the theorem can be concluded as long as we can show that

$$\sum_{\alpha,\beta=1}^{2}\boldsymbol{n}_\alpha\sqrt{\mathrm{g}_m}\,g_m^{\alpha\beta}\,\frac{\partial\psi_m}{\partial y_\beta}=(\mathbf{T}\times\mathbf{N})\circ\psi_m\Big|\frac{\partial\psi_m}{\partial y_1}\Big|\qquad\text{on}\quad B_m\cap\{y_2=0\}\,.\tag{4.19}$$

Let $\boldsymbol{\tau}_m=\sum\limits_{\alpha,\beta=1}^{2}\boldsymbol{n}_\alpha\sqrt{\mathrm{g}_m}g_m^{\alpha\beta}\dfrac{\partial\psi_m}{\partial y_\beta}$ on $B_m\cap\{y_2=0\}$. Since $\boldsymbol{n}_\alpha=-\delta_{2\alpha}$, we find that $\boldsymbol{\tau}_m\cdot\dfrac{\partial\psi_m}{\partial y_1}=0$ on $B_m\cap\{y_2=0\}$; thus

$$\boldsymbol{\tau}_m\cdot(\mathbf{T}\circ\psi_m)=0\qquad\text{on}\quad B_m\cap\{y_2=0\}\,.$$

Moreover, noting that $\boldsymbol{\tau}_m$ is a linear combination of tangent vectors $\dfrac{\partial\psi_m}{\partial y_\beta}$, we must have

$$\boldsymbol{\tau}_m\cdot(\mathbf{N}\circ\psi_m)=0\qquad\text{on}\quad B_m\cap\{y_2=0\}\,.$$

As a consequence,

$$\boldsymbol{\tau}_m\,/\!\!/\,(\mathbf{T}\times\mathbf{N})\circ\psi_m\qquad\text{on}\quad B_m\cap\{y_2=0\}\,.$$

Since $(\mathbf{T}\times\mathbf{N})$ points away from Σ, while $\dfrac{\partial\psi_m}{\partial y_2}\circ\psi_m^{-1}\Big|_{\partial\Sigma}$ points toward Σ, by the fact that

$$\boldsymbol{\tau}_m\cdot\frac{\partial\psi_m}{\partial y_2}=\sum_{\alpha,\beta=1}^{2}\boldsymbol{n}_\alpha\sqrt{\mathrm{g}_m}g_m^{\alpha\beta}\frac{\partial\psi_m}{\partial y_\beta}\cdot\frac{\partial\psi_m}{\partial y_2}=-\sqrt{\mathrm{g}_m}g_m^{22}<0\,,$$

we must have $\boldsymbol{\tau}_m \cdot (\mathbf{T} \times \mathbf{N}) \circ \psi_m > 0$ on $B_m \cap \{y_2 = 0\}$. In other words,

$$\boldsymbol{\tau}_m = |\boldsymbol{\tau}_m|(\mathbf{T} \times \mathbf{N}) \circ \psi_m \qquad \text{on} \quad B_m \cap \{y_2 = 0\}\,.$$

Finally, since

$$\boldsymbol{\tau}_m \cdot \boldsymbol{\tau}_m = \sum_{\alpha,\beta,\gamma,\delta=1}^{2} \mathrm{g}_m\, \boldsymbol{n}_\alpha\, \boldsymbol{n}_\gamma\, g_m^{\alpha\beta} g_m^{\gamma\delta} \frac{\partial \psi_m}{\partial y_\beta} \cdot \frac{\partial \psi_m}{\partial y_\delta} = \mathrm{g}_m g_m^{22} = g_{m11} = \Big|\frac{\partial \psi_m}{\partial y_1}\Big|^2,$$

we conclude that $\boldsymbol{\tau}_m = \Big|\dfrac{\partial \psi_m}{\partial y_1}\Big|(\mathbf{T} \times \mathbf{N}) \circ \psi_m$ on $\{y_2 = 0\}$; thus (4.19) is established. ▫

Remark 4.84. On $\partial\Sigma$, the vector $\mathbf{T} \times \mathbf{N}$ is "tangent" to Σ and points away from Σ. In other words, $\mathbf{T} \times \mathbf{N}$ can be treated as the "outward-pointing" unit "normal" of $\partial\Sigma$ which makes the divergence theorem on surfaces more intuitive.

