Fourier Analysis

富氏分析

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Chapter 4

Application on Signal Processing

In the study of signal processing, the Fourier transform and the inverse Fourier transform are often defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i2\pi x\cdot\xi} dx \quad \text{and} \quad \check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{i2\pi x\cdot\xi} d\xi \qquad \forall\, f \in L^1(\mathbb{R}^n)\,. \qquad (3.11).$$

Then for $T \in \mathscr{S}(\mathbb{R}^n)'$, the Fourier transform of T is defined again by

$$\langle \widehat{T}, \phi \rangle = \langle T, \widehat{\phi} \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

We also note that the definitions of the translation, dilation, and reflection of tempered distributions are independent of the Fourier transform, and are still defined by

$$\langle \tau_h T, \phi \rangle = \langle T, \tau_{-h}\phi \rangle\,, \quad \langle d_\lambda T, \phi \rangle = \langle T, \lambda^n d_{\lambda^{-1}}\phi \rangle \quad \text{and} \quad \langle \widetilde{T}, \phi \rangle = \langle T, \widetilde{\phi} \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

Concerning the convolution, when the Fourier transform is given by (3.11), we usually consider the $*$ convolution operator

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x-y)\, dy = \int_{\mathbb{R}^n} f(x-y) g(y)\, dy \qquad \forall\, f, g \in L^1(\mathbb{R}^n)\,.$$

instead of $*$ convolution operators. The convolution of T and $f \in \mathscr{S}(\mathbb{R}^n)$ is defined by

$$\langle T * f, \phi \rangle = \langle T, \widetilde{f} * \phi \rangle = \langle \widetilde{T}, f * \widetilde{\phi} \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

Then similar to Theorem 3.51, 3.53, and 3.58, we have

1. $\check{\widehat{T}} = \widehat{\check{T}} = T$ for all $T \in \mathscr{S}(\mathbb{R}^n)'$.

2. $\widehat{\tau_h T}(\xi) = \widehat{T}(\xi)e^{-2\pi i\xi\cdot h}$, $\widehat{d_\lambda T}(\xi) = \lambda^n \widehat{T}(\lambda\xi)$, and $\widehat{\widehat{T}}(\xi) = \check{T}(\xi)$ for all $T \in \mathscr{S}(\mathbb{R}^n)'$.

3. $\widehat{T * f} = \widehat{T}\widehat{f}$ and $\widehat{fT} = \widehat{f} * \widehat{T}$ for all $f \in \mathscr{S}(\mathbb{R}^n)$ and $T \in \mathscr{S}(\mathbb{R}^n)'$. Moreover, if $S \in \mathscr{S}(\mathbb{R}^n)'$ has the property that $S * \phi \in \mathscr{S}(\mathbb{R}^n)$ for all $\phi \in \mathbb{R}^n$, then $\widehat{T * S} = \widehat{T}\widehat{S}$ in $\mathscr{S}(\mathbb{R}^n)'$ for all $T \in \mathscr{S}(\mathbb{R}^n)'$.

Moreover,

1. $\widehat{\delta} = \check{\delta} = 1$ in $\mathscr{S}(\mathbb{R}^n)'$, and $\widehat{\delta_h}(\xi) = \widehat{\tau_h \delta}(\xi) = \check{\delta_{-h}} = \widecheck{\tau_{-h}\delta} = e^{-2\pi i h\cdot\xi}$ in $\mathscr{S}(\mathbb{R}^n)'$ for all $h \in \mathbb{R}^n$.

2. By Euler's identity, $\widehat{\cos(2\pi\omega x)}(\xi) = \dfrac{1}{2}(\delta_\omega + \delta_{-\omega})$ and $\widehat{\sin(2\pi\omega x)}(\xi) = \dfrac{1}{2i}(\delta_\omega - \delta_{-\omega})$.

3. $\delta * \delta = \delta$, and $\delta_a * \delta_b = \delta_{a+b}$ for all $a, b \in \mathbb{R}^n$.

4. $\delta * \phi = \phi$ and $(\delta_a * \phi)(x) = \phi(x-a)$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$.

5. Re-define the rect function $\Pi : \mathbb{R} \to \mathbb{R}$ by

$$\Pi(x) = \begin{cases} 1 & \text{if } |x| < \dfrac{1}{2}\,, \\ 0 & \text{if } |x| \geqslant \dfrac{1}{2}\,. \end{cases} \tag{4.1}$$

Then $\widehat{\Pi}(\xi) = \check{\Pi}(\xi) = \operatorname{sinc}(\xi)$, where sinc is the normalized sinc function given by (3.12).

6. Let $\Lambda : \mathbb{R} \to \mathbb{R}$ be the triangle function define by

$$\Lambda(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1\,, \\ 0 & \text{if } |x| \geqslant 1\,. \end{cases}$$

Then by the fact that Λ is an even function, if $\xi \neq 0$,

$$\begin{aligned} \widehat{\Lambda}(\xi) &= 2\int_0^1 (1-x)\cos(2\pi x\xi)\,dx = 2\Big[(1-x)\frac{\sin(2\pi x\xi)}{2\pi\xi}\Big|_{x=0}^{x=1} + \int_0^1 \frac{\sin(2\pi x\xi)}{2\pi\xi}\,dx\Big] \\ &= \frac{1-\cos(2\pi\xi)}{2\pi^2\xi^2} = \frac{\sin^2 \pi\xi}{\pi^2\xi^2}\,, \end{aligned}$$

while $\widehat{\Lambda}(0) = 1$. Therefore, $\widehat{\Lambda}(\xi) = \operatorname{sinc}^2(\xi)$. Using the property of convolution, we have $\Pi * \Pi = \Lambda$.

4.1 The Sampling Theorem and the Nyquist Rate

When a continuous function, $x(t)$, is sampled at a constant rate f_s samples per second（以每秒 f_s 次取樣）, there is always an unlimited number of other continuous functions that fit the same set of samples; however, only one of them is bandlimited to $\frac{1}{2}f_s$ cycles per second (hertz), which means that its Fourier transform, $\widehat{x}(f)$, is 0 for all $|f| \geqslant \frac{1}{2}f_s$.

Definition 4.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. f is said to be a ***bandlimited*** function if $\operatorname{spt}(\widehat{f})$ is bounded. The ***bandwidth*** of a bandlimited function f is the number $\sup \operatorname{spt}(\widehat{f})$. f is said to be ***timelimited*** if $\operatorname{spt}(f)$ is bounded.

Definition 4.2. In signal processing, the ***Nyquist rate*** is twice the bandwidth of a bandlimited function or a bandlimited channel.

In the field of digital signal processing, the ***sampling theorem*** is a fundamental bridge between continuous-time signals (often called "analog signals") and discrete-time signals (often called "digital signals"). It establishes a sufficient condition for a *sample rate*（取樣頻率）that permits a discrete sequence of samples to capture all the information from a continuous-time signal of finite bandwidth. To be more precise, Shannon's version of the theorem states that "if a function $x(t)$ contains no frequencies higher than B hertz, it is completely determined by giving its ordinates at a series of points spaced $\frac{1}{2B}$ seconds apart."

Let us start from the following famous Poisson summation formula to demonstrate why countable sampling is possible to reconstruct the full signal.

Lemma 4.3 (Poisson summation formula)**.** *Let the Fourier transform and the inverse Fourier transform be defined by* (3.11)*. Then*

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty} \widehat{f}(k)e^{2\pi ikx} \qquad \forall\, f \in \mathscr{S}(\mathbb{R})\,. \tag{4.2}$$

The convergences on both sides are uniform.

Proof. Let $f \in \mathscr{S}(\mathbb{R})$ be given. Then there exists $C > 0$ such that

$$\big|f(x)\big| + \big|f'(x)\big| \leqslant \frac{C}{1+|x|^2} \qquad \forall\, x \in \mathbb{R}.$$

Define $F(x) = \sum\limits_{n=-\infty}^{\infty} f(x+n)$. Then for $x \in [-1,1]$,

$$\big|f(x+n)\big| + \big|f'(x+n)\big| \leqslant \frac{2C}{1+(|n|-1)^2} \qquad \forall\, n \in \mathbb{Z}\,.$$

By the fact that

$$\sum_{n=0}^{\infty} \frac{C}{1+(|n|-1)^2} < \infty \quad \text{and} \quad \sum_{n=-\infty}^{-1} \frac{C}{1+|1+n|^2} < \infty\,,$$

the Weierstrass M-test implies that the series $\sum\limits_{n=-\infty}^{\infty} f(x+n)$ and $\sum\limits_{n=-\infty}^{\infty} f'(x+n)$ both converge uniformly on $[0,1]$. Therefore, $F : [0,1]$ is differentiable. Note that $F(x) = F(x+1)$, so F has period 1.

Since $F \in \mathscr{C}^1(\mathbb{R})$ and is periodic with period 1, Theorem 2.17 implies that

$$F(x) = \sum_{k=-\infty}^{\infty} \widehat{F}_k e^{2\pi ikx} \qquad \forall\, x \in \mathbb{R}\,, \tag{4.3}$$

where $\{\widehat{F}_k\}_{k=-\infty}^{\infty}$ are the Fourier coefficients of F given by $\widehat{F}_k = \displaystyle\int_0^1 F(x)e^{-2\pi ikx}\,dx$. By the uniform convergence of $\sum\limits_{n=-\infty}^{\infty} f(x+n)$ in $[0,1]$, we find that

$$\widehat{F}_k = \sum_{n=-\infty}^{\infty} \int_0^1 f(x+n)e^{-2\pi ikx}\,dx = \sum_{n=-\infty}^{\infty} \int_n^{n+1} f(x)e^{-2\pi ik(x-n)}\,dx = \int_{\mathbb{R}} f(x)e^{-2\pi ikx}\,dx = \widehat{f}(k)\,.$$

The Poisson summation formula (4.2) then follows from (4.3) and the identity above. □

Remark 4.4. Using Definition 3.3 of the Fourier transform, for $f \in \mathscr{S}(\mathbb{R})$ one has

$$\sum_{n=-\infty}^{\infty} f(x+2n\pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx}\,.$$

Corollary 4.5. *Let the Fourier transform and the inverse Fourier transform be defined by* (3.11). *Then*

$$\sum_{k=-\infty}^{\infty} \widehat{f}(\xi - \frac{k}{T}) = T \sum_{n=-\infty}^{\infty} f(nT)e^{-i2\pi nT\xi} \qquad \forall\, f \in \mathscr{S}(\mathbb{R})\,. \tag{4.4}$$

Proof. For a given $g \in \mathscr{S}(\mathbb{R})$, let $h = d_\lambda g$, where d_λ is a dilation operator. Then h is also a Schwartz function, and

$$\widehat{h}(\xi) = (\lambda d_{\lambda^{-1}} \widehat{g})(\xi) = \lambda \widehat{g}(\lambda \xi);$$

thus the Poisson summation formula (4.2) (with $x = 0$ and $f = g$) implies that

$$\sum_{n=-\infty}^{\infty} h(n\lambda) = \sum_{n=-\infty}^{\infty} g(n) = \sum_{k=-\infty}^{\infty} \widehat{g}(k) = \frac{1}{\lambda} \sum_{k=-\infty}^{\infty} \widehat{h}\big(\frac{k}{\lambda}\big). \tag{4.5}$$

