Fourier Analysis

富氏分析

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Contents

Chapter 5

Applications on Partial Differential Equations

5.1 Heat Conduction in a Rod

Consider the heat distribution on a rod of length L. Parameterize the rod by $[0, L]$, and let t be the time variable. Let $\rho(x)$, $s(x)$, $\kappa(x)$ denote the density, specific heat, and the thermal conductivity of the rod at position $x \in (0, L)$, respectively, and $u(x,t)$ denote the temperature at position x and time t. For $0 < x < L$, and $\Delta x, \Delta t \ll 1$,

$$\int_x^{x+\Delta x} \rho(y)s(y)\big[u(y,t+\Delta t)-u(y,t)\big]\,dy = \int_t^{t+\Delta t} \big[-\kappa(x)u_x(x,t')+\kappa(x+\Delta x)u_x(x+\Delta x,t')\big]\,dt',$$

where the left-hand side denotes the change of the total heat in the small section $(x, x+\Delta x)$, and the right-hand side denotes the heat flows from outside. Divide both sides by $\Delta x \Delta t$ and letting Δx and Δt approach zero, if all the functions appearing in the equation above are smooth enough, we find that

$$\rho(x)s(x)u_t(x,t) = \big[\kappa(x)u_x(x,t)\big]_x \qquad 0 < x < L\,, \quad t > 0\,. \tag{5.1}$$

Assuming uniform rod; that is, ρ, s, κ are constant, then (5.1) reduces to that

$$u_t(x,t) = \alpha^2 u_{xx}(x,t)\,, \qquad 0 < x < L\,, \quad t > 0\,, \tag{5.2a}$$

where $\alpha^2 = \dfrac{\kappa}{\rho s}$ is called the ***thermal diffusivity***. By re-scaling the length scale, we can assume that $\alpha = 1$.

To determine the state of the temperature, we need to impose that initial condition

$$u(x,0) = u_0(x) \qquad 0 < x < L \tag{5.2b}$$

for some given function $u_0 : [0, L] \to \mathbb{R}$ and a boundary condition. Usually one of the following four types of boundary conditions is imposed:

1. ***Dirichlet boundary condition***: The Dirichlet boundary condition is used to describe the phenomena that the temperature at the end points of the rod is known/ controable. Mathematically, it is expressed by

$$u(0,t) = a(t) \quad \text{and} \quad u(L,t) = b(t) \qquad \forall\, t > 0$$

   for some given functions $a(t)$ and $b(t)$.

2. ***Neumann boundary condition***: The Neumann boundary condition is used to describe the phenomena of insulation; that is, there is no heat flow at the end points. Mathematically, it is expressed by

$$u_x(0,t) = u_x(L,t) = 0 \qquad \forall\, t > 0\,.$$

   In general, we can consider the boundary condition

$$u_x(0,t) = a(t) \quad \text{and} \quad u_x(L,t) = b(t) \qquad \forall\, t > 0$$

   for some given functions $a(t)$ and $b(t)$.

3. ***Mixed type boundary condition***: We can also consider the case that at one end point the temperature is known while there is no heat flow on the other end point. In general, this is expressed by

$$u(0,t) = a(t) \quad \text{and} \quad u_x(L,t) = b(t) \qquad \forall\, t > 0$$

   or

$$u_x(0,t) = a(t) \quad \text{and} \quad u(L,t) = b(t) \qquad \forall\, t > 0$$

   for some given functions $a(t)$ and $b(t)$.

4. ***Periodic boundary condition***: Suppose that instead of rods we consider modelling the temperature distribution in a (big) ring (with perimeter L). Choosing a point on the ring as the "left-end" point and parameterizing the point of the ring by arc-length, we then have the "boundary" condition

$$u(0,t) = u(L,t) \qquad \forall\, t > 0\,.$$

   This is called the periodic boundary condition.