4.7 The Stokes Theorem

4.7.1 Measurements of the circulation - the curl operator

We consider the circulation or the speed of rotation of a vector field u about an axis in the direction $\mathbf{N}$. Let P be a plane passing thorough a point a and having normal $\mathbf{N}$, and C_r be a circle on the plane P centered at a with radius r. Pick the orientation of the unit tangent vector $\mathbf{T}$ which is compatible with the unit normal $\mathbf{N}$ (see Figure 4.6 for reference).

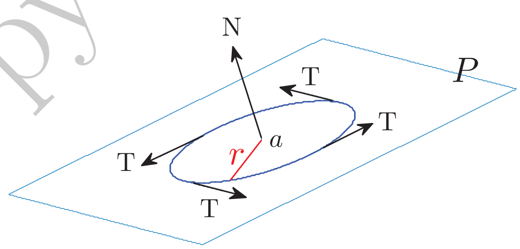


Figure 4.6: the circulation about an axis in direction $\mathbf{N}$

Since the instantaneous angular velocity of a vector field u along the circle C_r is measured by $\dfrac{\boldsymbol{u}\cdot\mathbf{T}}{r}$, it is quite reasonable to measure the circulation of u along C_r by averaging the angular velocity; that is, we consider the quantity

$$\frac{1}{2\pi r}\oint_{C_r} \frac{\boldsymbol{u}\cdot\mathbf{T}}{r}\, ds \tag{4.20}$$

as a (constant multiple of) measurement of the speed of rotation. The limit of the quantity above, as $r \to 0$, is then a good measurement of the rotation speed of $\boldsymbol{u}$ at the point a about the axis in the direction $\mathbf{N}$.

We start from the case that $\mathbf{N} = \mathbf{e}_3$ so that P be parallel to the x_1x_2-plane. With u_1, u_2, u_3 denoting respectively the first, the second and the third components of $\boldsymbol{u}$, by the change of variable $ds = rd\theta$ and the L'Hôspital rule (to obtain the second "$=$") we find that

$$\begin{aligned}
&\lim_{r\to 0} \frac{1}{2\pi r}\oint_{C_r} \frac{\boldsymbol{u}\cdot\mathbf{T}}{r}\,ds \\
&= \lim_{r\to 0} \frac{1}{2\pi r}\int_0^{2\pi} \big[u_2\big(\boldsymbol{a}+(r\cos\theta, r\sin\theta, 0)\big)\cos\theta - u_1\big(\boldsymbol{a}+(r\cos\theta, r\sin\theta, 0)\big)\sin\theta\big]\,d\theta \\
&= \frac{1}{2\pi}\frac{d}{dr}\Big|_{r=0}\int_0^{2\pi} \big[u_2\big(\boldsymbol{a}+(r\cos\theta, r\sin\theta, 0)\big)\cos\theta - u_1\big(\boldsymbol{a}+(r\cos\theta, r\sin\theta, 0)\big)\sin\theta\big]\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Big[\frac{\partial u_2}{\partial x_1}(\boldsymbol{a})\cos^2\theta + \frac{\partial u_2}{\partial x_2}(\boldsymbol{a})\cos\theta\sin\theta - \frac{\partial u_1}{\partial x_1}(\boldsymbol{a})\cos\theta\sin\theta - \frac{\partial u_1}{\partial x_2}(\boldsymbol{a})\sin^2\theta\Big]\,d\theta \\
&= \frac{1}{2}\Big[\frac{\partial u_2}{\partial x_1}(\boldsymbol{a}) - \frac{\partial u_1}{\partial x_2}(\boldsymbol{a})\Big] = \frac{1}{2}\sum_{i,j=1}^{2} \varepsilon_{3ij}\frac{\partial u_j}{\partial x_i}(\boldsymbol{a})\,. \qquad (4.21)
\end{aligned}$$

Now suppose the general case that $\mathbf{N} \neq \mathbf{e}_3$. Let $\widehat{\mathbf{e}}_3 = \mathbf{N}$ and choose $\widehat{\mathbf{e}}_1$ and $\widehat{\mathbf{e}}_2$ so that $\{\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3\}$ is an orthonormal basis following the right-hand rule (that is, $\widehat{\mathbf{e}}_1 \times \widehat{\mathbf{e}}_2 = \widehat{\mathbf{e}}_3$). Then the vector field $\boldsymbol{u}$ has two representations

$$\boldsymbol{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 = v_1\widehat{\mathbf{e}}_1 + v_2\widehat{\mathbf{e}}_2 + v_3\widehat{\mathbf{e}}_3\,. \qquad (4.22)$$