Now let $s = \tau_t h$ for some $t \in \mathbb{R}$, where τ_t is a translation operator. Then $s \in \mathscr{S}(\mathbb{R})$, and

$$\widehat{s}(\xi) = \widehat{h}(\xi) e^{-2\pi i t \xi}.$$

Therefore, (4.5) implies that

$$\sum_{n=-\infty}^{\infty} s(t + n\lambda) = \frac{1}{\lambda} \sum_{k=-\infty}^{\infty} \widehat{s}\big(\frac{k}{\lambda}\big) e^{\frac{2\pi i k t}{\lambda}}. \tag{4.6}$$

Finally, for $f \in \mathscr{S}(\mathbb{R})$, let $s = \check{f}$. Then using $\check{f} = \widetilde{\widehat{f}}$, $\lambda = \dfrac{1}{T}$ and $t = -\xi$ in the identity above, we obtain that

$$\sum_{k=-\infty}^{\infty} \widehat{f}(\xi - \frac{k}{T}) = \sum_{k=-\infty}^{\infty} \widehat{f}(\xi + \frac{k}{T}) = \sum_{k=-\infty}^{\infty} \check{f}(-\xi - \frac{k}{T}) = T \sum_{k=-\infty}^{\infty} f(kT) e^{-2\pi i k T \xi}$$

which shows (4.4). □

Remark 4.6. Identity (4.4) can be shown to hold for all continuous function f satisfying

$$\big|f(x)\big| + \big|\widehat{f}(x)\big| \leqslant \frac{C}{(1 + |x|)^{1+\delta}} \qquad \forall\, x \in \mathbb{R}$$

for some C, $\delta > 0$. Therefore, if $\widehat{f}$ has compact support, as long as the decay rate of f is bigger than 1, (4.4) is a valid identity.

A direct consequence of the corollary above is the following sampling theorem. Suppose that $f \in \mathscr{S}(\mathbb{R})$ and $\mathrm{spt}(\widehat{f}) \subseteq \big[0, \frac{1}{T}\big]$. Then (4.4) implies that

$$\widehat{f}(\xi) = \sum_{n=-\infty}^{\infty} f(nT) e^{-i2\pi n T \xi} \qquad \forall\, \xi \in \big[0, \frac{1}{T}\big].$$

This shows that if $\widehat{f}$ has compact support in $\big[0, \frac{1}{T}\big]$, f can be reconstructed based on partial knowledge of f, namely $f(nT)$.

Recall that the Fourier transform of the sine wave with frequency ω is "supported" in a symmetric domain $\{\omega, -\omega\}$. Therefore, in reality it is better to assume that the Fourier transform of a bandlimited signal is supported in a symmetric domain $[-B, B]$. In such a case we need $\left|\xi - \frac{k}{T}\right| \geqslant B$ for all $k \in \mathbb{Z}\backslash\{0\}$ and $\xi \in [-B, B]$, where T is the sampling frequency, in order to make use of (4.4) to gain all the information of the Fourier transform of the signal. Therefore, the sampling frequency T has to obey $\frac{1}{T} \geqslant 2B$ or $T \leqslant \frac{1}{2B}$ in order to gain the Fourier transform of the bandlimited signal.

Theorem 4.7 (Sampling theorem)**.** *If a (Schwartz) function f contains no frequencies higher than B hertz, it is completely determined by giving its ordinates at a series of points spaced $\frac{1}{2B}$ seconds apart.*

Alternative proof of Theorem 4.7. By the Fourier inversion formula,

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{2\pi i x \xi}\, d\xi\,, \qquad \text{where} \quad \widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi}\, dt\,.$$

By assumption, $\operatorname{spt}(\widehat{f}) \subseteq [-B, B]$; thus $f(x) = \int_{-B}^{B} \widehat{f}(\xi) e^{2\pi i x \xi}\, d\xi$ which implies that

$$f\Big(\frac{k}{2B}\Big) = \int_{-B}^{B} \widehat{f}(\xi) e^{\frac{i\pi k \xi}{B}}\, d\xi\,.$$

Treating $\widehat{f}$ as a function defined on $[-B, B]$, the identity above implies that $\Big\{\frac{1}{2B} f\Big(\frac{-k}{2B}\Big)\Big\}_{k=-\infty}^{\infty}$ is the Fourier coefficients of $\widehat{f}$ and

$$\widehat{f}(\xi) = \sum_{k=-\infty}^{\infty} \frac{1}{2B} f\Big(\frac{-k}{2B}\Big) e^{\frac{i\pi k \xi}{B}} = \sum_{k=-\infty}^{\infty} \frac{1}{2B} f\Big(\frac{k}{2B}\Big) e^{-\frac{i\pi k \xi}{B}} \qquad \forall\, \xi \in [-B, B] \tag{4.7}$$

which, together with the fact that $\widehat{f} = 0$ outside $[-B, B]$, allows us to reconstruct f using the Fourier inversion formula. □

Taking the Fourier inverse transform of $\widehat{f}(\xi)$ obtained by (4.7), we find that

$$\begin{aligned} f(x) &= \sum_{k=-\infty}^{\infty} \frac{1}{2B} f\Big(\frac{k}{2B}\Big) \int_{-B}^{B} e^{2\pi i x \xi - \frac{i\pi k \xi}{B}}\, d\xi = \sum_{k=-\infty}^{\infty} \frac{1}{2B} f\Big(\frac{k}{2B}\Big) \int_{-B}^{B} \cos\Big(\frac{2\pi B x - \pi k}{B}\xi\Big)\, d\xi \\ &= \sum_{k=-\infty}^{\infty} f\Big(\frac{k}{2B}\Big) \frac{\sin \pi(2Bx - k)}{\pi(2Bx - k)}\,. \end{aligned}$$

Using the normalized sinc function defined by (3.12), we recover the so-called ***Whittaker–Shannon interpolation formula***:

$$f(x) = \sum_{k=-\infty}^{\infty} f\big(\frac{k}{2B}\big)\operatorname{sinc}(2Bx-k) \qquad \forall\, f \in \mathscr{S}(\mathbb{R}) \text{ with } \operatorname{spt}(\widehat{f}) \subseteq [-B,B]\,. \tag{4.8}$$

In the following, we examine the Whittaker–Shannon interpolation formula (4.8) for the case that $f \notin \mathscr{S}(\mathbb{R})$. In fact, since

$$\int_{\mathbb{R}} \operatorname{sinc}(2Bx-k)\phi(x)\,dx = \int_{\mathbb{R}} \big(d_{\frac{1}{2B}}\operatorname{sinc}\big)\big(\frac{k}{2B}-x\big)\phi(x)\,dx = \big[\big(d_{\frac{1}{2B}}\operatorname{sinc}\big)*\phi\big]\big(\frac{k}{2B}\big)\,,$$

instead of (4.8) we show that

$$\langle f,\phi\rangle = \sum_{k=-\infty}^{\infty} f\big(\frac{k}{2B}\big)\big[\big(d_{\frac{1}{2B}}\operatorname{sinc}\big)*\phi\big]\big(\frac{k}{2B}\big) = \sum_{k=-\infty}^{\infty} \big\langle f\tau_{\frac{k}{2B}}\delta, \big(d_{\frac{1}{2B}}\operatorname{sinc}\big)*\phi\big\rangle\,. \tag{4.9}$$

Suppose that $1 \leqslant p \leqslant \infty$ and $g:\mathbb{R}\to\mathbb{R}$ is an L^p-function $\Big($that is, $\int_{\mathbb{R}} |g(x)|^p\,dx < \infty$ if $1 \leqslant p < \infty$ or g is bounded if $p=\infty\Big)$ supported in an open interval of length $2B$ (later we will let g be the Fourier transform of a bandlimited signal f; so it is reasonable to assume that g is compactly supported). Define

$$G(x) = \sum_{n=-\infty}^{\infty} g(x+2Bn) = \sum_{n=-\infty}^{\infty} (\tau_{-2Bn}g)(x)\,. \tag{4.10}$$

Let q satisfies $\frac{1}{p}+\frac{1}{q}=1$ and $\phi\in\mathscr{S}(\mathbb{R})$. The monotone convergence theorem shows that

$$\begin{aligned}\int_{\mathbb{R}} \sum_{n=-\infty}^{\infty} \big|(\tau_{-2Bn}g)(x)\phi(x)\big|\,dx &= \sum_{n=-\infty}^{\infty} \int_{-(2n+1)B}^{-(2n-1)B} \big|(\tau_{-2Bn}g)(x)\big|\langle x\rangle^{-2}\langle x\rangle^{2}\big|\phi(x)\big|\,dx \\ &\leqslant \sum_{n=-\infty}^{\infty} \int_{-(2n+1)B}^{-(2n-1)B} \big|g(x+2Bn)\big|\langle x\rangle^{-2} p_2(\phi)\,dx = p_2(\phi)\sum_{n=-\infty}^{\infty}\int_{-B}^{B}\big|g(x)\big|\langle x-2Bn\rangle^{-2}\,dx\,.\end{aligned}$$

If $1<p<\infty$, Hölder's inequality implies that

$$\begin{aligned}\int_{\mathbb{R}} \sum_{n=-\infty}^{\infty} \big|(\tau_{-2Bn}g)(x)\langle x\rangle^{-2}\big|\,dx &\leqslant \sum_{n=-\infty}^{\infty}\int_{-B}^{B}\big|g(x)\big|\langle x-2Bn\rangle^{-2}\,dx \\ &\leqslant \sum_{n=-\infty}^{\infty}\Big(\int_{-B}^{B}|g(x)|^p\,dx\Big)^{\frac{1}{p}}\Big(\int_{-B}^{B}\frac{dx}{(1+|x-2Bn|^2)^q}\Big)^{\frac{1}{q}} \\ &\leqslant \|g\|_{L^p(\mathbb{R})}\sum_{n=-\infty}^{\infty}\frac{(2B)^{\frac{1}{q}}}{1+(2|n|-1)^2B^2} < \infty\end{aligned}$$

while if $p = 1$,

$$\sum_{n=-\infty}^{\infty} \int_{-B}^{B} |g(x)| \langle x - 2Bn \rangle^{-2}\, dx \leqslant \|g\|_{L^1(\mathbb{R})} \sum_{n=-\infty}^{\infty} \frac{1}{1 + (2|n| - 1)^2 B^2} < \infty\,;$$

thus if $1 \leqslant p < \infty$,

$$\int_{\mathbb{R}} \sum_{n=-\infty}^{\infty} \big|(\tau_{-2Bn} g)(x)\phi(x)\big|\, dx \leqslant C_\ell \|g\|_{L^p(\mathbb{R})} p_\ell(\phi) \qquad \forall\, \ell \gg 1 \tag{4.11}$$

for some constant $C_\ell > 0$. On the other hand, if $p = \infty$,

$$\int_{\mathbb{R}} \sum_{n=-\infty}^{\infty} \big|(\tau_{-2Bn} g)(x)\phi(x)\big|\, dx \leqslant \|g\|_\infty \|\phi\|_{L^1(\mathbb{R})} \leqslant C_\ell \|g\|_\infty p_\ell(\phi) \qquad \forall\, \ell \gg 1\,. \tag{4.12}$$

