Fourier Analysis

富氏分析

鄭經斅

Contents

Chapter 3

Fourier Transforms

Before introducing the Fourier transform, let us "motivate" the idea a little bit. In Section 2.5 we show that $\{\mathbf{e}_k\}_{k=-\infty}^{\infty}$, where $\mathbf{e}_k(x) = e^{ikx}$, is a complete orthonormal set in $L^2(\mathbb{T})$. Similarly, let $L^2([-K,K])$ denote the inner-product space

$$L^2([-K,K]) = \big\{ f : [-K,K] \to \mathbb{C} \,\big|\, f \text{ is square integrable} \big\} / \sim$$

equipped with the inner product

$$\langle f, g \rangle = \frac{1}{2K} \int_{-K}^{K} f(x) \overline{g(x)} \, dx \,,$$

where $\sim$ denotes the equivalence relation $f \sim g$ if and only if $f - g = 0$ except on a set of measure zero. Then the set $\Big\{ \exp\big(\frac{ik\pi x}{K}\big) \Big\}_{k=-\infty}^{\infty}$ is a complete orthonormal set in $L^2([-K,K])$; that is, any functions $f \in L^2([-K,K])$ can be expressed as

$$f(x) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{\frac{ik\pi x}{K}} \,, \quad \text{where} \quad \widehat{f}(k) = \frac{1}{2K} \int_{-K}^{K} f(y) e^{-\frac{ik\pi y}{K}} \, dy \,. \tag{3.1}$$

Moreover, $\sum\limits_{k=-\infty}^{\infty} |\widehat{f}(k)|^2 = \dfrac{1}{2K} \displaystyle\int_{-K}^{K} |f(x)|^2 \, dx$. In other words, there is a one-to-one correspondence between $f \in L^2([-K,K])$ and $\widehat{f} \in \ell_2$, where ℓ^2 is the collection of square summable sequences; that is,

$$\ell^2 = \Big\{ \{a_k\}_{k=-\infty}^{\infty} \,\Big|\, \sum_{k=-\infty}^{\infty} |a_k|^2 < \infty \Big\} \,.$$

We look for a space X so that there is also a one-to-one correspondence between the square integrable functions on $\mathbb{R}$ and X. Intuitively, we can check what "might" happen by letting $K \to \infty$ in (3.1).

Making use of the Riemann sum to approximate the integral (by partition $[-K, K]$ into $2K^2$ intervals), we find that

$$
\begin{aligned}
f(x) &= \frac{1}{2K}\sum_{k=-\infty}^{\infty}\int_{-K}^{K} f(y)e^{\frac{ik\pi(x-y)}{K}}dy \approx \frac{1}{2K}\sum_{k=-K^2}^{K^2}\int_{-K}^{K} f(y)e^{\frac{ik\pi(x-y)}{K}}dy \\
&\approx \frac{1}{2K}\sum_{k=-K^2}^{K^2}\sum_{\ell=1}^{2K^2} f\big(-K+\frac{\ell}{K}\big)\exp\big(\frac{ik\pi(x+K-\frac{\ell}{K})}{K}\big)\frac{1}{K} \\
&\approx \frac{1}{2K}\sum_{k=-K^2}^{K^2}\sum_{\ell=-K^2}^{K^2} f\big(\frac{\ell}{K}\big)\exp\big(\frac{ik\pi(x-\frac{\ell}{K})}{K}\big)\frac{1}{K} \quad \big(y_\ell = \frac{\ell}{K}, \Delta y = \frac{1}{K}\big) \\
&= \frac{1}{2\pi}\sum_{\ell=-K^2}^{K^2}\sum_{k=-K^2}^{K^2} f\big(\frac{\ell}{K}\big)\exp\big(i\frac{k\pi}{K}(x-\frac{\ell}{K})\big)\frac{\pi}{K}\frac{1}{K} \\
&\approx \frac{1}{2\pi}\sum_{\ell=-K^2}^{K^2}\int_{-K\pi}^{K\pi} f\big(\frac{\ell}{K}\big)\exp\big(i\xi(x-\frac{\ell}{K})\big)d\xi\frac{1}{K} \quad \big(\xi_k = \frac{k\pi}{K}, \Delta\xi = \frac{\pi}{K}\big) \\
&\approx \frac{1}{2\pi}\int_{-K}^{K}\int_{-K\pi}^{K\pi} f(y)e^{i\xi(x-y)}d\xi dy = \frac{1}{2\pi}\int_{-K}^{K}\int_{-K\pi}^{K\pi} f(y)e^{i\xi(x-y)}dyd\xi \\
&\approx \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-i\xi y}dy\Big]e^{i\xi x}d\xi\,.
\end{aligned}
$$