Let $\mathrm{O} = \big[\widehat{\mathbf{e}}_1 \vdots \widehat{\mathbf{e}}_2 \vdots \widehat{\mathbf{e}}_3\big]$, and introduce a new Cartesian coordinate system $\boldsymbol{y} = \mathrm{O}^{\mathrm{T}}\boldsymbol{x}$. Note that $\boldsymbol{y}$ is the coordinate with coordinate axis parallel to the basis $\{\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3\}$. In this new Cartesian coordinate system, (4.21) implies that

$$\lim_{r\to 0} \frac{1}{2\pi r}\oint_{C_r} \frac{\boldsymbol{u}\cdot\mathbf{T}}{r}\,ds = \frac{1}{2}\Big[\frac{\partial v_2}{\partial y_1}(\boldsymbol{b}) - \frac{\partial v_1}{\partial y_2}(\boldsymbol{b})\Big]\,,$$

where $\boldsymbol{b} = \mathrm{O}^{\mathrm{T}}\boldsymbol{a}$.

Now we transform the result above back to the original coordinate system (so that the limit is in terms of derivatives of u_j w.r.t. x_i). Note that (4.22) implies that $\boldsymbol{v} = \mathrm{O}^{\mathrm{T}}\boldsymbol{u}$ so that $v_j = \widehat{\mathbf{e}}_j \cdot \boldsymbol{u}$. Moreover, with e_{jk} denoting the k-th component (w.r.t. the ordered basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$) of $\widehat{\mathbf{e}}_j$; that is, $\widehat{\mathbf{e}}_j = e_{j1}\mathbf{e}_1 + e_{j2}\mathbf{e}_2 + e_{j3}\mathbf{e}_3$, the chain rule provides that

$$\frac{\partial}{\partial y_1} = e_{11}\frac{\partial}{\partial x_1} + e_{12}\frac{\partial}{\partial x_2} + e_{13}\frac{\partial}{\partial x_3} \qquad \text{and} \qquad \frac{\partial}{\partial y_2} = e_{21}\frac{\partial}{\partial x_1} + e_{22}\frac{\partial}{\partial x_2} + e_{23}\frac{\partial}{\partial x_3}\,;$$

thus

$$\begin{aligned}\lim_{r\to 0}\frac{1}{2\pi r}\oint_{C_r}\frac{\boldsymbol{u}\cdot\mathbf{T}}{r}\,ds &= \frac{1}{2}\sum_{j=1}^{3}\Big[e_{1j}\frac{\partial(\boldsymbol{u}\cdot\widehat{\mathbf{e}}_2)}{\partial x_j}(\boldsymbol{a}) - e_{2j}\frac{\partial(\boldsymbol{u}\cdot\widehat{\mathbf{e}}_1)}{\partial x_j}(\boldsymbol{a})\Big]\\ &= \frac{1}{2}\sum_{j,k=1}^{3}\big(e_{1j}e_{2k}-e_{2j}e_{1k}\big)\frac{\partial u_k}{\partial x_j}(\boldsymbol{a}) = \frac{1}{2}\sum_{j,k,r,s=1}^{3}(\delta_{jr}\delta_{ks}-\delta_{js}\delta_{kr})e_{1r}e_{2s}\frac{\partial u_k}{\partial x_j}(\boldsymbol{a}),\end{aligned}$$

where $\delta_{\cdot\cdot}$'s are the Kronecker deltas. Using (4.9), we further conclude that

$$\lim_{r\to 0}\frac{1}{2\pi r}\oint_{C_r}\frac{\boldsymbol{u}\cdot\mathbf{T}}{r}\,ds = \frac{1}{2}\sum_{i,j,k,r,s=1}^{3}\varepsilon_{ijk}\varepsilon_{irs}e_{1r}e_{2s}\frac{\partial u_k}{\partial x_j}(\boldsymbol{a}).$$