Therefore, $G \in \mathscr{S}(\mathbb{R})'$ since

$$\big|\langle G, \phi \rangle\big| \leqslant \int_{\mathbb{R}} \sum_{n=-\infty}^{\infty} \big|(\tau_{-2Bn} g)(x)\phi(x)\big|\, dx \leqslant C_\ell p_\ell(\phi) \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}) \text{ and } \ell \gg 1\,.$$

Moreover, it follows from (4.11) and (4.12) that $\sum\limits_{n=-\infty}^{\infty} \big|(\tau_{-2Bn} g)\phi\big| \in L^1(\mathbb{R})$. By the fact that

$$\Big|\sum_{n=-k}^{k} (\tau_{-2Bn} g)(x)\phi(x)\Big| \leqslant \sum_{n=-\infty}^{\infty} \big|(\tau_{-2Bn} g)(x)\phi(x)\big| \qquad \forall\, x \in \mathbb{R}\,,$$

the dominated convergence theorem implies that

$$\begin{aligned}\langle G, \phi \rangle &= \int_{\mathbb{R}} \lim_{k\to\infty} \sum_{n=-k}^{k} (\tau_{-2Bn} g)(x)\phi(x)\, dx = \lim_{k\to\infty} \int_{\mathbb{R}} \sum_{n=-k}^{k} (\tau_{-2Bn} g)(x)\phi(x)\, dx \\ &= \sum_{n=-\infty}^{\infty} \langle \tau_{-2Bn} g, \phi \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R})\,.\end{aligned}$$

Suppose that $\text{spt}(g) \subseteq (a - B, a + B)$. Then $G = g$ on $(a - B, a + B)$. In addition, if $x \in [a + (k-1)B, a + (k+1)B]$, then $G(x) = g(x - kB)$; thus $G(x + 2B) = G(x)$ for all $x \in \mathbb{R}$. In other words, G can be viewed as the $2B$-periodic extension of non-vanishing part of g.

Let $\phi \in \mathscr{S}(\mathbb{R})$. By the definition of the inverse Fourier transform of tempered distributions,

$$\langle \check{G}, \phi \rangle = \langle G, \check{\phi} \rangle = \sum_{n=-\infty}^{\infty} \langle \tau_{-2Bn} g, \check{\phi} \rangle = \sum_{n=-\infty}^{\infty} \langle g, \tau_{2Bn} \check{\phi} \rangle\,.$$

By the Poisson summation formula,

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(\tau_{2Bn}\check{\phi})(x) &= \sum_{n=-\infty}^{\infty}\check{\phi}(x-2Bn) = \sum_{n=-\infty}^{\infty}\check{\phi}(x+2Bn) = \sum_{n=-\infty}^{\infty}(d_{\frac{1}{2B}}\check{\phi})\big(\frac{x}{2B}+n\big)\\ &= \sum_{k=-\infty}^{\infty}\widehat{d_{\frac{1}{2B}}\check{\phi}}(k)e^{\frac{\pi ikx}{B}} = \sum_{k=-\infty}^{\infty}\frac{1}{2B}\phi\big(\frac{k}{2B}\big)e^{\frac{\pi ikx}{B}}\end{aligned}$$

and the convergence is uniform. Therefore,

$$\begin{aligned}\langle\check{G},\phi\rangle &= \sum_{n=-\infty}^{\infty}\langle g,\tau_{2Bn}\check{\phi}\rangle = \sum_{k=-\infty}^{\infty}\frac{1}{2B}\phi\big(\frac{k}{2B}\big)\int_{\mathrm{spt}(g)}g(x)e^{\frac{\pi ikx}{B}}\,dx = \sum_{k=-\infty}^{\infty}\frac{1}{2B}\phi\big(\frac{k}{2B}\big)\int_{\mathbb{R}}g(x)e^{\frac{\pi ikx}{B}}\,dx\\ &= \sum_{k=-\infty}^{\infty}\frac{1}{2B}\phi\big(\frac{k}{2B}\big)\check{g}\big(\frac{k}{2B}\big) = \frac{1}{2B}\sum_{k=-\infty}^{\infty}\check{g}\big(\frac{k}{2B}\big)\langle\tau_{\frac{k}{2B}}\delta,\phi\rangle.\end{aligned}$$

Similarly, $\langle\widehat{G},\phi\rangle = \frac{1}{2B}\sum_{k=-\infty}^{\infty}\widehat{g}\big(\frac{k}{2B}\big)\langle\tau_{\frac{k}{2B}}\delta,\phi\rangle$ or one can use the formula that $\widehat{G} = \tilde{\check{G}}$ to deduce that

$$\langle\widehat{G},\phi\rangle = \langle\check{G},\tilde{\phi}\rangle = \sum_{k=-\infty}^{\infty}\frac{1}{2B}\tilde{\phi}\big(\frac{k}{2B}\big)\check{g}\big(\frac{k}{2B}\big) = \sum_{k=-\infty}^{\infty}\frac{1}{2B}\phi\big(\frac{k}{2B}\big)\widehat{g}\big(\frac{k}{2B}\big) = \frac{1}{2B}\sum_{k=-\infty}^{\infty}\widehat{g}\big(\frac{k}{2B}\big)\langle\tau_{\frac{k}{2B}}\delta,\phi\rangle.$$

Symbolically, we can write $\widehat{G} = \frac{1}{2B}\sum_{k=-\infty}^{\infty}\widehat{g}\big(\frac{k}{2B}\big)\tau_{\frac{k}{2B}}\delta$ and $\check{G} = \frac{1}{2B}\sum_{k=-\infty}^{\infty}\check{g}\big(\frac{k}{2B}\big)\tau_{\frac{k}{2B}}\delta$ in $\mathscr{S}(\mathbb{R})'$.

Remark 4.8. Let Ш denote the tempered distribution

$$\langle Ш,\phi\rangle = \sum_{n=-\infty}^{\infty}\phi(n) \qquad \forall\,\phi\in\mathscr{S}(\mathbb{R}).$$

We note that the sum above makes sense if $\phi\in\mathscr{S}(\mathbb{R})$, and

$$\sum_{n=-\infty}^{\infty}\phi(n) = \sum_{n=-\infty}^{\infty}\langle n\rangle^{-k}\langle n\rangle^{k}\phi(n) \leqslant \Big(\sum_{n=-\infty}^{\infty}\langle n\rangle^{-k}\Big)p_k(\phi) = C_k p_k(\phi) \qquad \forall\,k\geqslant 2.$$

Therefore, Ш is indeed a tempered distribution. Since $\phi(n) = \langle\tau_n\delta,\phi\rangle$, symbolically we also write $Ш = \sum_{n=-\infty}^{\infty}\tau_n\delta$.

By the definition of the Fourier transform of tempered distributions,

$$\langle\widehat{Ш},\phi\rangle = \langle Ш,\widehat{\phi}\rangle = \sum_{n=-\infty}^{\infty}\widehat{\phi}(n) \qquad \forall\,\phi\in\mathscr{S}(\mathbb{R}),$$

and the Poisson summation formula implies that

$$\langle \widehat{\text{Ш}}, \phi \rangle = \sum_{k=-\infty}^{\infty} \phi(k) = \langle \text{Ш}, \phi \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R})\,.$$

Therefore, Theorem 3.51 implies that $\widehat{\text{Ш}} = \check{\text{Ш}} = \text{Ш}$ in $\mathscr{S}(\mathbb{R})'$. Define $\text{Ш}_p = \dfrac{1}{p} d_p \text{Ш}$, where d_p is a dilation operator. Then

$$\langle \text{Ш}_p, \phi \rangle = \langle \text{Ш}, d_{p^{-1}}\phi \rangle = \sum_{n=-\infty}^{\infty} (d_{p^{-1}}\phi)(n) = \sum_{n=-\infty}^{\infty} \phi(pn) = \sum_{n=-\infty}^{\infty} \langle \tau_{pn}\delta, \phi \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R})\,. \tag{4.13}$$

Symbolically, $\text{Ш}_p = \sum\limits_{n=-\infty}^{\infty} \tau_{pn}\delta$. Moreover, $\widehat{\text{Ш}_p} = \widecheck{\text{Ш}_p} = d_{p^{-1}}\text{Ш} = \dfrac{1}{p}\text{Ш}_{\frac{1}{p}}$ which is the same as saying that

$$\langle \widecheck{\text{Ш}_p}, \phi \rangle = \langle d_{p^{-1}}\text{Ш}, \varphi \rangle = p^{-1}\langle \text{Ш}, d_p\varphi \rangle = \frac{1}{p}\sum_{n=-\infty}^{\infty} \langle \tau_{\frac{n}{p}}\delta, \phi \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R})\,.$$

Symbolically, $\widecheck{\text{Ш}_p} = \dfrac{1}{p}\sum\limits_{n=-\infty}^{\infty} \tau_{\frac{n}{p}}\delta$.

Formally speaking, G given by (4.10) can be expressed as $G = \text{Ш}_{2B} * g$. Using this representation, $\check{G} = \widecheck{\text{Ш}_{2B}}\check{g} = \dfrac{1}{2B}\sum\limits_{n=-\infty}^{\infty} \check{g}\tau_{\frac{n}{2B}}\delta$. Therefore, by the fact that $g = (d_{2B}\Pi)G$ in $\mathscr{S}(\mathbb{R})'$, we find that

$$\begin{aligned}
\check{g}(x) = (\widecheck{d_{2B}\Pi} * \check{G})(x) = \big[(2Bd_{\frac{1}{2B}}\check{\Pi}) * \check{G}\big](x) = 2B\int_{\mathbb{R}} \check{G}(y)\check{\Pi}(2B(x-y))\,dy \\
= \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} \check{g}(y)\tau_{\frac{n}{2B}}(y)\operatorname{sinc}(2B(x-y))\,dy = \sum_{n=-\infty}^{\infty} \check{g}\Big(\frac{n}{2B}\Big)\operatorname{sinc}(2Bx-n)\,.
\end{aligned}$$

The Whittaker–Shannon interpolation formula (4.8) then follows from letting $g = \widehat{f}$ in the identity above.