Therefore, if we define $\widehat{f}(\xi) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{\mathbb{R}} f(y)e^{-i\xi y}dy$, then the formal computation above suggests that

$$
f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{f}(\xi)e^{i\xi x}d\xi\,. \tag{3.2}
$$

In the rest of this section, we are going to verify the identity above rigorously (for functions f with certain properties).

3.1 The Definition and Basic Properties of the Fourier Transform

For notational convenience, we **abuse** the following notion from real analysis.

Definition 3.1. The space $L^1(\mathbb{R}^n)$ consists of all functions that are integrable on $\mathbb{R}^n$ and whose integrals are absolute convergent. In other words,

$$
L^1(\mathbb{R}^n) = \Big\{f : \mathbb{R}^n \to \mathbb{C} \,\Big|\, \int_{\mathbb{R}^n} |f(x)|\,dx < \infty\Big\}\,;
$$

that is, $f \in L^1(\mathbb{R}^n)$ if the limit $\displaystyle\lim_{R\to\infty} \int_{B(0,R)} \big|f(x)\big|\,dx = \|f\|_{L^1(\mathbb{R}^n)}$ exists.

Remark 3.2. Even though we have not defined the integral for complex-valued function, the definition of $L^1(\mathbb{R}^n)$ should be clear: when f is complex-valued function, the absolute integrability of f is equivalent to that the real part and the imaginary part of f are both absolutely integrable, and

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\,dx &= \int_{\mathbb{R}^n} \mathrm{Re}(f)(x)\,dx + i\int_{\mathbb{R}^n} \mathrm{Im}(f)(x)\,dx \\ &= \int_{\mathbb{R}^n} \frac{f(x)+\overline{f(x)}}{2}\,dx + \int_{\mathbb{R}^n} \frac{f(x)-\overline{f(x)}}{2}\,dx\,,\end{aligned}$$

where $\overline{f(x)}$ is the complex conjugate of $f(x)$.

Definition 3.3. For all $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f, denoted by $\mathscr{F}f$ or $\widehat{f}$, is defined by

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}^{\,n}} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx \qquad \forall\, \xi \in \mathbb{R}^n\,,$$

where $x\cdot\xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$.

3.2 Some Further Properties of the Fourier Transform

Proposition 3.4. $\mathscr{F} : L^1(\mathbb{R}^n) \to \mathscr{C}_b(\mathbb{R}^n;\mathbb{C})$, *and*

$$\|\mathscr{F}f\|_\infty \equiv \sup_{\xi\in\mathbb{R}^n} \big|(\mathscr{F}f)(\xi)\big| \leqslant \|f\|_{L^1(\mathbb{R}^n)}\,. \tag{3.3}$$

Proof. First we show that $\mathscr{F}f$ is continuous if $f \in L^1(\mathbb{R}^n)$. Let $\xi \in \mathbb{R}^n$ and $\varepsilon > 0$ be given. Since $f \in L^1(\mathbb{R}^n)$, there exists $R > 0$ such that

$$\int_{B(0,r)^\complement} \big|f(x)\big|\,dx < \frac{\varepsilon}{3} \qquad \forall\, r \geqslant R\,.$$