Since $\widehat{\mathbf{e}}_1\times\widehat{\mathbf{e}}_2=\widehat{\mathbf{e}}_3$, we have $e_{3i}=\sum\limits_{r,s=1}^{3}\varepsilon_{irs}e_{1r}e_{2s}$; thus the identity above shows that

$$\lim_{r\to 0}\frac{1}{2\pi r}\oint_{C_r}\frac{\boldsymbol{u}\cdot\mathbf{T}}{r}\,ds = \frac{1}{2}\sum_{i,j,k=1}^{3}\varepsilon_{ijk}e_{3i}\frac{\partial u_k}{\partial x_j}(\boldsymbol{a}) = \frac{1}{2}\sum_{i=1}^{3}\Big(\sum_{j,k=1}^{3}\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}(\boldsymbol{a})\Big)e_{3i}.$$

(The blue expression of) (4.21) and the identity above motivate the following

Definition 4.85 (The curl operator)**.** Let $\boldsymbol{u}:\Omega\subseteq\mathbb{R}^{\rm n}\to\mathbb{R}^{\rm n}$, ${\rm n}=2$ or ${\rm n}=3$, be a vector field.

1. For ${\rm n}=2$, the curl of $\boldsymbol{u}$ is a scalar function defined by

$$\operatorname{curl}\boldsymbol{u}=\sum_{i,j=1}^{2}\varepsilon_{3ij}\,\boldsymbol{u}^{j}_{,i}\,.$$

2. For ${\rm n}=3$, the curl of $\boldsymbol{u}$ is a vector-valued function defined by

$$(\operatorname{curl}\boldsymbol{u})^{i}=\sum_{j,k=1}^{3}\varepsilon_{ijk}\,\boldsymbol{u}^{k}_{,j}\,.$$

The function curl$\boldsymbol{u}$ is also called the *vorticity* of $\boldsymbol{u}$, and is usually denoted by one single Greek letter ω.

Having the curl operator defined, for the three-dimensional case the circulation of a vector field $\boldsymbol{u}$ on the plane with normal $\mathbf{N}$ is given by $\dfrac{\operatorname{curl}\boldsymbol{u}\cdot\mathbf{N}}{2}$.

4.7.2 The Stokes theorem

The path we choose to circle around the point a does not have to be a circle. However, in such a case the average of the angular velocity no longer makes sense (since $\boldsymbol{u}\cdot\mathbf{T}$ might not contribute to the motion in the angular direction), and we instead consider the limit of the following quantity

$$\lim_{\mathrm{A}\to 0}\frac{1}{\mathrm{A}}\oint_C \boldsymbol{u}\cdot\mathbf{T}\,ds,$$

where A is the area enclosed by C. This limit is always $\mathrm{curl}\boldsymbol{u}\cdot\mathbf{N}$ because of the famous Stokes' theorem.

Theorem 4.86 (The Stokes theorem)**.** *Let $\boldsymbol{u}:\Omega\subseteq\mathbb{R}^3\to\mathbb{R}^3$ be a smooth vector field, and Σ be a $\mathscr{C}^1$-surface with $\mathscr{C}^1$-boundary $\partial\Sigma$ in Ω. Then*

$$\int_{\partial\Sigma}\boldsymbol{u}\cdot\mathbf{T}\,ds=\int_\Sigma \mathrm{curl}\boldsymbol{u}\cdot\mathbf{N}\,dS,$$

where $\mathbf{N}$ and $\mathbf{T}$ are compatible normal and tangent vector fields.

To prove the Stokes theorem, we first establish the following

Lemma 4.87. *Let $\Omega\subseteq\mathbb{R}^3$ be a bounded Lipschitz domain, and $\boldsymbol{w}:\Omega\to\mathbb{R}^{\mathrm{n}}$ be a mooth vector-valued function. If $\Sigma\subseteq\Omega$ is an oriented $\mathscr{C}^1$-surface with normal $\mathbf{N}$, then*

$$\mathrm{curl}\boldsymbol{w}\cdot\mathbf{N}=\mathrm{div}_\Sigma(\boldsymbol{w}\times\mathbf{N})\qquad\text{on}\quad\Sigma. \tag{4.23}$$