Example 4.9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function supported in $[0, T]$ (thus one can view f as a signal recorded in the time interval $[0, T]$). Define $F(x) = \sum\limits_{n=-\infty}^{\infty} f(x+nT)$. Then

$$\widehat{F}(\xi) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \widehat{f}\Big(\frac{k}{T}\Big)(\tau_{\frac{k}{T}}\delta)(\xi)\,.$$

On the other hand, the Fourier series of $\sum\limits_{n=-\infty}^{\infty} f(x+nT)$, the T-periodic extension of $f\mathbf{1}_{[0,T]}$, is

$$s(f,x) = \sum_{k=-\infty}^{\infty} \widehat{f}_k e^{\frac{2\pi ikx}{T}}\,, \text{ where } \widehat{f}_k = \frac{1}{T}\int_0^T f(x)e^{-\frac{2\pi ikx}{T}}\,dx = \frac{1}{T}\widehat{f}\Big(\frac{k}{T}\Big)\,.$$

Therefore, $\widehat{F} = \sum\limits_{k=-\infty}^{\infty} \widehat{f}_k \tau_{\frac{k}{T}} \delta$, and accordingly, $F(x) = \sum\limits_{k=-\infty}^{\infty} \widehat{f}_k e^{\frac{2\pi i k x}{T}}$ in $\mathscr{S}(\mathbb{R})'$.

Now suppose that the signal is sampled with sampling rate F_s (times per second). Then in total there are $N = TF_s$ samples of the signal. Write these samples as $\{x_0, x_1, \cdots, x_{N-1}\}$. Then $x_\ell = f\big(\frac{\ell}{F_s}\big)$. We remark that the set $\{x_0, x_1, \cdots, x_{N-1}\}$ resembles a digitalized version of the signal and is usually called a digital signal. The DFT of the digital signal is given by

$$X_k = \sum_{\ell=0}^{N-1} x_\ell e^{\frac{-2\pi i k \ell}{N}} = \sum_{\ell=0}^{N-1} f\big(\frac{\ell}{F_s}\big) e^{\frac{-2\pi i k}{T} \cdot \frac{\ell}{F_s}} \qquad \forall\, k \in \mathbb{Z}$$

and the inverse DFT of $\{X_k\}_{k\in\mathbb{Z}}$ is given by

$$x_\ell = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{\frac{2\pi i k \ell}{N}} = \frac{1}{N} \sum_{\ell=0}^{N-1} X_k e^{\frac{2\pi i k}{T} \cdot \frac{\ell}{F_s}} \qquad \forall\, \ell \in \mathbb{Z}\,.$$

4.1.1 The inner-product point of view

Let $e_k(x) = \text{sinc}(x-k) = (\tau_k \text{sinc})(x)$. Then $e_k \in L^2(\mathbb{R})$ since

$$\int_{\mathbb{R}} \text{sinc}^2(x-k)\, dx = \int_{\mathbb{R}} \text{sinc}^2 x dx = \int_{\mathbb{R}} \frac{\sin^2 \pi x}{\pi^2 x^2} dx = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\sin^2 x}{x^2} dx < \infty\,.$$

By the Plancherel formula (3.10),

$$(e_k, e_\ell)_{L^2(\mathbb{R})} s = \big(\widehat{\tau_k \text{sinc}}, \widehat{\tau_\ell \text{sinc}}\big)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \Pi(\xi) e^{2\pi i k \xi} \overline{\Pi(\xi) e^{2\pi i \ell \xi}}\, d\xi = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i (k-\ell)\xi}\, d\xi$$

which is 0 if $k \neq \ell$ and is 1 is $k = \ell$. Therefore, we find that $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{R})$.

Now suppose that $f \in L^2(\mathbb{R})$ such that $\text{spt}(\widehat{f}) \subseteq \big[-\frac{1}{2}, \frac{1}{2}\big]$. Then

$$\begin{aligned}(f, e_k)_{L^2(\mathbb{R})} &= \big(\widehat{f}, \widehat{\tau_k \sin c}\big)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \widehat{f}(\xi) \overline{\Pi(-\xi) e^{-2\pi i k \xi}}\, d\xi = \int_{-\frac{1}{2}}^{\frac{1}{2}} \widehat{f}(\xi) e^{2\pi i k \xi}\, d\xi \\ &= \int_{\mathbb{R}} \widehat{f}(\xi) e^{2\pi i k \xi}\, d\xi = \check{\widehat{f}}(k) = f(k)\end{aligned}$$

if f is continuous at k. In other words, if $f \in L^2(\mathbb{R}) \cap \mathscr{C}(\mathbb{R})$ such that $\text{spt}(\widehat{f}) \subseteq \big[-\frac{1}{2}, \frac{1}{2}\big]$, then

$$\sum_{k=-\infty}^{\infty} f(k) \text{sinc}(x-k) = \sum_{k=-\infty}^{\infty} (f, e_k)_{L^2(\mathbb{R})} e_k(x)$$

which, by the Whittaker–Shannon interpolation formula (4.8), further shows that

$$f = \sum_{k=-\infty}^{\infty} (f, e_k)_{L^2(\mathbb{R})} e_k \qquad \text{in} \quad \mathscr{S}(\mathbb{R})' .$$

In other words, one can treat $\{e_k\}_{k\in\mathbb{Z}}$ as an "orthonormal basis" in the space

$$\Big\{ f \in L^2(\mathbb{R}) \big(\cap \mathscr{C}(\mathbb{R}) \big) \,\Big|\, \mathrm{spt}(\widehat{f}) \subseteq \big[-\frac{1}{2}, \frac{1}{2} \big] \Big\} .$$

4.1.2 Sampling periodic functions

A bandlimited signal cannot be timelimited; thus when applying the sampling theorem, it always requires infinitely many sampling to construct the signal perfectly. On the other hand, it is possible that to construct a bandlimited signal perfectly using finitely many sampling provided that the bandlimited signal is periodic. In the following, we discuss why this is true.

Suppose that f is a q-periodic bandlimited signals such that $\mathrm{spt}(f) \subseteq [-B, B]$. Then the Whittaker–Shannon interpolation formula implies that

$$f(x) = \sum_{k=-\infty}^{\infty} f\big(\frac{k}{p}\big) \mathrm{sinc}(px - k)$$

as long as $p > 2B$. If $pq \in \mathbb{N}$, then by the fact that $f\big(\frac{k}{p}\big) = f\big(\frac{k + mpq}{p}\big)$ for all $m \in \mathbb{Z}$, the sum above can be re-grouped as

$$\begin{aligned} f(x) &= \sum_{\ell=0}^{pq-1} \sum_{m=-\infty}^{\infty} f\big(\frac{\ell + mpq}{p}\big) \mathrm{sinc}(px - \ell - mpq) \\ &= \sum_{\ell=0}^{pq-1} f\big(\frac{\ell}{p}\big) \sum_{m=-\infty}^{\infty} \mathrm{sinc}(px - \ell - mpq) \, ; \end{aligned} \tag{4.14}$$

thus if we can find the sum $\sum\limits_{m=-\infty}^{\infty} \mathrm{sinc}(px - \ell - mpq)$, f can be rewritten as finite sum.

Lemma 4.10. *If $p, q > 0$ and pq is an odd number, then*

$$\sum_{m=-\infty}^{\infty} \mathrm{sinc}(px - mpq) = \frac{\mathrm{sinc}(px)}{\mathrm{sinc}\frac{x}{q}} \qquad \forall\, x \in \mathbb{R} .$$

Proof. Note that

$$\sum_{m=-\infty}^{\infty} \operatorname{sinc}(px - mpq) = \sum_{m=-\infty}^{\infty} (d_{\frac{1}{p}}\operatorname{sinc})(x - mq) = (\text{Ш}_q * d_{\frac{1}{p}}\operatorname{sinc})(x)\,,$$

where $\text{Ш}_q = \frac{1}{q} d_q \text{Ш}$ is defined in Remark 4.8. Taking the Fourier transform, we find that

$$\mathscr{F}\big[(\text{Ш}_q * d_{\frac{1}{p}}\operatorname{sinc})\big](\xi) = \big(\widehat{\text{Ш}_q}\, \widehat{d_{\frac{1}{p}}\operatorname{sinc}}\big)(\xi) = \frac{1}{pq}\big(\text{Ш}_{\frac{1}{q}} d_p \Pi\big)(\xi) = \frac{1}{pq}(d_p\Pi)(\xi) \sum_{k=-\infty}^{\infty} \big(\tau_{\frac{k}{q}}\delta\big)(\xi)\,.$$

If pq is odd, then $\frac{p}{2} \neq \frac{k}{q}$ for all $k \in \mathbb{N}$; thus by the fact that $\operatorname{spt}(d_p\Pi) \subseteq \big[-\frac{p}{2}, \frac{p}{2}\big]$, we have

$$(d_p\Pi)(\xi) \sum_{k=-\infty}^{\infty} \big(\tau_{\frac{k}{q}}\delta\big)(\xi) = \sum_{-\frac{p}{2} < \frac{k}{q} < \frac{p}{2}} \big(\tau_{\frac{k}{q}}\delta\big)(\xi) = \sum_{k=-\frac{pq-1}{2}}^{\frac{pq-1}{2}} \big(\tau_{\frac{k}{q}}\delta\big)(\xi)\,.$$

Therefore,

$$\mathscr{F}\big[(\text{Ш}_q * d_{\frac{1}{p}}\operatorname{sinc})\big](\xi) = \frac{1}{pq} \sum_{k=-\frac{pq-1}{2}}^{\frac{pq-1}{2}} \big(\tau_{\frac{k}{q}}\delta\big)(\xi)\,.$$

Taking the inverse Fourier transform,

$$\sum_{m=-\infty}^{\infty} \operatorname{sinc}(px - mpq) = \mathscr{F}^*\mathscr{F}\big[(\text{Ш}_q * d_{\frac{1}{p}}\operatorname{sinc})\big](x) = \frac{1}{pq} \sum_{k=-\frac{pq-1}{2}}^{\frac{pq-1}{2}} e^{\frac{2\pi i k x}{q}} = \frac{1}{pq}\frac{\sin \pi p x}{\sin \frac{\pi x}{q}} = \frac{\operatorname{sinc}(px)}{\operatorname{sinc}\frac{x}{q}}\,. \quad \square$$

By Lemma 4.10, (4.14) implies that

$$\sum_{m=-\infty}^{\infty} \operatorname{sinc}(px - \ell - mpq) = \frac{\operatorname{sinc}(p(x - \frac{\ell}{p}))}{\operatorname{sinc}\frac{x - \frac{\ell}{p}}{q}} = \frac{\operatorname{sinc}(px - \ell)}{\operatorname{sinc}\frac{px-\ell}{pq}}\,;$$

thus we obtain that

$$f(x) = \sum_{\ell=0}^{pq-1} f\big(\frac{\ell}{p}\big) \frac{\operatorname{sinc}(px - \ell)}{\operatorname{sinc}\frac{px-\ell}{pq}}\,. \tag{4.15}$$

Example 4.11. Let $f(x) = \cos(2\pi x)$. Then f is 1-periodic and $\operatorname{spt}(\widehat{f}) \subseteq (-1-\epsilon, 1+\epsilon)$ for all $\epsilon > 0$. Letting $p = 3$ in (4.15), we find that

$$\cos(2\pi x) = \frac{\operatorname{sinc}(3x)}{\operatorname{sinc} x} + \cos\frac{2\pi}{3}\frac{\operatorname{sinc}(3x-1)}{\operatorname{sinc}\frac{3x-1}{3}} + \cos\frac{4\pi}{3}\frac{\operatorname{sinc}(3x-2)}{\operatorname{sinc}\frac{3x-2}{3}}\,.$$

4.2 Necessary Conditions for Sampling of Entire Functions

The sampling theorem provides a way of reconstructing signals based on sampled signals with sampling frequency larger than twice of the bandwidth of bandlimited signals. It is natural to ask whether we can reduce the sampling frequency for perfect reconstruction of bandlimited signals or not. Moreover, it is also possible that the support of the Fourier transform of a signal (usually called spectrum of the signal) is contained in a "small" portion of the interval $[-B, B]$, and in this case we hope to reduce the sampling frequency for the reconstruction of the signal.
Question: Is there a lower bound of the sampling frequency for perfect reconstruction of bandlimited signals?