5.1.1 The Dirichlet problem

In this sub-section we consider the heat equation with Dirichlet boundary condition:

$$u_t - u_{xx} = 0 \qquad \text{in} \quad (0, L) \times (0, \infty)\,, \tag{5.3a}$$

$$u = u_0 \qquad \text{on} \quad (0, L) \times \{t = 0\}\,, \tag{5.3b}$$

$$u(0, t) = a\,, \quad u(L, t) = b \qquad \text{for all } t > 0\,, \tag{5.3c}$$

where a and b are given constants. Let $v(x, t) = u(x, t) - \dfrac{b - a}{L}x - a$. Then v satisfies

$$v_t - v_{xx} = 0 \qquad \text{in} \quad (0, L) \times (0, \infty)\,, \tag{5.4a}$$

$$v = v_0 \qquad \text{on} \quad (0, L) \times \{t = 0\}\,, \tag{5.4b}$$

$$v(0, t) = v(L, t) = 0 \qquad \text{for all } t > 0\,, \tag{5.4c}$$

where $v_0 : [0, L] \to \mathbb{R}$ is given by $v_0(x) = u_0(x) - \dfrac{b - a}{L}x - a$. As long as the solution v to (5.4) is found, the solution u to (5.3) can be constructed using $u(x, t) = v(x, t) + \dfrac{b - a}{L}x + a$. Therefore, we focus on solving (5.4) (using the Fourier series method).

The idea of using the Fourier series to solve (5.4) is that for each fixed $t > 0$ we express v in terms of its Fourier series representation (using proper "basis"). Recall that for a function $f : [0, L] \to \mathbb{R}$, we have the Fourier representation

$$f(x) \text{ ``=''} \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\frac{2\pi kx}{L} + s_k \sin\frac{2\pi kx}{L} \qquad x \in [0, L]\,,$$

where $c_k = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{2\pi kx}{L}\,dx$ and $s_k = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{2\pi kx}{L}\,dx$, so

$$v(x, t) \text{ ``=''} \frac{c_0(t)}{2} + \sum_{k=1}^{\infty} c_k(t) \cos\frac{2\pi kx}{L} + s_k(t) \sin\frac{2\pi kx}{L} \qquad x \in [0, L]\,,$$

for some sequence of functions $\big\{c_k(t)\big\}_{k=0}^{\infty}$ and $\big\{s_k(t)\big\}_{k=1}^{\infty}$. However, this particular Fourier series of v is not a good choice of solving (5.4) since it is difficult to validate the boundary condition (5.4c).

Note that for $f : [0, L] \to \mathbb{R}$ instead of the Fourier series representation above we can also consider the "cosine" series or "sine" series that are obtained by treating f as the restriction of an even or an odd function defined on $[-L, L]$ to $[0, L]$. In other words, define $f_e, f_o : [-L, L] \to \mathbb{R}$, called the ***even*** and ***odd extension*** of f respectively, by

$$f_e(x) = \begin{cases} f(x) & \text{if } x \in [0, L]\,, \\ f(-x) & \text{if } x \in [-L, 0)\,, \end{cases} \qquad \text{and} \qquad f_o(x) = \begin{cases} f(x) & \text{if } x \in [0, L]\,, \\ -f(-x) & \text{if } x \in [-L, 0)\,, \end{cases}$$

then $f = f_o = f_e$ on $[0, L]$. Since

$$f_e(x) \text{ ``}=\text{''} \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\frac{\pi k x}{L} \qquad \text{and} \qquad f_o(x) \text{ ``}=\text{''} \sum_{k=1}^{\infty} s_k \sin\frac{\pi k x}{L} \qquad x \in [-L, L]\,,$$

where $c_k = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{\pi k x}{L}\,dx$ and $s_k = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{\pi k x}{L}\,dx$, we have

$$f(x) \text{ ``}=\text{''} \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\frac{\pi k x}{L} \qquad \text{and} \qquad f(x) \text{ ``}=\text{''} \sum_{k=1}^{\infty} s_k \sin\frac{\pi k x}{L} \qquad x \in [0, L]\,.$$

Using the sine series, for each $t > 0$ $v(x, t)$ can be expressed as

$$v(x,t) \text{ ``}=\text{''} \sum_{k=1}^{\infty} d_k(t) \sin\frac{\pi k x}{L} \qquad x \in [0, L]\,.$$

for some sequence of function $\big\{d_k(t)\big\}_{k=1}^{\infty}$ to be determined. We note that using this particular representation of v the boundary condition (5.4c) automatically holds. Therefore, it suffices to find $\big\{d_k(t)\big\}_{k=1}^{\infty}$ such that (5.4a,b) hold.