Proof. Let $\mathcal{O}\subseteq\Omega$ be a $\mathscr{C}^1$-domain such that $\Sigma\subseteq\partial\mathcal{O}$ and $\mathbf{N}$ is the outward-pointing unit normal on $\partial\mathcal{O}$. In other words, Σ is part of the boundary of $\mathcal{O}$. Since

$$(\nabla\varphi)^i=\frac{\partial\varphi}{\partial\mathbf{N}}\mathbf{N}^i+(\nabla_{\partial\mathcal{O}}\varphi)^i\qquad\text{on}\quad\partial\mathcal{O},$$

by the divergence theorem we conclude that for all $\varphi\in\mathscr{C}^1(\overline{\mathcal{O}})$,

$$\begin{aligned}\int_{\partial\mathcal{O}}(\mathrm{curl}\boldsymbol{w}\cdot\mathbf{N})\varphi\,dS&=\int_{\mathcal{O}}\mathrm{curl}\boldsymbol{w}\cdot\nabla\varphi\,dx=\int_{\partial\mathcal{O}}(\mathbf{N}\times\boldsymbol{w})\cdot\nabla\varphi\,dS\\&=\int_{\partial\mathcal{O}}(\mathbf{N}\times\boldsymbol{w})\cdot\nabla_{\partial\mathcal{O}}\varphi\,dS=\int_{\partial\mathcal{O}}\mathrm{div}_{\partial\mathcal{O}}(\boldsymbol{w}\times\mathbf{N})\varphi\,dS.\end{aligned}$$

Identity (4.23) is concluded since φ can be chosen arbitrarily on Σ. □

Proof of the Stokes theorem. Using (4.23) and then applying the divergence theorem on surfaces with boundary (Theorem 4.83), we find that

$$\int_\Sigma \operatorname{curl}\boldsymbol{u}\cdot\mathbf{N}\,dS = \int_\Sigma \operatorname{div}_\Sigma(\boldsymbol{u}\times\mathbf{N})\,dS = \int_{\partial\Sigma}(\boldsymbol{u}\times\mathbf{N})\cdot(\mathbf{T}\times\mathbf{N})\,ds = \int_{\partial\Sigma}(\boldsymbol{u}\cdot\mathbf{T})\,ds$$

in which the identity $(\boldsymbol{u}\times\mathbf{N})\cdot(\mathbf{T}\times\mathbf{N}) = \boldsymbol{u}\cdot\mathbf{T}$ is used. □

Example 4.88. Let Σ be the surface given in Example 4.51, and $\boldsymbol{F}:\mathbb{R}^3\to\mathbb{R}^3$ be a vector-valued function given by $\boldsymbol{F}(x,y,z)=(y,-x,0)$. Then by the definition of line integral,

$$\begin{aligned}\oint_C \boldsymbol{F}\cdot dr &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin^2 t, -\cos t\sin t, 0)\cdot\big(\cos^2 t-\sin^2 t, 2\sin t\cos t, -\sin t\big)dt\\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\big(\sin^2 t\cos^2 t - \sin^4 t - 2\sin^2 t\cos^2 t\big)dt\\ &= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 t dt = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1-\cos 2t}{2}dt = -\Big(\frac{t}{2}-\frac{\sin 2t}{4}\Big)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -\frac{\pi}{2}\,.\end{aligned}$$

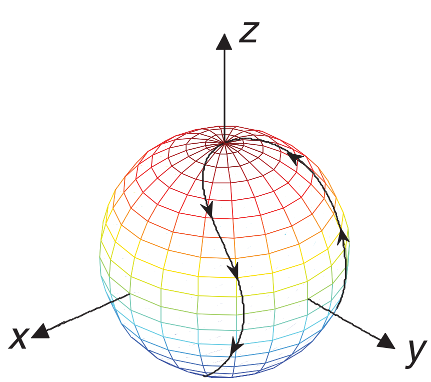
while by the fact that $\operatorname{curl}\boldsymbol{F}=(0,0,-2)$, the Stokes theorem implies that

$$\begin{aligned}\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} &= \int_\Sigma (0,0,-2)\cdot\mathbf{N}\,dS = \int_{\psi^{-1}(\Sigma)} -2\cos\phi\sin\phi\,d(\theta,\phi) = -2\int_0^{\frac{\pi}{2}}\int_\phi^{\pi-\phi}\sin\phi\cos\phi\,d\theta d\phi\\ &= -\int_0^{\frac{\pi}{2}}(\pi-2\phi)\sin 2\phi\,d\phi = \Big(\frac{\pi}{2}\cos 2\phi - \phi\cos 2\phi + \frac{1}{2}\sin 2\phi\Big)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}}\\ &= -\frac{\pi}{2}-\frac{\pi}{2}+\frac{\pi}{2} = -\frac{\pi}{2}\,.\end{aligned}$$

Example 4.89. Let C be a smooth curve parameterized by

$$\boldsymbol{r}(t) = \big(\cos(\sin t)\sin t, \sin(\sin t)\sin t, \cos t\big)\,,\qquad t\in[0,2\pi]\,.$$

Then the curve C is a closed curve on $\mathbb{S}^2$, and divide $\mathbb{S}^2$ into two parts. Let Σ denote the part with smaller area.