Moreover, there exists $M > 0$ such that

$$\int_{\mathbb{R}^n} \big|f(x)\big|\,dx \leqslant M < \infty\,.$$

Since $\phi(x,y) = e^{-ix\cdot y}$ is uniformly continuous on $A \equiv B(0,R)\times B(\xi,1)$, there exists $0 < \delta < 1$ such that

$$\big|\phi(x_1,y_1) - \phi(x_2,y_2)\big| < \frac{\varepsilon}{3M} \quad \text{whenever} \quad \big|(x_1,y_1)-(x_2,y_2)\big| < \delta \text{ and } (x_1,y_1),(x_2,y_2) \in A\,.$$

In particular, for all $x \in B(0,R)$ and $\eta \in B(\xi,\delta)$,

$$\left|e^{-ix\cdot\xi} - e^{-ix\cdot\eta}\right| < \frac{\varepsilon}{3M}\,.$$

Therefore, for $\eta \in B(\xi,\delta)$,

$$\begin{aligned}\left|\widehat{f}(\eta) - \widehat{f}(\xi)\right| &\leqslant \frac{1}{\sqrt{2\pi}^{\,n}}\int_{\mathbb{R}^n}\left|f(x)\right|\left|e^{-ix\cdot\eta} - e^{-ix\cdot\xi}\right| dx \\ &\leqslant \frac{2}{\sqrt{2\pi}^{\,n}}\int_{B(0,R)^{\complement}}\left|f(x)\right| dx + \frac{1}{\sqrt{2\pi}^{\,n}}\int_{B(0,R)}\left|f(x)\right|\left|e^{-ix\cdot\eta} - e^{-ix\cdot\xi}\right| dx \\ &\leqslant \frac{1}{\sqrt{2\pi}^{\,n}}\Big[\frac{2\varepsilon}{3} + \frac{\varepsilon}{3M}\int_{B(0,R)}\left|f(x)\right| dx\Big] < \varepsilon\,;\end{aligned}$$

thus $\mathscr{F}f$ is continuous. The validity of (3.3) should be clear, and is left as an exercise. ▫

Definition 3.5. A function f on $\mathbb{R}^n$ is said to have rapid decrease/decay if for all integers $N \geqslant 0$, there exists a_N such that

$$|x|^N|f(x)| \leqslant a_N, \qquad \text{as } x \to \infty.$$

Definition 3.6. The *Schwartz space* $\mathscr{S}(\mathbb{R}^n)$ is the collection of all (complex-valued) smooth functions f on $\mathbb{R}^n$ such that f and all of its derivatives have rapid decrease. In other words,

$$\mathscr{S}(\mathbb{R}^n) = \big\{u \in \mathscr{C}^\infty(\mathbb{R}^n)\,\big|\,|\cdot|^N D^k u \text{ is bounded for all } k, N \in \mathbb{N} \cup \{0\}\big\}.$$

Elements in $\mathscr{S}(\mathbb{R}^n)$ are called Schwartz functions.

The prototype element of $\mathscr{S}(\mathbb{R}^n)$ is $e^{-|x|^2}$ which is not compactly supported, but has rapidly decreasing derivatives.

The reader is encouraged to verify the following basic properties of $\mathscr{S}(\mathbb{R}^n)$:

1. $\mathscr{S}(\mathbb{R}^n)$ is a vector space.

2. $\mathscr{S}(\mathbb{R}^n)$ is an algebra under the pointwise product of functions.

3. $\mathcal{P}u \in \mathscr{S}(\mathbb{R}^n)$ for all $u \in \mathscr{S}(\mathbb{R}^n)$ and all polynomial functions $\mathcal{P}$.

4. $\mathscr{S}(\mathbb{R}^n)$ is closed under differentiation.

5. $\mathscr{S}(\mathbb{R}^n)$ is closed under translations and multiplication by complex exponentials $e^{ix\cdot\xi}$.