Assume that the differentiation of the series can be obtained by term-by-term differentiation; that is,

$$\frac{\partial}{\partial t}\sum_{k=1}^{\infty} d_k(t)\sin\frac{\pi k x}{L} = \sum_{k=1}^{\infty}\frac{\partial}{\partial t}\Big(d_k(t)\sin\frac{\pi k x}{L}\Big) = \sum_{k=1}^{\infty} d_k'(t)\sin\frac{\pi k x}{L}$$

and

$$\frac{\partial^2}{\partial x^2}\sum_{k=1}^{\infty} d_k(t)\sin\frac{\pi k x}{L} = \sum_{k=1}^{\infty}\frac{\partial^2}{\partial x^2}\Big(d_k(t)\sin\frac{\pi k x}{L}\Big) = -\sum_{k=1}^{\infty}\frac{k^2\pi^2}{L^2} d_k(t)\sin\frac{\pi k x}{L}\,.$$

Then (5.4a) implies that

$$\sum_{k=1}^{\infty}\Big[d_k'(t) + \frac{k^2\pi^2}{L^2} d_k(t)\Big]\sin\frac{\pi k x}{L} = 0 \qquad x \in [0, L]\,.$$

As a consequence,

$$d_k'(t) + \frac{k^2\pi^2}{L^2} d_k(t) = 0 \qquad \forall\, k \in \mathbb{N}\,. \tag{5.5a}$$

To determine d_k uniquely, an initial condition for d_k has to be imposed. Noting that (5.4b) implies that

$$v_0(x) \text{ ``}=\text{''} \sum_{k=1}^{\infty} d_k(0) \sin \frac{\pi k x}{L} \qquad x \in [0, L]\,;$$

thus

$$d_k(0) = \widehat{v}_{0k} \equiv \frac{2}{L} \int_0^L v_0(x) \sin \frac{\pi k x}{L}\, dx\,. \tag{5.5b}$$

Solving the initial value problem (5.5), we find that

$$d_k(t) = \widehat{v}_{0k} e^{-\frac{k^2\pi^2}{L^2} t} \qquad \forall\, k \in \mathbb{N}\,;$$

thus the solution to (5.4) can be written as

$$v(x,t) = \sum_{k=1}^{\infty} \widehat{v}_{0k} e^{-\frac{k^2\pi^2}{L^2} t} \sin \frac{k\pi x}{L}\,.$$

Therefore, the solution to (5.3) can be written as

$$u(x,t) = \sum_{k=1}^{\infty} \widehat{v}_{0k} e^{-\frac{k^2\pi^2}{L^2} t} \sin \frac{k\pi x}{L} + \frac{b-a}{L} x + a\,. \tag{5.6}$$

• the long time behavior: Suppose that the temperature at the left-end and right-end points are fixed as a and b (as described in the boundary condition (5.3c)). Then we expect that no matter what the temperature distribution is given initially, the temperature distribution approaches a linear distribution; that is, we expect that $u(x,t) \to \dfrac{b-a}{L} x + a$ as $t \to \infty$ for all $x \in [0, L]$. This expectation is in fact true, and we try to prove this here.

Using (5.6), we obtain that

$$\Big|u(x,t) - \frac{b-a}{L} x - a\Big| \leqslant \sum_{k=1}^{\infty} \big|\widehat{v}_{0k}\big| e^{-\frac{k^2\pi^2}{L^2} t}\,.$$

By the fact that

$$\big|\widehat{v}_{0k}\big| \leqslant \frac{2}{L} \int_0^L \big|v_0(x)\big|\, dx = \frac{2}{L} \|v_0\|_{L^1(0,L)}\,,$$

we find that

$$\begin{aligned} \Big|u(x,t) - \frac{b-a}{L} x - a\Big| &\leqslant \frac{2}{L} \|v_0\|_{L^1(0,L)} \sum_{k=1}^{\infty} e^{-\frac{k^2\pi^2}{L^2} t} \leqslant \frac{2}{L} \|v_0\|_{L^1(0,L)} \sum_{k=1}^{\infty} e^{-\frac{k^2\pi^2}{L^2}(t-1)} e^{-\frac{k^2\pi^2}{L^2}} \\ &\leqslant \frac{2}{L} \|v_0\|_{L^1(0,L)} e^{-\frac{\pi^2}{L^2}(t-1)} \sum_{k=1}^{\infty} e^{-\frac{k^2\pi^2}{L^2}}\,; \end{aligned}$$