As in Example 4.51 and Example 4.88, we would like to find the area of Σ, and verify the Stokes theorem for the special case that $\boldsymbol{F}: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\boldsymbol{F}(x, y, z) = (y, -x, 0)\,.$$

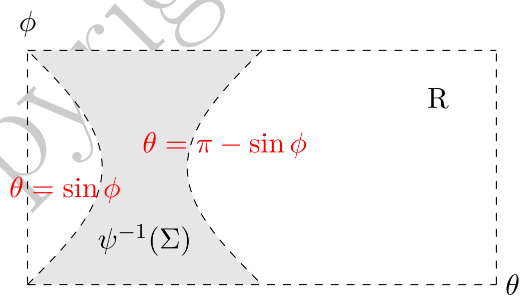
To find the surface area of Σ, we need to parameterize Σ. As in Example 4.51, we look for $\gamma(t) = \big(\theta(t), \phi(t)\big)$, $t \in [0, 2\pi]$, such that $\psi\big(\gamma(t)\big) = \boldsymbol{r}(t)$, where $\psi : \mathrm{R} \equiv (0, 2\pi) \times (0, \pi)$ is given by $\psi(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)\,.$

For $t \in (0, \pi)$, since $\cos t = \cos\phi(t)$ and $\phi(t) \in (0, \pi)$, we must have $\phi(t) = t$; thus the two identities $\cos(\sin t)\sin t = \cos\theta(t)\sin\phi(t)$ and $\sin(\sin t)\sin t = \sin\theta(t)\sin\phi(t)$ further imply that $\theta(t) = \sin t$. Therefore, the curve $\boldsymbol{r}\big((0, \pi)\big)$ corresponds to $\theta = \sin\phi$, $\phi \in (0, \pi)$, on R.

On the other hand, for $t \in (\pi, 2\pi)$, the identity $\cos\phi(t) = \cos t$ implies that $\phi(t) = 2\pi - t$. The two identities $\cos(\sin t)\sin t = \cos\theta(t)\sin\phi(t)$ and $\sin(\sin t)\sin t = \sin\theta(t)\sin\phi(t)$ further imply that

$$\cos(\sin t) = -\cos\theta(t) \quad \text{and} \quad \sin(\sin t) = -\sin\theta(t) \qquad t \in (\pi, 2\pi)\,.$$

Therefore, $\theta(t) = \pi + \sin t$ which implies that the curve $\boldsymbol{r}\big((\pi, 2\pi)\big)$ corresponds to $\theta = \pi - \sin\phi$, $\phi \in (0, \pi)$, on R.



Therefore, the surface area of Σ is

$$\int_0^\pi \int_{\sin\phi}^{\pi - \sin\phi} \sin\phi \, d\theta d\phi = \int_0^\pi (\pi - 2\sin\phi)\sin\phi \, d\phi = -\Big(\pi\cos\phi + \phi - \frac{\sin(2\phi)}{2}\Big)\Big|_{\phi=0}^{\phi=\pi} = \pi\,.$$

Next, we compute the line integral $\displaystyle\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$. First, we note that

$$\boldsymbol{r}'(t) = (-\sin(\sin t)\sin t\cos t + \cos(\sin t)\cos t, \cos(\sin t)\sin t\cos t + \sin(\sin t)\cos t, -\sin t)\,;$$

thus

$$\begin{aligned}(\boldsymbol{F}\circ\boldsymbol{r})(t)\cdot\boldsymbol{r}'(t) &= -\sin^2(\sin t)\sin^2 t\cos t + \sin(\sin t)\cos(\sin t)\sin t\cos t \\ &\quad - \cos^2(\sin t)\sin^2 t\cos t - \sin(\sin t)\cos(\sin t)\sin t\cos t \\ &= -\sin^2 t\cos t\,.\end{aligned}$$

As a consequence,

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = -\int_0^{2\pi}\sin^2 t\cos t\,dt = -\frac{1}{3}\sin^3 t\Big|_{t=0}^{t=2\pi} = 0\,.$$