Remark 3.7. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and $\mathscr{C}_c^\infty(\Omega)$ denote the collection of all smooth functions with compact support in Ω; that is,

$$\mathscr{C}_c^\infty(\Omega) \equiv \left\{ u \in \mathscr{C}^\infty(\Omega) \,\middle|\, \{x \in \Omega \,|\, f(x) \neq 0\} \subset\subset \Omega \right\},$$

then $\mathscr{C}_c^\infty(\mathbb{R}^n) \subseteq \mathscr{S}(\mathbb{R}^n)$. The set $\text{cl}\big(\{x \in \Omega \,|\, f(x) \neq 0\}\big)$ is called the ***support*** of f.

The following lemma allows us to take the Fourier transform of Schwartz functions.

Lemma 3.8. *If $f \in \mathscr{S}(\mathbb{R}^n)$, then $f \in L^1(\mathbb{R}^n)$.*

Proof. If $f \in \mathscr{S}(\mathbb{R}^n)$, then $(1+|x|)^{n+1}|f(x)| \leqslant C$ for some $C > 0$. Therefore, with ω_{n-1} denoting the the surface area of the $(n-1)$-dimensional unit sphere,

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x)|\, dx &\leqslant \int_{\mathbb{R}^n} \frac{C}{(1+|x|)^{n+1}} dx = \int_{\mathbb{S}^{n-1}} \int_0^\infty \frac{C}{(1+r)^{n+1}} r^{n-1} dr dS \\ &\leqslant C\omega_{n-1} \int_0^\infty (1+r)^{-2} dr = C\omega_n \end{aligned}$$

which is a finite number. □

Now we check if $\widehat{f}$ is differentiable if $f \in \mathscr{S}(\mathbb{R}^n)$. Note that if $f \in \mathscr{S}(\mathbb{R}^n)$, then the function $y_j = x_j f(x)$ belongs to $\mathscr{S}(\mathbb{R}^n)$ for all $1 \leqslant j \leqslant n$.

Lemma 3.9. *If $f \in \mathscr{S}(\mathbb{R}^n)$, then $\widehat{f}$ is differentiable, and for each $j \in \{1, \cdots, n\}$, $\dfrac{\partial \widehat{f}}{\partial \xi_j}$ exists is given by*

$$\frac{\partial \widehat{f}}{\partial \xi_j}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} (-ix_j) f(x) e^{-ix\cdot\xi} dx = \Big[\frac{1}{i} x_j f(x)\Big]^\wedge(\xi)\,. \tag{3.4}$$

Proof. Let g_j be defined by $g_j(x) = -ix_j f(x)$. Since f and g_j are both Schwartz functions,

$$\lim_{k\to\infty} \int_{B(0,k)^\complement} |f(x)|\, dx = 0 \quad \text{and} \quad \lim_{k\to\infty} \int_{B(0,k)^\complement} |g_j(x)|\, dx = 0\,.$$

Let $\chi : \mathbb{R}_+ \to \mathbb{R}$ be a smooth decreasing function such that

$$\chi(r) = \begin{cases} 1 & \text{if } 0 \leqslant r \leqslant 1\,, \\ 0 & \text{if } r > 2\,. \end{cases}$$

Define $f_k(x) = \chi\big(\frac{|x|}{k}\big) f(x)$. We first show that

$$\frac{\partial \widehat{f_k}}{\partial \xi_j}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \chi\big(\frac{|x|}{k}\big) g_j(x) e^{-ix\cdot\xi}\, dx\,. \tag{3.5}$$