thus with C denoting the constant $\dfrac{2}{L}\|v_0\|_{L^1(0,L)}e^{\frac{\pi^2}{L^2}}\sum\limits_{k=1}^{\infty}e^{-\frac{k^2\pi^2}{L^2}}$, we have

$$\sup_{x\in[0,L]}\Big|u(x,t)-\frac{b-a}{L}x-a\Big|\leqslant Ce^{-\frac{\pi^2}{L^2}t}\,. \tag{5.7}$$

Since $C<\infty$, we conclude that the function $u(\cdot,t)$ converges to the function $\dfrac{b-a}{L}x+a$ uniformly on $[0,L]$ as $t\to\infty$.

5.1.2 The Neumann problem

In this sub-section we consider the heat equation with Neumann boundary condition:

$$u_t-u_{xx}=0 \quad \text{in} \quad (0,L)\times(0,\infty)\,, \tag{5.8a}$$

$$u=u_0 \quad \text{on} \quad (0,L)\times\{t=0\}\,, \tag{5.8b}$$

$$u_x(0,t)=a\,,\quad u_x(L,t)=b \quad \text{for all } t>0\,, \tag{5.8c}$$

where a and b are given constants. Let $v(x,t)=u(x,t)-\dfrac{b-a}{2L}(x^2+2t)-ax$. Then v satisfies

$$v_t-v_{xx}=0 \quad \text{in} \quad (0,L)\times(0,\infty)\,, \tag{5.9a}$$

$$v=v_0 \quad \text{on} \quad (0,L)\times\{t=0\}\,, \tag{5.9b}$$

$$v_x(0,t)=v_x(L,t)=0 \quad \text{for all } t>0\,, \tag{5.9c}$$

where $v_0:[0,L]\to\mathbb{R}$ is given by $v_0(x)=u_0(x)-\dfrac{b-a}{2L}x^2-ax$. As long as the solution v to (5.9) is found, the solution u to (5.8) can be constructed using $u(x,t)=v(x,t)+\dfrac{b-a}{2L}x^2+ax$. Therefore, we focus on solving (5.9) (using the Fourier series method). We look for $\{d_k(t)\}_{k=0}^{\infty}$ such that

$$v(x,t)=\frac{d_0(t)}{2}+\sum_{k=1}^{\infty}d_k(t)\cos\frac{\pi kx}{L}$$

validates (5.9a,b).

Assume that the differentiation of the series can be obtained by term-by-term differentiation; that is,

$$\frac{\partial}{\partial t}\sum_{k=1}^{\infty}d_k(t)\cos\frac{\pi kx}{L}=\sum_{k=1}^{\infty}\frac{\partial}{\partial t}\Big(d_k(t)\cos\frac{\pi kx}{L}\Big)=\sum_{k=1}^{\infty}d_k'(t)\cos\frac{\pi kx}{L}\,,$$

$$\frac{\partial}{\partial x}\sum_{k=1}^{\infty}d_k(t)\cos\frac{\pi kx}{L}=\sum_{k=1}^{\infty}\frac{\partial}{\partial x}\Big(d_k(t)\cos\frac{\pi kx}{L}\Big)=-\sum_{k=1}^{\infty}\frac{k\pi}{L}d_k(t)\sin\frac{\pi kx}{L}\,,$$

and
$$\frac{\partial^2}{\partial x^2}\sum_{k=1}^{\infty} d_k(t)\cos\frac{\pi k x}{L} = \sum_{k=1}^{\infty}\frac{\partial^2}{\partial x^2}\big(d_k(t)\cos\frac{\pi k x}{L}\big) = -\sum_{k=1}^{\infty}\frac{k^2\pi^2}{L^2}d_k(t)\cos\frac{\pi k x}{L}\,.$$
Then (5.9c) holds automatically, and (5.9a) implies that
$$\frac{d_0'(t)}{2} + \sum_{k=1}^{\infty}\Big[d_k'(t) + \frac{k^2\pi^2}{L^2}d_k(t)\Big]\cos\frac{\pi k x}{L} = 0\,.$$