On the other hand,

$$\begin{aligned}\int_\Sigma \mathrm{curl}\boldsymbol{F}\cdot\mathbf{N}\,dS &= \int_0^\pi\int_{\sin\phi}^{\pi-\sin\phi}(0,0,-2)\cdot(\cos\theta\sin\phi,\sin\theta\sin\phi,\cos\phi)\sin\phi\,d\theta d\phi \\ &= -2\int_0^\pi \sin\phi\cos\phi(\pi-2\sin\phi)\,d\phi \\ &= \Big(\frac{\pi}{2}\cos 2\phi + \frac{4}{3}\sin^3\phi\Big)\Big|_{\phi=0}^{\phi=\pi} = 0\,.\end{aligned}$$

4.8 Green's Theorem

In most of materials Green's theorem is introduced prior to the divergence theorem and the Stokes theorem; however, we treat Green's theorem as a corollary of the divergence theorem (Theorem 4.75), the Stokes theorem (Theorem 4.86) and Theorem 4.83.

Theorem 4.90 (Green's theorem)**.** *Let $\mathcal{D}$ be a bounded domain whose boundary $\partial\mathcal{D}$ is piecewise smooth, and $M,N:\mathcal{D}\to\mathbb{R}$ be of class $\mathscr{C}^1$. Then*

$$\oint_{\partial\mathcal{D}}(M,N)\cdot d\boldsymbol{r} = \int_{\mathcal{D}}(N_x - M_y)\,d\mathbb{A}\,,$$

where the line integral (on the left-hand side of the identity above) is taken so that the curve is counter-clockwise oriented.

Proof 1. Let $\boldsymbol{u}(x,y) = \big(N(x,y),-M(x,y)\big)$ be a vector-valued function defined on the 2-dimensional domain $\mathcal{D}$. Suppose that $\partial\mathcal{D}$ is parameterized by $\boldsymbol{r}(t) = \big(x(t),y(t)\big)$ for $t\in[a,b]$, where $\boldsymbol{r}'$ points in the counter-clockwise direction. Then with $\mathbf{N}$ denoting the outward-pointing unit normal of $\partial\mathcal{D}$, the divergence theorem implies that

$$\oint_{\partial\mathcal{D}}(M,N)\cdot d\boldsymbol{r} = \oint_{\partial\mathcal{D}}\boldsymbol{u}\cdot\mathbf{N}\,ds = \int_{\mathcal{D}}\mathrm{div}\boldsymbol{u}\,d\mathbb{A} = \int_{\mathcal{D}}(N_x - M_y)\,d\mathbb{A}\,. \qquad \square$$

Proof 2. Let $\boldsymbol{F}(x,y,z) = \big(M(x,y), N(x,y), 0\big)$ be a vector-valued function defined in a subset of $\mathbb{R}^3$. Then

$$\text{curl}\boldsymbol{F} = (0,0,N_x - M_y)\,;$$

thus the Stokes theorem implies that

$$\begin{aligned}\oint_{\partial\mathcal{D}} (M,N)\cdot d\boldsymbol{r} &= \int_{\partial\mathcal{D}} \boldsymbol{F}\cdot\mathbf{T}\,ds = \int_{\mathcal{D}} \text{curl}\boldsymbol{F}\cdot\mathbf{N}\,dS = \int_{\mathcal{D}} (0,0,N_x - M_y)\cdot(0,0,1)\,d\mathbb{A} \\ &= \int_{\mathcal{D}} (N_x - M_y)\,d\mathbb{A}\,.\end{aligned}$$ □

Proof 3. Let $\Sigma = \mathcal{D}\times\{z=0\}$. Then Σ is a surface with boundary and the upward-pointing unit normal $\mathbf{N} = (0,0,1)$. Let $\boldsymbol{F}:\Sigma\to\mathbb{R}^3$ and $\boldsymbol{u}:\mathcal{D}\to\mathbb{R}^2$ be vector-valued functions defined by $\boldsymbol{F}(x,y,z) = \big(N(x,y), -M(x,y), 0\big)$ and $\boldsymbol{u}(x,y) = \big(N(x,y), -M(x,y)\big)$, respectively. We note that if $\partial\mathcal{D}$ is parameterized by $\boldsymbol{r}(t) = (x(t), y(t), 0)$, then

$$\mathbf{T}\times\mathbf{N} = \frac{1}{\|\boldsymbol{r}'(t)\|_{\mathbb{R}^3}}\big(x'(t), y'(t), 0\big)\times(0,0,1) = \frac{1}{\|\boldsymbol{r}'(t)\|_{\mathbb{R}^3}}\big(y'(t), -x'(t), 0\big)\,;$$