Too see this, we note that

$$\begin{aligned}&\frac{\widehat{f}_k(\xi+he_j)-\widehat{f}_k(\xi)}{h}-\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\chi\big(\frac{|x|}{k}\big)g_j(x)e^{-ix\cdot\xi}\,dx\\&=\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\chi\big(\frac{|x|}{k}\big)f(x)e^{-ix\cdot\xi}\Big[\frac{e^{-ihx_j}-1}{h}+ix_j\Big]\,dx\\&=\frac{1}{\sqrt{2\pi}^n}\int_{B(0,2k)}\chi\big(\frac{|x|}{k}\big)f(x)e^{-ix\cdot\xi}\Big[\frac{e^{-ihx_j}-1}{h}+ix_j\Big]\,dx\,;\end{aligned}$$

thus by the fact that $\dfrac{e^{-ihx_j}-1}{h}+ix_j\to 0$ uniformly on $B(0,2k)$ as $h\to 0$, Theorem 1.6 implies that

$$\lim_{h\to 0}\frac{\widehat{f}_k(\xi+he_j)-\widehat{f}_k(\xi)}{h}-\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\chi\big(\frac{|x|}{k}\big)g_j(x)e^{-ix\cdot\xi}\,dx=0\,;$$

hence (3.5) is established. Therefore, for each $k\in\mathbb{N}$,

$$\sup_{\xi\in\mathbb{R}^n}\Big|\frac{\partial\widehat{f}_k}{\partial\xi_j}(\xi)-\widehat{g}_j(\xi)\Big|\leqslant\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\big|1-\chi\big(\frac{|x|}{k}\big)\big|\big|g_j(x)\big|\,dx\leqslant\frac{1}{\sqrt{2\pi}^n}\int_{B(0,k)^\complement}\big|g_j(x)\big|\,dx$$

which converges to zero as $k\to\infty$. In other words, $\dfrac{\partial\widehat{f}_k}{\partial\xi_j}\to\widehat{g}_j$ uniformly on $\mathbb{R}^n$ as $k\to\infty$.

Similarly,

$$\sup_{\xi\in\mathbb{R}^n}\big|\widehat{f}_k(\xi)-\widehat{f}(\xi)\big|\leqslant\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\big|1-\chi\big(\frac{|x|}{k}\big)\big|\big|f(x)\big|\,dx\leqslant\frac{1}{\sqrt{2\pi}^n}\int_{B(0,k)^\complement}\big|f(x)\big|\,dx$$

which converges to zero as $k\to\infty$. Therefore, $\widehat{f}_k\to\widehat{f}$ uniformly on $\mathbb{R}^n$. By Theorem 1.5, $\dfrac{\partial\widehat{f}}{\partial\xi_j}=\widehat{g}_j$ so the lemma is concluded. □

Corollary 3.10. *For $f\in\mathscr{S}(\mathbb{R}^n)$, $\widehat{f}\in\mathscr{C}^\infty(\mathbb{R}^n)$ and*

$$D_\xi^\alpha\widehat{f}(\xi)=\frac{1}{i^{|\alpha|}}\Big[x_1^{\alpha_1}\cdots x_n^{\alpha_n}f(x)\Big]^\wedge(\xi)\,,$$

*where for a **multi-index** $\alpha=(\alpha_1,\cdots,\alpha_n)$, $|\alpha|\equiv\alpha_1+\cdots+\alpha_n$ and $D_\xi^\alpha\equiv\dfrac{\partial^{\alpha_1}}{\partial\xi_1^{\alpha_1}}\cdots\dfrac{\partial^{\alpha_n}}{\partial\xi_n^{\alpha_n}}=\dfrac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\cdots\partial\xi_n^{\alpha_n}}$.*

Lemma 3.11. *If $f\in\mathscr{S}(\mathbb{R}^n)$, then for $j\in\{1,2,\cdots,n\}$, $\mathscr{F}_x\Big[\dfrac{1}{i}\dfrac{\partial f}{\partial x_j}(x)\Big](\xi)=\xi_j\widehat{f}(\xi)$.*