Generally speaking, the way of sampling does not have to be uniform as long as the samples from a signal are enough to reconstruct the signal. A good choice of sampled set should obey

1. the signal is uniquely determined by the set of sampled signals - the ***uniqueness*** of the reconstruction of signals;

2. each set of sampled signals should come from a possible bandlimited signal - the ***existence*** of the reconstruction of signals.

These two requirements for sets on which the signals are sampled, together with the idea that the sampled set is not necessary uniform, induce the following

Definition 4.12. Let $S \subseteq \mathbb{R}^n$ be a measurable set in $\mathbb{R}^n$, and $\mathcal{B}(S)$ denote the subspace of $L^2(\mathbb{R}^n)$ consisting of those functions whose Fourier transform (given by (3.11)) is supported on S; that is,

$$\mathcal{B}(S) \equiv \left\{ f \in L^2(\mathbb{R}^n) \,\middle|\, \mathrm{spt}(\widehat{f}) \subseteq S \right\}.$$

A subset Λ of $\mathbb{R}^n$ is said to be ***uniformly discrete*** if the distance between any two distinct points of Λ exceeds some positive quantity; that is, there exists $\lambda_0 > 0$ such that $\|x - y\|_{\mathbb{R}^n} \geqslant \lambda_0$ for all $x, y \in \Lambda$ and $x \neq y$. Such a λ_0 is called a ***separation number***. A uniformly discrete set Λ is said to be

1. a ***set of sampling*** for $\mathcal{B}(S)$ if there exists a constant K such that

$$\|f\|^2_{L^2(\mathbb{R}^n)} \leqslant K \sum_{\lambda \in \Lambda} \left| f(\lambda) \right|^2 \qquad \forall\, f \in \mathcal{B}(S)\,;$$

2. a ***set of interpolation*** for $\mathcal{B}(S)$ if for each square-summable collection of complex numbers $\{a_\lambda\}_{\lambda\in\Lambda}$ there exists $f \in \mathcal{B}(S)$ with $f(\lambda) = a_\lambda$ for all $\lambda \in \Lambda$.

Example 4.13. Let I be an interval of length 1. Then $\mathbb{Z}$ is a set of sampling and interpolation for $\mathcal{B}(I)$:

1. If $f \in \mathcal{B}(I)$, then $\widehat{f}$ has the following Fourier series representation

$$\widehat{f}(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx} \qquad \text{for almost every } x \in I\,,$$

where $c_k = \displaystyle\int_I \widehat{f}(x) e^{-2\pi ikx}\,dx$. By the fact that $\text{spt}(\widehat{f}) \subseteq I$, the Fourier inversion formula implies that

$$c_k = \int_{\mathbb{R}} \widehat{f}(x) e^{-2\pi ikx}\,dx = f(-k)\,;$$

thus $\big\{f(-k)\big\}_{k=-\infty}^{\infty}$ is the Fourier coefficients of $\widehat{f}$. The Plancherel identity and the Parseval identity then imply that

$$\|f\|^2_{L^2(\mathbb{R})} = \|\widehat{f}\|^2_{L^2(\mathbb{R})} = \|\widehat{f}\|^2_{L^2(I)} = \sum_{k=-\infty}^{\infty} |c_k|^2 = \sum_{k=-\infty}^{\infty} \|f(k)\|^2\,;$$

thus $\mathbb{Z}$ is a set of sampling for $\mathcal{B}(I)$.

2. Let $\{c_k\}_{k=-\infty}^{\infty}$ be a square-summable sequence. Define

$$g(x) = \mathbf{1}_I(x) \sum_{k=-\infty}^{\infty} c_k e^{-2\pi ikx}\,.$$

Then the Parseval identity implies that

$$\|g\|^2_{L^2(\mathbb{R})} = \|g\|^2_{L^2(I)} = \sum_{k=-\infty}^{\infty} |c_k|^2 < \infty\,;$$

thus $g \in L^2(\mathbb{R})$. Let $f = \check{g}$. Then $f \in L^2(\mathbb{R})$ and the Fourier inversion formula implies that for all $k \in \mathbb{Z}$,

$$f(k) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{2\pi ik\xi}\,d\xi = \int_{\mathbb{R}} g(\xi) e^{2\pi ik\xi}\,d\xi = \int_I g(\xi) e^{2\pi ik\xi}\,d\xi = \widehat{g}_{-k} = c_k\,;$$

thus $\mathbb{Z}$ is a set of interpolation for $\mathcal{B}(I)$.

Remark 4.14. Suppose that f is an L^2-signal satisfying $\text{spt}(\widehat{f}) \subseteq (B-1, B)$ for some $B \gg 1$. Then certainly $\text{spt}(\widehat{f}) \subseteq (-B, B)$ and the sample theorem implies that to perfectly reconstruct the signal one can consider sampling f every $\dfrac{1}{2B}$ seconds. On the other hand, Example 4.13 shows that one can reconstruct the signal by sampling the signal once per second. This is a huge amount of reduction of sampling if $B \gg 1$. Therefore, the sampling rate provided by the sampling theorem is only a sufficient condition for perfect reconstruction of bandlimited signals, but possibly can be reduced for specific cases.

For $n = 1$, Landau in his paper "Necessary density conditions for sampling and interpolation of certain entire functions" shows the following

Theorem 4.15. *Let S be the union of a finite number of intervals of total measure $|S|$.*

1. *If Λ is a set of sampling for $\mathcal{B}(S)$, then there exist generic constants A, B such that*
$$n^-(r) \equiv \inf_{y\in\mathbb{R}} \#\big(\Lambda \cap [y, y+r]\big) \geqslant |S|r - A\log^+ r - B \qquad \forall\, r > 0\,. \tag{4.16}$$
2. *If Λ is a set of interpolation for $\mathcal{B}(S)$, then there exist generic constants A, B such that*
$$n^+(r) \equiv \sup_{y\in\mathbb{R}} \#\big(\Lambda \cap [y, y+r]\big) \leqslant |S|r + A\log^+ r + B \qquad \forall\, r > 0\,.$$

In the following, we only focus on the proof of the first case in Theorem 4.15.

Before proceeding, we need to introduce some terminologies. Let $Q, S \subseteq \mathbb{R}^n$, and $\mathcal{D}(Q)$ be the subspace of $L^2(\mathbb{R}^n)$ consisting of functions supported on Q. Let D_Q and B_S denote the orthogonal projection of $L^2(\mathbb{R}^n)$ onto $\mathcal{D}(S)$ and $\mathcal{B}(S)$, respectively. Then
$$B_S = \mathscr{F}^*\chi_S\mathscr{F} \qquad \text{and} \qquad D_Q = \chi_Q\,, \tag{4.17}$$
where χ_A denotes the operator defined by multiplying by the characteristic function of A.

Proposition 4.16. *Let $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be square integrable, $K(x, y) = \overline{K(y, x)}$ for all $x, y \in \mathbb{R}^n$, and $K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be an operator defined by*
$$(Kf)(x) = \int_{\mathbb{R}^n} k(x, y) f(y)\, dy\,.$$
Then

1. $k(x, y) = \sum\limits_{k=1}^{\infty} \mu_k \varphi_k(x)\overline{\varphi_k(y)}$, *where $\{\varphi_k\}_{k=1}^{\infty}$ denotes the orthonormal sequence of eigenfunctions, and $\{\mu_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ denotes the sequence of corresponding eigenvalues of K;*

2. $\sum_{k=1}^{\infty} \mu_k = \int_{\mathbb{R}^n} k(x,x)\,dx;$ 3. $\sum_{k=1}^{\infty} \mu_k^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|k(x,y)\right|^2 dxdy.$

Theorem 4.17. *Let $Q, S \subseteq \mathbb{R}^n$ be bounded measurable sets, and D_Q, B_S be the projection operators of $L^2(\mathbb{R}^n)$ defined in (4.17). Denoting the eigenvalues of $B_S D_Q B_S$, arranged in non-increasing order, by $\lambda_k(S,Q)$, where $k \in \mathbb{N} \cup \{0\}$. Then*

(i) $\lambda_k(S,Q) = \lambda_k(Q,S)$.

(ii) $\lambda_k(S,Q) = \lambda_k(S+\sigma, Q+\tau) = \lambda_k(\alpha S, \alpha^{-1} Q)$ *for all* $\sigma, \tau \in \mathbb{R}^n$ *and* $\alpha > 0$.

(iii) $\sum_{k=0}^{\infty} \lambda_k(S,Q) = |S||Q|$.

(iv) $\sum_{k=0}^{\infty} \lambda_k^2(S,Q) \geqslant \sum_{k=0}^{\infty} \lambda_k^2(S,Q) + \sum_{k=0}^{\infty} \lambda_k^2(S,Q)$ *if* $Q = Q_1 \cup Q_2$ *and* $Q_1 \cap Q_2 = \varnothing$.

(v) $\sum_{k=0}^{\infty} \lambda_k^2(S,Q) \geqslant \left(sq - \frac{2}{\pi^2}\log^+(sq) - \frac{6}{\pi^2}\right)^n$, *where S and Q are cubes with edges parallel to the coordinate axes with* $|S| = s^n$, $|Q| = q^n$, *and* $\log^+ x = \max\{0, \log x\}$.