Therefore, d_0 is a constant and d_k satisfies (5.5) as well. Moreover, expressing v_0 in terms of cosine series, (5.9b) implies that
$$d_k(0) = \widehat{v}_{0k} \equiv \frac{2}{L}\int_0^L v_0(x)\cos\frac{\pi k x}{L}\,dx \qquad \forall\, k \in \mathbb{N}\cup\{0\}\,.$$
Solving (5.5) with the initial condition above, we obtain that
$$d_k(t) = \widehat{v}_{0k}e^{-\frac{k^2\pi^2}{L^2}t} \qquad \forall\, k \in \mathbb{N};$$
thus the solution to (5.9) can be written as
$$v(x,t) = \frac{1}{L}\int_0^L v_0(x)\,dx + \sum_{k=1}^{\infty}\widehat{v}_{0k}e^{-\frac{k^2\pi^2}{L^2}t}\cos\frac{k\pi x}{L}\,.$$
Therefore, the solution to (5.3) can be written as
$$u(x,t) = \frac{1}{L}\int_0^L v_0(x)\,dx + \sum_{k=1}^{\infty}\widehat{v}_{0k}e^{-\frac{k^2\pi^2}{L^2}t}\cos\frac{k\pi x}{L} + \frac{b-a}{2L}(x^2+2t) + ax\,.$$

• the long time behavior: Suppose that the rod is insulated at the end-points; that is, the temperature u satisfies $u_x(0,t) = u_x(L,t) = 0$ for all $t > 0$. Then $v_0 = u_0$ and we expect that no matter what the temperature distribution is given initially, the temperature distribution approaches the average temperature; that is, we expect that $u(x,t) \to \dfrac{1}{L}\displaystyle\int_0^L u_0(x)\,dx$ as $t \to \infty$ for all $x \in [0,L]$. Similar to the derivation of (5.7),
$$\begin{aligned}\Big|u(x,t) - \frac{1}{L}\int_0^L u_0(x)\,dx\Big| &\leqslant \frac{2}{L}\|v_0\|_{L^1(0,L)}\sum_{k=1}^{\infty}e^{-\frac{k^2\pi^2}{L^2}t} \leqslant \frac{2}{L}\|v_0\|_{L^1(0,L)}\sum_{k=1}^{\infty}e^{-\frac{k^2\pi^2}{L^2}(t-1)}e^{-\frac{k^2\pi^2}{L^2}}\\ &\leqslant \frac{2}{L}\|v_0\|_{L^1(0,L)}e^{-\frac{\pi^2}{L^2}(t-1)}\sum_{k=1}^{\infty}e^{-\frac{k^2\pi^2}{L^2}}\,;\end{aligned}$$

thus with C denoting the constant $\dfrac{2}{L}\|v_0\|_{L^1(0,L)}e^{\frac{\pi^2}{L^2}}\sum\limits_{k=1}^{\infty} e^{-\frac{k^2\pi^2}{L^2}}$, we have

$$\sup_{x\in[0,L]}\Big|u(x,t)-\frac{1}{L}\int_0^L u_0(x)\,dx\Big| \leqslant Ce^{-\frac{\pi^2}{L^2}t}\,. \tag{5.10}$$

Since $C<\infty$, we conclude that the function $u(\cdot,t)$ converges to the function $\dfrac{1}{L}\displaystyle\int_0^L u_0(x)\,dx$ uniformly on $[0,L]$ as $t\to\infty$.