Proof. W.L.O.G., we assume that $j = n$. Write $x = (x', x_n)$. Since $f \in \mathscr{S}(\mathbb{R}^n)$, there exists $C > 0$ such that

$$(1 + |x'|)^n |x_n| \big|f(x', x_n)\big| \leqslant C \qquad \forall\, x = (x', x_n) \in \mathbb{R}^n .$$

Then

1. For each $x' \in \mathbb{R}^{n-1}$, $f(x', \pm R) \to 0$ as $R \to \infty$.
2. The function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ defined by $g(x') = \dfrac{1}{(1 + |x'|)^n}$ is integrable on $\mathbb{R}^{n-1}$ (see the proof of Lemma 3.8), and $\big|f(x', \pm R)\big| \leqslant g(x')$ for each $x' \in \mathbb{R}^{n-1}$ and $R > 1$.

Therefore, the Dominated Convergence Theorem implies that

$$\lim_{R\to\infty} \int_{[-R,R]^{n-1}} f(x', \pm R) e^{-i(x',R)\cdot\xi}\, dx' = 0 ;$$

thus Fubini's Theorem and integrating by parts formula imply that

$$\begin{aligned}
\mathscr{F}\Big[\frac{1}{i}\frac{\partial f}{\partial x_n}(x)\Big](\xi) &= \frac{1}{i}\frac{1}{\sqrt{2\pi}^n} \lim_{R\to\infty} \int_{[-R,R]^n} \frac{\partial f}{\partial x_n}(x) e^{-ix\cdot\xi} dx \\
&= \frac{1}{i}\frac{1}{\sqrt{2\pi}^n} \lim_{R\to\infty} \int_{[-R,R]^{n-1}} \Big(\int_{-R}^{R} \frac{\partial f}{\partial x_n}(x) e^{-ix\cdot\xi} dx_n\Big) dx' \\
&= \frac{1}{i}\frac{1}{\sqrt{2\pi}^n} \lim_{R\to\infty} \Big[\Big(\int_{[-R,R]^{n-1}} f(x', x_n) e^{-i(x',x_n)\cdot\xi}\, dx'\Big)\Big|_{x_n=-R}^{x_n=R} + i\xi_n \int_{[-R,R]^n} f(x) e^{-ix\cdot\xi} dx\Big] \\
&= \xi_n \frac{1}{\sqrt{2\pi}^n} \lim_{R\to\infty} \int_{[-R,R]^n} f(x) e^{-ix\cdot\xi} dx = \xi_k \widehat{f}(\xi) .
\end{aligned}$$

□

Corollary 3.12. $\mathcal{P}(\xi_1, \cdots, \xi_n)\widehat{f}(\xi) = \mathscr{F}_x\Big[\mathcal{P}\Big(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_n}\Big) f(x)\Big](\xi)$ *for all* $f \in \mathscr{S}(\mathbb{R}^n)$ *and polynomial* $\mathcal{P}$.

Corollary 3.13. *The Fourier transform of a Schwartz function is a Schwartz function; that is,* $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$.

Proof. Let $\mathcal{P}$ be a polynomial and $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index. By Corollary 3.10 and 3.12,

$$\begin{aligned}
\mathcal{P}(\xi) D^\alpha \widehat{f}(\xi) &\equiv \mathcal{P}(\xi_1, \cdots, \xi_n) \frac{\partial^{|\alpha|} \widehat{f}}{\partial \xi_1^{\alpha_1} \cdots \partial \xi_n^{\alpha_n}}(\xi) \\
&= \frac{1}{i^{|\alpha|}} \mathscr{F}_x\Big[\mathcal{P}\big(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_n}\big)\big[x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} f(x)\big]\Big](\xi) ;
\end{aligned}$$

thus $\mathcal{P}D^\alpha \widehat{f}$ is the Fourier transform of a Schwartz function g defined by

$$g(x) = \frac{1}{i^{|\alpha|}}\mathcal{P}\Big(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_n}\Big)\big[x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} f(x)\big] .$$

By Proposition 3.4 and Lemma 3.8, $\mathcal{P}D^\alpha \widehat{f}$ is bounded. ▫

Remark 3.14. There exists a duality under $\wedge$ between differentiability and rapid decrease: the more differentiability f possesses, the more rapid decrease $\widehat{f}$ has and vice versa.