thus by the fact that the surface divergence operator div_Σ is the same as the 2-d divergence operator (since Σ is flat), Theorem 4.83 implies that

$$\oint_{\partial\mathcal{D}} (M,N)\cdot d\boldsymbol{r} = \oint_{\partial\mathcal{D}} \boldsymbol{F}\cdot(\mathbf{T}\times\mathbf{N})\,ds = \int_{\Sigma} \text{div}_\Sigma\boldsymbol{F}\,dS = \int_{\mathcal{D}} \text{div}\,\boldsymbol{u}\,d\mathbb{A} = \int_{\mathcal{D}} (N_x - M_y)\,d\mathbb{A}\,.$$ □

Corollary 4.91. *Let* $\mathrm{R}\subseteq\mathbb{R}^2$ *be a domain enclosed by a simple closed curve* C *which is parameterized by* $\mathbf{r}(t) = \big(x(t), y(t)\big)$ *for* $t\in[a,b]$*. Suppose* $\mathbf{r}'$ *points in the counter-clockwise direction. Then*

$$\textit{the area of } \mathrm{R} = \frac{1}{2}\int_a^b \big[x(t)y'(t) - y(t)x'(t)\big]\,dt\,.$$

Proof. The corollary is concluded by applying Green's theorem to the special case: $M(x,y) = -y$ and $N(x,y) = x$. □

Example 4.92. Compute the area enclosed by the Cardioid which has a polar representation $r = (1-\sin\theta)$ with $\theta\in[0,2\pi]$.

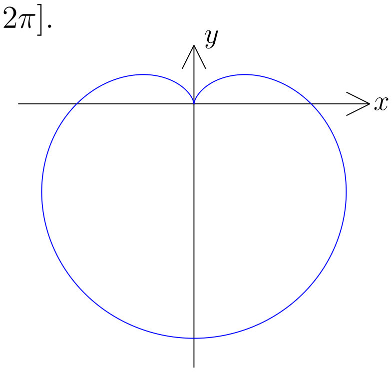


Figure 4.7: The Cardioid

Given the polar representation $r = (1 - \sin\theta)$, a parametrization of the Cardioid is

$$\boldsymbol{r}(t) = \big(x(t), y(t)\big) = \big((1-\sin t)\cos t, (1-\sin t)\sin t\big) \qquad t \in [0, 2\pi]\,.$$

Then Corollary 4.91 implies that the area enclosed by the Cardioid is

$$\begin{aligned}
&\frac{1}{2}\int_0^{2\pi} \big[(1-\sin t)\cos t\big(-\cos t\sin t + (1-\sin t)\cos t\big) \\
&\qquad - (1-\sin t)\sin t\big(-\cos^2 t - (1-\sin t)\sin t\big)\big]\,dt \\
&\quad = \frac{1}{2}\int_0^{2\pi} (1-\sin t)\big[\cos^2 t - 2\sin t\cos^2 t + \sin t\cos^2 t + \sin^2 t - \sin^3 t\big]\,dt \\
&\quad = \frac{1}{2}\int_0^{2\pi} (1-\sin t)(1-\sin t\cos^2 t - \sin^3 t)dt = \frac{1}{2}\int_0^{2\pi} (1-\sin t)^2 dt = \frac{3\pi}{2}\,.
\end{aligned}$$

Before finishing this chapter, we would like to establish an unproven theorem: Theorem 4.33. We recall Theorem 4.33 as follows.

Theorem 4.33. Let $\mathcal{D} \subseteq \mathbb{R}^2$ be simply connected, and $\boldsymbol{F} = (M, N) : \mathcal{D} \to \mathbb{R}^2$ be of class $\mathscr{C}^1$. If $M_y = N_x$, then $\boldsymbol{F}$ is conservative.

Proof of Theorem 4.33. By Theorem 4.30, it suffices to show that $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = 0$ for all piecewise smooth closed curve $C \in \mathcal{D}$. Nevertheless, if C is a piecewise closed curve and R is the region enclosed by C, by the fact that $\mathcal{D}$ is simply connected, we must have $\partial\mathrm{R} = C$. Therefore, Green's theorem implies that

$$\oint_C (M, N)\cdot d\boldsymbol{r} = \int_{\mathrm{R}} (N_x - M_y)\, d\mathbb{A} = 0\,. \qquad \square$$