(vi) *For any k-dimensional subspace C_k of $L^2(\mathbb{R}^n)$,*

$$\lambda_k(S,Q) \leqslant \sup_{\substack{f \in \mathcal{B}(S) \\ f \perp C_k, f \neq 0}} \frac{\|D_Q f\|^2_{L^2(\mathbb{R}^n)}}{\|f\|^2_{L^2(\mathbb{R}^n)}} \quad \text{and} \quad \lambda_{k-1}(S,Q) \geqslant \inf_{\substack{f \in \mathcal{B}(S) \cap C_k \\ f \neq 0}} \frac{\|D_Q f\|^2_{L^2(\mathbb{R}^n)}}{\|f\|^2_{L^2(\mathbb{R}^n)}}.$$

Proof. For two (completely continuous) operators A and B, we write $A \sim B$ if A and B has the same nonzero eigenvalues, including multiplicities. Suppose that $\lambda \neq 0$ is an eigenvalue of $B_S D_Q B_S$. Then $B_S D_Q B_S \varphi = \lambda\varphi$ for some $\varphi \neq 0$. By the fact that B_S is a projection, we have

$$\lambda B_S \varphi = B_S B_S D_Q B_S \varphi = B_S D_Q B_S \varphi = \lambda \varphi$$

which implies that $B_S \varphi = \varphi$. Moreover, $D_Q B_S \varphi \neq 0$. Applying D_Q to the equation above, we find that

$$D_Q B_S D_Q D_Q B_S \varphi = D_Q B_S D_Q B_S \varphi = \lambda D_Q B_S \varphi$$

which, by the fact that $D_Q B_S \varphi \neq 0$, implies that λ is also a eigenvalue of $D_Q B_S D_Q$. As a consequence,

$$B_S D_Q B_S \sim D_Q B_S D_Q\,. \tag{4.18}$$

Therefore, to study the nonzero eigenvalues of the operator $B_S D_Q B_S$, it suffices to study the operator $D_Q B_S D_Q$.

Let C denoted the complex conjugate operator; that is, $Cf = \bar{f}$. Then $C\mathscr{F}C = \mathscr{F}^{-1}$ and $C\mathscr{F}^{-1}C = \mathscr{F}$. By the fact that $\mathscr{F}$ is unitary and $D_Q B_S D_Q$ is symmetric (so the eigenvalues are real),

$$\begin{aligned} D_Q B_S D_Q &\sim C D_Q B_S D_Q C = \chi_Q C\mathscr{F}^{-1}C\chi_S C\mathscr{F}C\chi_Q = \chi_Q\mathscr{F}\chi_S\mathscr{F}^{-1}\chi_Q \\ &\sim \mathscr{F}^{-1}\chi_Q\mathscr{F}\chi_S\mathscr{F}^{-1}\chi_Q\mathscr{F} = B_Q D_S B_Q\,. \end{aligned}$$

This proves (ii). Since S and Q are bounded, the Fubini theorem implies that

$$\begin{aligned} (D_Q B_S D_Q f)(x) &= \chi_Q(x)\Big[\int_{\mathbb{R}^n}\chi_S(\xi)\Big(\int_{\mathbb{R}^n}(\chi_Q f)(y)^{-2\pi i y\cdot\xi}\,dy\Big)e^{2\pi i x\cdot\xi}\,d\xi\Big] \\ &= \Big[\int_{\mathbb{R}^n}\chi_Q(x)\chi_Q(y)f(y)\Big(\int_{\mathbb{R}^n}\chi_S(\xi)e^{-2\pi i(y-x)\cdot\xi}\,d\xi\Big)dy\Big] \\ &= \int_{\mathbb{R}^n}\chi_Q(x)\chi_Q(y)\widehat{\chi_S}(y-x)f(y)\,dy\,. \end{aligned}$$

Using (4.18), the change of variables formula together with (i) shows (ii).

Let $k(x,y) = \chi_Q(x)\chi_Q(y)\widehat{\chi_S}(y-x)$ and K be the operator defined by $(Kf)(x) = \int_{\mathbb{R}^n} k(x,y)f(y)\,dy$. Then $k(x,y) = \overline{k(y,x)}$; thus Proposition 4.16 implies that

$$\sum_{k=0}^{\infty}\lambda_k(S,Q) = \int_{\mathbb{R}^n} k(x,x)\,dx = \int_Q \widehat{\chi_S}(0)\,d\xi = |S||Q|$$

which establishes (iii).

To prove (iv), we make use of Proposition 4.16 and find that

$$\sum_{k=0}^{\infty}\lambda_k(S,Q)^2 = \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\big|k(x,y)\big|^2\,dx\Big)dy = \int_{Q\times Q}\big|\widehat{\chi_S}(y-x)\big|^2\,d(x,y)\,.$$

Since $Q\times Q \supseteq (Q_1\times Q_1)\cup(Q_2\times Q_2)$ and $(Q_1\times Q_1)\cap(Q_2\times Q_2) = \varnothing$, by the identity above we conclude that

$$\begin{aligned} \sum_{k=0}^{\infty}\lambda_k(S,Q)^2 &\geqslant \int_{Q_1\times Q_1}\big|\widehat{\chi_S}(y-x)\big|^2\,d(x,y) + \int_{Q_2\times Q_2}\big|\widehat{\chi_S}(y-x)\big|^2\,d(x,y) \\ &= \lambda_k(S,Q_1) + \lambda_k(S,Q_2)\,. \end{aligned}$$

Let S and Q be cubes with volume s^n and q^n. Using (ii) we can assume that S and Q are centered at the origin; that is, $S = \big[-\frac{s}{2},\frac{s}{2}\big]^n$ and $Q = \big[-\frac{q}{2},\frac{q}{2}\big]^n$. Then

$$\widehat{\chi_S}(y-x)=\int_{[-\frac{s}{2},\frac{s}{2}]^n} e^{2\pi i(x-y)\cdot\xi}\,d\xi=\prod_{i=1}^{n}\frac{\sin\pi(x_i-y_i)s}{\pi(x_i-y_i)}\,;$$

thus Proposition 4.16 provides that

$$\begin{aligned}\sum_{k=0}^{\infty}\lambda_k(S,Q)^2&=\int_{[-\frac{q}{2},\frac{q}{2}]^n}\Big(\int_{[-\frac{q}{2},\frac{q}{2}]^n}\prod_{i=1}^{n}\frac{\sin^2(\pi|x_i-y_i|s)}{\pi^2|x_i-y_i|^2}\,dx\Big)dy\\&=\Big(\int_{-\frac{q}{2}}^{\frac{q}{2}}\int_{-\frac{q}{2}}^{\frac{q}{2}}\frac{\sin^2(\pi|x-y|s)}{\pi^2|x-y|^2}\,dxdy\Big)^n.\end{aligned}$$

By the fact that $\displaystyle\int_{-\infty}^{\infty}\frac{\sin^2 t}{t^2}\,dt=\pi$,

$$\begin{aligned}\int_{-\frac{q}{2}}^{\frac{q}{2}}\int_{-\frac{q}{2}}^{\frac{q}{2}}\frac{\sin^2(\pi|x-y|s)}{\pi^2|x-y|^2}\,dxdy&=\frac{s}{\pi}\int_{-\frac{q}{2}}^{\frac{q}{2}}\Big(\int_{\pi(-\frac{q}{2}-y)s}^{\pi(\frac{q}{2}-y)s}\frac{\sin^2 t}{t^2}\,dt\Big)dy\\&=\frac{s}{\pi}\int_{-\frac{q}{2}}^{\frac{q}{2}}\Big(\int_{-\infty}^{\infty}\frac{\sin^2 t}{t^2}\,dt-\int_{\pi(\frac{q}{2}-y)s}^{\infty}\frac{\sin^2 t}{t^2}\,dt-\int_{-\infty}^{\pi(-\frac{q}{2}-y)s}\frac{\sin^2 t}{t^2}\,dt\Big)dy\\&=sq-\frac{2s}{\pi}\int_{-\frac{q}{2}}^{\frac{q}{2}}\Big(\int_{\pi(\frac{q}{2}-y)s}^{\infty}\frac{\sin^2 t}{t^2}\,dt\Big)dy\\&=sq-\frac{q}{\pi}\int_{-1}^{1}\Big(\int_{\frac{q\pi}{2}(1-y)}^{\infty}\frac{\sin^2(st)}{t^2}\,dt\Big)dy\,.\end{aligned}$$

Note that

$$\begin{aligned}&\frac{q}{\pi}\int_{-1}^{1}\Big(\int_{\frac{q\pi}{2}(1-y)}^{\infty}\frac{\sin^2(st)}{t^2}\,dt\Big)dy\\&=\frac{q}{\pi}\int_{q\pi}^{\infty}\Big(\int_{-1}^{1}\frac{\sin^2(st)}{t^2}\,dy\Big)dt+\frac{q}{\pi}\int_{0}^{q\pi}\Big(\int_{1-\frac{2t}{q\pi}}^{1}\frac{\sin^2(st)}{t^2}\,dy\Big)dt\\&=\frac{2}{\pi}\int_{\pi}^{\infty}\frac{\sin^2(sqt)}{t^2}\,dt+\frac{2}{\pi^2}\int_{0}^{q\pi}\frac{\sin^2(st)}{t}\,dt\\&\leqslant\frac{2}{\pi}\int_{\pi}^{\infty}\frac{1}{t^2}\,dt+\frac{2}{\pi^2}\int_{0}^{sq}\frac{\sin^2(\pi t)}{t}\,dt\\&\leqslant\frac{2}{\pi^2}\Big[1+\int_{0}^{1}\frac{\sin^2(\pi t)}{t}\,dt+\int_{1}^{sq}\frac{\sin^2(\pi t)}{t}\,dt\Big]\leqslant\frac{2}{\pi^2}\Big[3+\log^+(sq)\Big]\,,\end{aligned}$$

so (v) is established.

For a given k-dimensional subspace C_k, the subspace B_SC_k has dimension $d\leqslant k$. Moreover, $f\perp B_SC_k$ if and only if $B_Sf\perp C_k$. By the fact that $\|B_Sf\|_{L^2(\mathbb{R}^n)}\leqslant\|f\|_{L^2(\mathbb{R}^n)}$ and

$$\lambda_k(S,Q)\leqslant\sup_{f\perp C_k,f\neq 0}\frac{(B_SD_QB_Sf,f)_{L^2(\mathbb{R}^n)}}{\|f\|^2_{L^2(\mathbb{R}^n)}}$$

for any k-dimensional subspace C_k of $L^2(\mathbb{R}^n)$, we conclude that

$$\lambda_k(S,Q) \leqslant \lambda_d(S,Q) \leqslant \sup_{\substack{f\perp B_S C_k\\ f\neq 0, B_S f\neq 0}} \frac{(B_S D_Q B_S f, f)_{L^2(\mathbb{R}^n)}}{\|f\|^2_{L^2(\mathbb{R}^n)}} \leqslant \sup_{\substack{B_S f\perp C_k\\ B_S f\neq 0}} \frac{(D_Q B_S f, B_S f)_{L^2(\mathbb{R}^n)}}{\|B_S f\|^2_{L^2(\mathbb{R}^n)}}$$
$$\leqslant \sup_{\substack{f\in\mathcal{B}(S)\\ f\perp C_k, f\neq 0}} \frac{(D_Q f, f)_{L^2(\mathbb{R}^n)}}{\|f\|^2_{L^2(\mathbb{R}^n)}} = \sup_{\substack{f\in\mathcal{B}(S)\\ f\perp C_k, f\neq 0}} \frac{(D_Q f, D_Q f)_{L^2(\mathbb{R}^n)}}{\|f\|^2_{L^2(\mathbb{R}^n)}} = \sup_{\substack{f\in\mathcal{B}(S)\\ f\perp C_k, f\neq 0}} \frac{\|D_Q f\|^2_{L^2(\mathbb{R}^n)}}{\|f\|^2_{L^2(\mathbb{R}^n)}}.$$