5.2 Heat Conduction on $\mathbb{R}^n$

Consider the heat equation on $\mathbb{R}^n$

$$u_t-\Delta u=0 \qquad \text{in} \quad \mathbb{R}^n\times(0,\infty)\,, \tag{5.11a}$$

$$u=u_0 \qquad \text{on} \quad \mathbb{R}^n\times\{t=0\}\,, \tag{5.11b}$$

where Δ is the Laplace operator, called Laplacian, defined by

$$\Delta u=\sum_{k=1}^n \frac{\partial^2 u}{\partial x_k^2}\,(=\operatorname{div}\nabla u)\,.$$

For a function f of x (and probably also t), let $\mathscr{F}(f)=\widehat{f}$ denote the Fourier transform of f in x; that is,

$$\mathscr{F}(f)(\xi,t)=\widehat{f}(\xi,t)=\frac{1}{\sqrt{2\pi}^{\,n}}\int_{\mathbb{R}^n} f(x,t)e^{-ix\cdot\xi}\,dx\,.$$

Then by assuming that $\widehat{u}(\cdot,t)$ exists for all $t>0$, Lemma 3.11 implies that

$$\begin{aligned}\mathscr{F}(\Delta u)(\xi,t)&=\mathscr{F}\Big(\sum_{k=1}^n\frac{\partial}{\partial x_k}\frac{\partial u}{\partial x_k}\Big)(\xi,t)=\sum_{k=1}^n\mathscr{F}\Big(\frac{\partial}{\partial x_k}\frac{\partial u}{\partial x_k}\Big)(\xi,t)\\&=\sum_{k=1}^n i\xi_k\mathscr{F}\Big(\frac{\partial u}{\partial x_k}\Big)(\xi,t)=\sum_{k=1}^n(i\xi_k)^2\widehat{u}(\xi,t)=-|\xi|^2\widehat{u}(\xi,t)\,.\end{aligned}$$

Assume further that

$$\begin{aligned}\frac{\partial}{\partial t}\widehat{u}(\xi,t)&=\frac{\partial}{\partial t}\frac{1}{\sqrt{2\pi}^{\,n}}\int_{\mathbb{R}^n}u(x,t)e^{-ix\cdot\xi}\,dx=\frac{1}{\sqrt{2\pi}^{\,n}}\int_{\mathbb{R}^n}\frac{\partial}{\partial t}u(x,t)e^{-ix\cdot\xi}\,dx\\&=\frac{1}{\sqrt{2\pi}^{\,n}}\int_{\mathbb{R}^n}u_t(x,t)e^{-ix\cdot\xi}\,dx=\widehat{u_t}(\xi,t)\,.\end{aligned}$$

Taking the Fourier transform of (5.11a), we find that

$$\frac{\partial}{\partial t}\widehat{u}(\xi,t) + |\xi|^2\widehat{u}(\xi,t) = \mathscr{F}(u_t - \Delta u)(\xi,t) = 0\,. \tag{5.12}$$

Since $\widehat{u}(\xi,0) = \mathscr{F}(u(\cdot,0))(\xi) = \widehat{u_0}(\xi)$, solving the ODE (5.12) with this initial condition we obtain that

$$\widehat{u}(\xi,t) = \widehat{u_0}(\xi)e^{-|\xi|^2 t} = \widehat{u_0}(\xi)\widehat{P_{2t}}(\xi)\,,$$

where $P_t(x) = \dfrac{1}{\sqrt{t}^n}e^{-\frac{|x|^2}{2t}}$. By Theorem 3.26, we conclude that

$$u(x,t) = (u_0 * P_{2t})(x) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\frac{1}{\sqrt{2t}^n}e^{-\frac{|x-y|^2}{4t}}u_0(y)\,dy = \frac{1}{\sqrt{4\pi t}^n}\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{4t}}u_0(y)\,dy\,. \tag{5.13}$$

This induces the following

Definition 5.1. The function $\mathcal{H}(x,t) = \dfrac{1}{\sqrt{4\pi t}^n}e^{-\frac{|x|^2}{4t}}$ is called the ***heat kernel***.