Definition 3.15. For all $f \in L^1(\mathbb{R}^n)$, we define operator $\mathscr{F}^*$ by

$$(\mathscr{F}^* f)(x) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}d\xi .$$

The function $\mathscr{F}^* f$ sometimes is also denoted by $\check{f}$.

Before proceeding, we establish a special case of the Fubini theorem for improper integrals which will be used in the following discussion.

Proposition 3.16 (Fubini theorem - special case)**.** *Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be absolutely integrable, and $g, h \in L^1(\mathbb{R}^n)$. If $|f(x,y)| \leqslant |g(x)||h(y)|$ for all $x, y \in \mathbb{R}^n$, then*

$$\begin{aligned}\int_{\mathbb{R}^{2n}} f(x,y)d(x,y) &\equiv \lim_{R\to\infty}\int_{[-R,R]^{2n}} f(x,y)d(x,y)\\ &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(x,y)\,dy\Big)dx = \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(x,y)\,dx\Big)dy .\end{aligned}$$

Proof. Let $\varepsilon > 0$ be given. Since $g, h \in L^1(\mathbb{R}^n)$, there exists $R_0 > 0$ such that

$$\int_{([-R,R]^n)^\complement} \big[|g(x)| + |h(x)|\big]\,dx < \frac{\varepsilon}{1 + \|g\|_{L^1(\mathbb{R}^n)} + \|h\|_{L^1(\mathbb{R}^n)}} \qquad \text{whenever} \quad R > R_0.$$

Therefore, the Fubini theorem for Riemann integral implies that

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x,y)\,dydx &= \int_{[-R,R]^n}\int_{\mathbb{R}^n} f(x,y)\,dydx + \int_{([-R,R]^n)^\complement}\int_{\mathbb{R}^n} f(x,y)\,dydx\\ &= \int_{[-R,R]^n}\Big(\int_{[-R,R]^n} + \int_{([-R,R]^n)^\complement}\Big) f(x,y)\,dydx + \int_{([-R,R]^n)^\complement}\int_{\mathbb{R}^n} f(x,y)\,dydx\\ &= \int_{[-R,R]^{2n}} f(x,y)d(x,y) + \int_{[-R,R]^n}\int_{([-R,R]^n)^\complement} f(x,y)\,dydx + \int_{([-R,R]^n)^\complement}\int_{\mathbb{R}^n} f(x,y)\,dydx ;\end{aligned}$$

thus by the fact that $|f(x,y)| \leqslant |g(x)||h(y)|$,

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(x,y)\,dy\Big)dx - \int_{[-R,R]^{2n}} f(x,y)d(x,y)\Big| \\
&\qquad \leqslant \int_{[-R,R]^n}\Big(\int_{([-R,R]^n)^\complement} |g(x)||h(y)|\,dy\Big)dx + \int_{([-R,R]^n)^\complement}\Big(\int_{\mathbb{R}^n} |g(x)||h(y)|\,dy\Big)dx \\
&\qquad \leqslant \|g\|_{L^1(\mathbb{R}^n)} \int_{([-R,R]^n)^\complement} |h(y)|\,dy + \|h\|_{L^1(\mathbb{R}^n)} \int_{([-R,R]^n)^\complement} |g(x)|\,dx \\
&\qquad < \frac{\big(\|g\|_{L^1(\mathbb{R}^n)} + \|h\|_{L^1(\mathbb{R}^n)}\big)\varepsilon}{1+\|g\|_{L^1(\mathbb{R}^n)} + \|h\|_{L^1(\mathbb{R}^n)}} < \varepsilon
\end{aligned}
$$

whenever $R > R_0$. □

Lemma 3.17. *If f and $g \in \mathscr{S}(\mathbb{R}^n)$, then*

$$(\check{f} * g)(x) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}\widehat{g}(\xi)\,d\xi\,.$$