On the other hand, by the fact that

$$\lambda_{k-1}(S,Q) \geqslant \inf_{f\in C_k, f\neq 0} \frac{(B_S D_Q B_S f, f)_{L^2(\mathbb{R}^n)}}{\|f\|^2_{L^2(\mathbb{R}^n)}}$$

for any k-dimensional subspace of $L^2(\mathbb{R}^n)$, choosing $C_k \subseteq \mathcal{B}(S)$ we obtain that

$$\lambda_{k-1}(S,Q) \geqslant \inf_{f\in C_k, f\neq 0} \frac{(B_S D_Q B_S f, f)_{L^2(\mathbb{R}^n)}}{\|f\|^2_{L^2(\mathbb{R}^n)}} = \inf_{\substack{f\in\mathcal{B}(S)\cap C_k\\ f\neq 0}} \frac{(D_Q B_S f, B_S f)_{L^2(\mathbb{R}^n)}}{\|f\|^2_{L^2(\mathbb{R}^n)}}$$
$$= \inf_{\substack{f\in\mathcal{B}(S)\cap C_k\\ f\neq 0}} \frac{(D_Q f, f)_{L^2(\mathbb{R}^n)}}{\|f\|^2_{L^2(\mathbb{R}^n)}} = \inf_{\substack{f\in\mathcal{B}(S)\cap C_k\\ f\neq 0}} \frac{(D_Q f, D_Q f)_{L^2(\mathbb{R}^n)}}{\|f\|^2_{L^2(\mathbb{R}^n)}} = \inf_{\substack{f\in\mathcal{B}(S)\cap C_k\\ f\neq 0}} \frac{\|D_Q f\|^2_{L^2(\mathbb{R}^n)}}{\|f\|^2_{L^2(\mathbb{R}^n)}};$$

thus (vi) is established. □

Lemma 4.18. *For any bounded measurable set $S \subseteq \mathbb{R}^n$ and $d > 0$, there exists a Schwartz function $h : \mathbb{R}^n \to \mathbb{C}$ such that* $\mathrm{spt}(h) \subseteq B(0,d)$ *and* $|\widehat{h}(\xi)| \geqslant 1$ *for all $\xi \in S$.*

Proof. Since S is bounded, $S \subseteq B(0,R)$ for some $R > 0$. Let $f \in \mathscr{S}(\mathbb{R}^n)$ be such that $f > 2$ on $B(0,R)$. Since $\check{f} \in \mathscr{S}(\mathbb{R}^n)$, there exists $g \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ such that $\|\check{f} - g\|_{L^1(\mathbb{R}^n)} < 1$. Choose $r > d$ such that $\mathrm{spt}(g) \subseteq B(0,r)$, and defined the function h by

$$h(x) \equiv \frac{r^n}{d^n} g\Big(\frac{rx}{d}\Big).$$

Then h is supported in $B(0,d)$. Moreover,

$$\widehat{h}(\xi) = \int_{\mathbb{R}^n} \frac{r^n}{d^n} g\Big(\frac{rx}{d}\Big) e^{2\pi i x\cdot\xi}\, dx = \widehat{g}\Big(\frac{d\xi}{r}\Big) \qquad \forall\, \xi \in \mathbb{R}^n,$$

and the Fourier inversion formula implies that

$$\sup_{\xi\in\mathbb{R}^n} \Big|f\Big(\frac{d\xi}{r}\Big) - \widehat{h}(\xi)\Big| = \sup_{\xi\in\mathbb{R}^n} \Big|f\Big(\frac{d\xi}{r}\Big) - \widehat{g}\Big(\frac{d\xi}{r}\Big)\Big| = \|f - \widehat{g}\|_{L^\infty(\mathbb{R}^n)} \leqslant \|\check{f} - g\|_{L^1(\mathbb{R}^n)} < 1.$$

Therefore, if $|\xi| < R$, we must have $\dfrac{d|\xi|}{r} < R$; hence

$$\big|\widehat{h}(\xi)\big| \geqslant \Big|f\Big(\frac{d\xi}{r}\Big)\Big| - 1 \geqslant 1 \qquad \forall\, |\xi| \leqslant R.$$

Since $S \subseteq B(0,R)$, $|\widehat{h}| \geqslant 1$ on S. □

Lemma 4.19. *Let $S \subseteq \mathbb{R}$ be a bounded set and Λ be a uniformly discrete set of sampling for $\mathcal{B}(S)$ with separation number d and counting function n. For a compact set I, I^+ denotes the set of points whose distance to I is less than $\frac{d}{2}$. Then*

$$\lambda_{n(I^+)}(S, I) \leqslant \gamma < 1 \tag{4.19}$$

for some γ depending on S, Λ but not on I.

Proof. By Lemma 4.18, there exists a Schwartz function h such that h vanishes outside $B(0, \frac{d}{2})$ and $|\widehat{h}| \geqslant 1$ on S. Let C be the subspace of $L^2(\mathbb{R})$ spanned by the functions $\overline{h(\lambda - \cdot)}$ for $\lambda \in \Lambda \cap I^+$. Since

$$\big(\overline{h(\lambda_i - \cdot)}, \overline{h(\lambda_j - \cdot)}\big)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \overline{h(\lambda_i - x)} h(\lambda_j - x)\, dx = 0 \quad \text{if } \lambda_i \neq \lambda_j\,,$$

the dimension of C is $n(I^+)$.

For a given $f \in \mathcal{B}(S)$ be given, we define $g = f * h$; that is,

$$g(x) = \int_{\mathbb{R}} f(y) h(x - y)\, dy = \int_{|y-x|<\frac{d}{2}} f(y) h(x-y)\, dy\,.$$

Then $\widehat{g} = \widehat{f}\,\widehat{h}$ which further implies that $g \in \mathcal{B}(S)$. Therefore, by the fact that Λ is a set of sampling for $\mathcal{B}(S)$,

$$\|g\|^2_{L^2(\mathbb{R})} \leqslant K \sum_{\lambda \in \Lambda} \big|g(\lambda)\big|^2\,.$$

Moreover, the Plancherel identity shows that

$$\|g\|_{L^2(\mathbb{R})} = \|\widehat{g}\|_{L^2(\mathbb{R})} = \|\widehat{f}\|_{L^2(\mathbb{R})} \|\widehat{h}\|_{L^2(\mathbb{R})} \geqslant \|\widehat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})} \tag{4.20}$$

and the Cauchy-Schwarz inequality shows that

$$|g(x)|^2 \leqslant \|h\|^2_{L^2(\mathbb{R})} \int_{|y-x|<\frac{d}{2}} f(y)^2\, dy\,.$$

Therefore, if $f \in \mathcal{B}(S)$ and $f \perp C$, we have

$$\begin{aligned}
\|f\|^2_{L^2(\mathbb{R})} \leqslant \|g\|^2_{L^2(\mathbb{R})} &\leqslant K \sum_{\lambda \in \Lambda} \big|g(\lambda)\big|^2 = K \sum_{\lambda \in \Lambda, \lambda \notin I^+} \big|g(\lambda)\big|^2 \\
&\leqslant K \|h\|^2_{L^2(\mathbb{R})} \sum_{\lambda \in \Lambda, \lambda \notin I^+} \int_{|y-\lambda|<\frac{d}{2}} \big|f(y)\big|^2\, dy \\
&\leqslant K \|h\|^2_{L^2(\mathbb{R})} \int_{I^\complement} \big|f(y)\big|^2\, dy = K \|h\|^2_{L^2(\mathbb{R})} \Big[\|f\|^2_{L^2(\mathbb{R})} - \int_{\mathbb{R}} \big|D_I f(y)\big|^2\, dy\Big]\,.
\end{aligned}$$

As a consequence, letting $\gamma \equiv 1 - \dfrac{1}{K\|h\|^2_{L^2(\mathbb{R})}}$, we have

$$\frac{\|D_I f\|^2_{L^2(\mathbb{R})}}{\|f\|^2_{L^2(\mathbb{R})}} \leqslant 1 - \frac{1}{K\|h\|^2_{L^2(\mathbb{R})}} = \gamma < 1\,.$$

Inequality (4.19) then follows from (vi) of Theorem 4.17. □

Lemma 4.20. *Let $S \subseteq \mathbb{R}$ be a bounded set and Λ be a uniformly discrete set of interpolation for $\mathcal{B}(S)$ with separation number d and counting function n. For a compact set I, I^- denotes the set of points whose distance to $I^{\complement}$ exceeds $\frac{d}{2}$. Then*

$$\lambda_{n(I^-)-1}(S,I) \geqslant \delta > 0$$

for some δ depending on S and Λ but not on I.

Proof. Again by Lemma 4.18, there exists a Schwartz function h such that h vanishes outside $B(0,\frac{d}{2})$ and $|\widehat{h}| \geqslant 1$ on S.

Define a bounded linear operator A on $\mathcal{B}(S)$ by $Ag = \big\{g(\lambda)\big\}_{\lambda\in\Lambda}$ if $g \in \mathcal{B}(S)$. To see the boundedness of A, let $g \in \mathcal{B}(S)$ be given, and let $f \in \mathcal{B}(S)$ be such that $\widehat{g} = \widehat{f}\,\widehat{h}$; that is,

$$f(x) = \int_{\mathbb{R}} \frac{\widehat{g}(\xi)}{\widehat{h}(\xi)} e^{2\pi i x\xi}\, d\xi\,.$$

The same as (4.20), we have $\|f\|_{L^2(\mathbb{R})} \leqslant \|g\|_{L^2(\mathbb{R})}$, and the Cauchy-Schwarz inequality implies that

$$\big|g(x)\big|^2 \leqslant \|h\|^2_{L^2(\mathbb{R})} \int_{|y-x|<\frac{d}{2}} |f(y)|^2\, dy\,.$$

Since Λ is uniformly discrete with separation number d, by the fact that $g = f * h$, we have

$$\sum_{\lambda\in\Lambda} \big|g(\lambda)\big|^2 \leqslant \|h\|^2_{L^2(\mathbb{R})} \sum_{\lambda\in\Lambda} \int_{|y-\lambda|<\frac{d}{2}} \big|f(y)\big|^2\, dy \leqslant \|h\|^2_{L^2(\mathbb{R})}\|f\|^2_{L^2(\mathbb{R})} \leqslant \|h\|^2_{L^2(\mathbb{R})}\|g\|^2_{L^2(\mathbb{R})}\,. \tag{4.21}$$

Therefore, $A : \mathcal{B}(S) \to \ell^2$ is bounded.