Having introduced the heat kernel, the solution to (5.11), given by (5.13) can be expressed by

$$u(x,t) = \big(\mathcal{H}(\cdot,t)*u_0\big)(x)\,.$$

• Non-uniqueness of solutions: The Fourier transform method only picks up solutions whose Fourier transform is defined, and it is possible that there are other solutions to (5.11). Consider the function

$$u(x,t) = \sum_{k=0}^{\infty}\frac{g^{(k)}(t)}{(2k)!}x^{2k}\,, \tag{5.14}$$

where g is given by

$$g(t) = \begin{cases} \exp(-t^{-2}) & \text{if } t > 0\,,\\ 0 & \text{if } t = 0\,.\end{cases}$$

Then there exists $\theta > 0$ such that

$$\big|g^{(k)}(t)\big| \leqslant \frac{k!}{(\theta t)^k}\exp\big(-\frac{1}{2}t^{-2}\big) \qquad \forall\, t > 0\,. \tag{5.15}$$

In fact, using the Cauchy integral formula,

$$g^{(k)}(t) = \frac{k!}{2\pi i}\oint_{|z-t|=\theta t}\frac{g(z)}{(z-t)^{k+1}}\,dz$$

where $\theta \in (0,1)$ is chosen so small such that $\big|g(z)\big| \leqslant \exp\big(-\frac{1}{2}t^{-2}\big)$ on $|z-t| = \theta t$. The choice of such a θ is possible since by writing $z = x + iy$,

$$\big|g(x+iy)\big| = \Big|\exp\big(\frac{y^2-x^2+2ixy}{(x^2+y^2)^2}\big)\Big| = \exp\big(\frac{y^2-x^2}{(x^2+y^2)^2}\big) \leqslant \exp\big(\frac{\theta^2-(1-\theta)^2}{((1+\theta)^2+\theta^2)^2}t^{-2}\big) \text{ if } \theta < \frac{1}{2}\,.$$

By the fact that $\dfrac{k!}{(2k)!} \leqslant \dfrac{1}{k!}$, we find that

$$\sum_{k=0}^{\infty} \Big| \frac{g^{(k)}(t)}{(2k)!} x^{2k} \Big| \leqslant \sum_{k=0}^{\infty} \frac{x^{2k}}{k!(\theta t)^k} \exp\big(-\frac{1}{2}t^{-2}\big) = \exp\Big[\frac{1}{t}\big(\frac{x^2}{\theta} - \frac{1}{2}t^{-2}\big)\Big] \quad \forall\, t > 0, x \in \mathbb{R}\,. \tag{5.16}$$

Therefore, by comparison the series in (5.14) converges for all $t > 0$, and trivially also converges for $t = 0$. Moreover, (5.16) shows that for each $t \in \mathbb{R}$, (5.14) is a convergent power series; thus

$$u_{xx}(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} \frac{\partial^2 x^{2k}}{\partial x^2} = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2} \qquad \forall\, t > 0, x \in \mathbb{R}\,.$$

On the other hand, by the fact that $\dfrac{(k+1)!}{(2k)!} \leqslant \dfrac{1}{(k-1)!}$ if $k \geqslant 1$, using (5.15) we find that for all $t > 0$ and $x \in \mathbb{R}$,

$$\begin{aligned}\sum_{k=0}^{\infty} \Big| \frac{g^{(k+1)}(t)}{(2k)!} x^{2k} \Big| &\leqslant |g'(t)| + \sum_{k=1}^{\infty} \frac{x^{2k}}{(k-1)!(\theta t)^{k+1}} \exp\big(-\frac{1}{2}t^{-2}\big) \\ &= |g'(t)| + \frac{x^2}{(\theta t)^2} \exp\Big[\frac{1}{t}\big(\frac{x^2}{\theta} - \frac{1}{2}t^{-2}\big)\Big]\,.\end{aligned}$$

Therefore, the series $\sum\limits_{k=0}^{\infty} \dfrac{g^{(k+1)}(t)}{(2k)!} x^{2k}$ converges uniformly on any bounded set of $\mathbb{R}$; thus

$$u_t(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)}{(2k)!} x^{2k} = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2} = u_{xx}(x,t) \qquad \forall\, t > 0, x \in \mathbb{R}\,.$$

This implies that u satisfies the heat equation

$$\begin{aligned} u_t - u_{xx} &= 0 \qquad \text{in} \quad \mathbb{R} \times (0,\infty)\,, \\ u &= 0 \qquad \text{on} \quad \mathbb{R} \times \{t = 0\}\,. \end{aligned}$$

Note that using the Fourier transform method to solve the PDE above we obtain trivial solution. The reason for not seeing the solution given by (5.14) using the Fourier transform method is that the Fourier transform of the function u given by (5.14) does not exists.

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