Define $\mathcal{E}(S) \equiv \big\{f \in \mathcal{B}(S) \,\big|\, f(\lambda) = 0 \text{ for all } \lambda \in \Lambda\big\}$. For $f \in \mathcal{B}(S)$, the Cauchy-Schwarz inequality and the Plancherel identity imply that

$$|f(x)|^2 \leqslant \Big(\int_S \big|\widehat{f}(y)\big|\, dy\Big)^2 \leqslant |S|\|\widehat{f}\|^2_{L^2(\mathbb{R})} = |S|\|f\|^2_{L^2(\mathbb{R})}\,,$$

so if $\{f_k\}_{k=1}^{\infty} \subseteq \mathcal{B}(S)$ converges to f in $L^2(\mathbb{R})$ (that means $\|f_k - f\|_{L^2(\mathbb{R})} \to 0$ as $k \to \infty$), $\{f_k\}_{k=1}^{\infty}$ also converges to f uniformly on S. In particular, if $\{f_k\}_{k=1}^{\infty} \subseteq \mathcal{E}(S)$ converges to f in L^2 sense, then for $\lambda \in \Lambda$,

$$\big|f(\lambda)\big| = \lim_{k\to\infty} \big|f(\lambda) - f_k(\lambda)\big| \leqslant \lim_{k\to\infty} \sqrt{|S|}\|f_k - f\|_{L^2(\mathbb{R})} = 0$$

which implies that $f \in \mathcal{E}(S)$. In other words, $\mathcal{E}(S)$ is a closed subspace.

Let $\mathcal{E}^{\perp}(S)$ denote the orthogonal complement of $\mathcal{E}(S)$, and $\{a_\lambda\}_{\lambda\in\Lambda} \in \ell^2$ be given. Since Λ is a set of interpolation for $\mathcal{B}(S)$, there exists $f \in \mathcal{B}(S)$ such that

$$f(\lambda) = a_\lambda \qquad \forall\, \lambda \in \Lambda\,.$$

By the fact that $\mathcal{B}(S) = \mathcal{E}(S) \oplus \mathcal{E}^{\perp}(S)$, there exist (unique) $f_1 \in \mathcal{E}(S)$ and $f_2 \in \mathcal{E}^{\perp}(S)$ such that $f = f_1 + f_2$. Therefore, since $f_1(\lambda) = 0$ for all $\lambda \in \Lambda$, we have

$$f_2(\lambda) = f_1(\lambda) + f_2(\lambda) = f(\lambda) = a_\lambda \qquad \forall\, \lambda \in \Lambda\,.$$

Therefore, Λ is a set of interpolation for $\mathcal{E}^{\perp}(S)$. This also implies that $A : \mathcal{E}^{\perp}(S) \to \ell^2$ is surjective.

Moreover, noting that $A : \mathcal{E}^{\perp}(S) \to \ell^2$ is one-to-one, we find that $A : \mathcal{E}^{\perp}(S) \to \ell^2$ is a bounded linear bijective operator. Therefore, the bounded inverse theorem (from functional analysis) implies that $A^{-1} : \ell^2 \to \mathcal{E}^{\perp}(S)$ is also bounded linear; thus there exists $K > 0$ such that

$$\|g\|^2_{L^2(\mathbb{R})} \leqslant K \sum_{\lambda\in\Lambda} \big|g(\lambda)\big|^2 \qquad \forall\, g \in \mathcal{E}^{\perp}(S)\,. \tag{4.22}$$

In other words, Λ is a set of sampling for $\mathcal{E}^{\perp}(S)$ as well.

For each $\lambda \in \Lambda$, let $\varphi_\lambda \in \mathcal{E}^{\perp}(S)$ be the function whose value is 1 at λ and 0 at other point of Λ. We remark that such a φ_λ exists since Λ is a set of interpolation for $\mathcal{E}^{\perp}(S)$. Clearly $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ is a set of linear independent functions. Let $\psi_\lambda \in \mathcal{B}(S)$ be such that $\widehat{\varphi_\lambda} = \widehat{\psi_\lambda}\widehat{h}$; that is,

$$\psi_\lambda(x) = \int_S \frac{\widehat{\varphi_\lambda}(\xi)}{\widehat{h}(\xi)} e^{2\pi i x\cdot\xi}\, d\xi\,.$$

Then $\{\psi_\lambda\}_{\lambda\in\Lambda}$ is also a set of linear independent functions. Let C be the subspace of $\mathcal{B}(S)$ spanned by $\{\psi_\lambda\}_{\lambda\in\Lambda\cap I^-}$. Then $\dim(C) = n(I^-) = \#(\Lambda \cap I^-)$. For a given function $f \in C$, $f = \sum\limits_{\lambda\in\Lambda\cap I^-} c_\lambda\psi_\lambda$ for some $\{c_\lambda\}_{\lambda\in\Lambda\cap I^-}$; thus

$$\widehat{f * h} = \widehat{f}\,\widehat{h} = \sum_{\lambda\in\Lambda\cap I^-} c_\lambda \widehat{\psi_\lambda}\widehat{h} = \sum_{\lambda\in\Lambda\cap I^-} c_\lambda \widehat{\varphi_\lambda}$$

which shows that $f*h$ is a linear combination of $\{\varphi_\lambda\}_{\lambda\in\Lambda\cap I^-}$. This further implies that

$$f*h \in \mathcal{E}^\perp(S) \quad \text{and} \quad (f*h)(\lambda) = 0 \quad \forall\, \lambda \notin \Lambda\cap I^- \qquad \text{whenever} \quad f\in C\,.$$

As a consequence, using (4.20) and (4.22), we obtain that if $f \in C$,

$$\begin{aligned} K^{-1}\|f\|^2_{L^2(\mathbb{R})} \leqslant K^{-1}\|(f*h)\|^2_{L^2(\mathbb{R})} \leqslant \sum_{\lambda\in\Lambda}\big|(f*h)(\lambda)\big|^2 &= \sum_{\lambda\in\Lambda\cap I^-}\big|(f*h)(\lambda)\big|^2 \\ \leqslant \|h\|^2_{L^2(\mathbb{R})}\sum_{\lambda\in\Lambda\cap I^-}\int_{|y-\lambda|<\frac{d}{2}}\big|f(y)\big|^2\,dy &\leqslant \|h\|^2_{L^2(\mathbb{R})}\|f\|^2_{L^2(I)} = \|h\|^2_{L^2(\mathbb{R})}\|D_I f\|^2_{L^2(\mathbb{R})}\,; \end{aligned}$$

thus for $f \in C$,

$$\frac{\|D_I f\|^2_{L^2(\mathbb{R})}}{\|f\|^2_{L^2(\mathbb{R})}} \geqslant \frac{1}{K\|h\|^2_{L^2(\mathbb{R})}} = \delta > 0\,,$$

where we note that δ depends only on S (due to the dependence on h) and Λ but not on I. The lemma is then concluded by (vii) of Theorem 4.17. □

Proof of (4.16). Let d be a separation number of Λ, $I = [-\frac{1}{2}, \frac{1}{2}]$ be a unit interval, and J be an interval of length r such that $n^-(r) = n(J) = \#(\Lambda\cap J)$. Since J is a single interval, then J^+, the set of points whose distance to J is less than $\frac{d}{2}$, satisfies $n(J^+) \leqslant n(J)+2$; thus (ii) of Theorem 4.17 and Lemma 4.19 imply that

$$\lambda_{n(J)+2}(S, rI) \leqslant \lambda_{n(J^+)}(S, J) \leqslant \gamma < 1 \tag{4.23}$$

for some γ independent of r.

Suppose that S consists of p disjoint intervals $J_1, \cdots, J_p$. By Example 4.13, the set of integers $\mathbb{Z}$ is a uniformly discrete set of sampling and interpolation of $\mathcal{B}(I)$ with separation number 1. The set $(rS)^-$, the collection of points whose distance to $(rS)^\complement$ exceeds $\frac{1}{2}$, consists of at most p disjoint intervals, so

$$\#((rS)^-\cap\mathbb{Z}) \geqslant |(rS)^-| - p = r|S| - 2p\,.$$

By (i) and (ii) of Theorem 4.17 and Lemma 4.20, we find that

$$\lambda_{r|S|-2p-1}(S, rI) = \lambda_{r|S|-2p-1}(I, rS) \geqslant \lambda_{\#((rS)^-\cap\mathbb{Z})-1}(I, rS) \geqslant \delta > 0 \tag{4.24}$$

for some δ independent of r.

Let $\mu(S, rI) = \sum\limits_{k=0}^{\infty}\lambda_k(S, rI)\big(1-\lambda_k(S, rI)\big)$. By (iii)-(v) of Theorem 4.17,

$$\begin{aligned}\mu(S,rI) &= r|S| - \sum_{k=0}^{\infty}\lambda_k^2(S,rI) \leqslant r|S| - \sum_{j=1}^{p}\sum_{k=0}^{\infty}\lambda_k^2(J_j,rI)\\ &\leqslant r|S| - \sum_{j=1}^{p}\Big(r|J_j| - \frac{2}{\pi^2}\log^+(r|J_j|) - \frac{6}{\pi^2}\Big)\\ &\leqslant \frac{2}{\pi^2}\sum_{j=1}^{p}\log^+ r|J_j| + \frac{6p}{\pi^2} \leqslant A\log^+ r + B\end{aligned}$$

for some constants A, B depending only on S.

Now suppose that $n(J) + 2 \leqslant r|S| - 2p - 1$, then (4.23) and (4.24) imply that

$$0 < \delta \leqslant \lambda_k(S,rI) \leqslant \gamma < 1 \qquad \forall\, k \in [n(J)+2, r|S|-2p-1]\,.$$

Therefore,

$$\big(r|S| - 2p - 1 - n(J) - 2 + 1\big)\min\big\{\delta(1-\delta),\gamma(1-\gamma)\big\} \leqslant \mu(S,rI) \leqslant A\log^+ r + B$$

which shows that

$$n(J) \geqslant r|S| - A\log^+ r - B \tag{4.25}$$

for some constants A, B depending on S and Λ but not r. On the other hand, if $n(J)+2 > r|S| - 2p - 1$, (4.25) holds automatically (for proper choices of A and B); thus (4.16) is established. ▫

We can measure the density of a uniformly discrete set Λ in terms of function $n^{\pm}(r)$.

Definition 4.21. The ***Beurling upper and lower uniform densities*** of a uniformly discrete set Λ, denoted by $D^+(\Lambda)$ and $D^-(\Lambda)$, respectively, are the numbers defined by

$$D^{\pm}(\Lambda) = \lim_{r\to\infty}\frac{n^{\pm}(r)}{r}\,.$$

The Beurling density reduces to the usual concept of average sampling rate for uniform and periodic non-uniform sampling.

Corollary 4.22. *Let $S \subseteq \mathbb{R}$ be a bounded set with measure $|S|$ and Λ be a uniformly discrete set.*

1. *If Λ is a set of sampling for $\mathcal{B}(S)$, then $D^-(\Lambda) \geqslant |S|$.*

2. *If Λ is a set of interpolation for $\mathcal{B}(S)$, then $D^+(\Lambda) \leqslant |S|$.*

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