Proof. By definition of $\check{f}$ and convolution,

$$(\check{f} \star g)(x) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} \check{f}(x-y)g(y)\,dy = \Big(\frac{1}{2\pi}\Big)^n \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(\xi)e^{i(x-y)\cdot\xi}g(y)\,d\xi\Big)dy\,.$$

The Fubini theorem then implies that

$$
\begin{aligned}
(\check{f} \star g)(x) &= \Big(\frac{1}{2\pi}\Big)^n \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}e^{-iy\cdot\xi}g(y)\,dy\Big)d\xi \\
&= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}\Big(\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\xi}g(y)\,dy\Big)d\xi = \Big(\frac{1}{2\pi}\Big)^n \int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}\widehat{g}(\xi)\,d\xi\,.
\end{aligned}
$$
□

The operator $\mathscr{F}^*$, indicated implicitly by the way it is written, is the formal adjoint of $\mathscr{F}$. To be more precise, we have the following

Lemma 3.18. *$(\mathscr{F}u, v)_{L^2(\mathbb{R}^n)} = (u, \mathscr{F}^*v)_{L^2(\mathbb{R}^n)}$ for all $u, v \in \mathscr{S}(\mathbb{R}^n)$, where $(\cdot,\cdot)_{L^2(\mathbb{R}^n)}$ is an inner product on $\mathscr{S}(\mathbb{R}^n)$ given by*

$$(u,v)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x)\overline{v(x)}\,dx\,.$$

Proof. Since $u, v \in \mathscr{S}(\mathbb{R}^n)$, by Fubini's Theorem,

$$
\begin{aligned}
(\mathscr{F}u, v)_{L^2(\mathbb{R}^n)} &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} u(x)e^{-ix\cdot\xi}dx\Big)\overline{v(\xi)}\,d\xi \\
&= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x)\overline{e^{ix\cdot\xi}v(\xi)}\,d\xi\,dx \\
&= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} u(x)\int_{\mathbb{R}^n} \overline{e^{ix\cdot\xi}v(\xi)}\,d\xi\,dx = (u, \mathscr{F}^*v)_{L^2(\mathbb{R}^n)}\,.
\end{aligned}
$$
□

3.3 The Fourier Inversion Formula

We remind the readers that our goal is to prove (3.2), while having introduced operators $\mathscr{F}$ and $\mathscr{F}^*$, it is the same as showing that $\mathscr{F}$ and $\mathscr{F}^*$ are inverse to each other; that is, we want to show that

$$\mathscr{F}\mathscr{F}^* = \mathscr{F}^*\mathscr{F} = \mathrm{Id} \qquad \text{on} \quad \mathscr{S}(\mathbb{R}^n)\,.$$

For $t > 0$ and $x \in \mathbb{R}$, let $P_t(x) = \dfrac{1}{\sqrt{t}}\, e^{-\frac{x^2}{2t}}$. Note that $P_t \in \mathscr{S}(\mathbb{R})$ and P_t is normalized so that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} P_t(x)\, dx = 1\,.$$

Now we compute the Fourier transform of P_t. By Lemma 3.9, we find that

$$\frac{d\widehat{P}_t}{d\xi}(\xi) = \frac{-i}{\sqrt{2\pi t}} \int_{\mathbb{R}} xP_t(x)e^{-ix\xi}\, dx = \frac{-i}{\sqrt{2\pi t}} \int_{\mathbb{R}} xP_t(x)\cos(\xi x)\, dx - \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} xP_t(x)\sin(\xi x)\, dx\,.$$

Since the functions $y = xP_t(x)$ is absolutely integrable on $\mathbb{R}$ for each fixed $t > 0$, the integral $\int_{\mathbb{R}} xP_t(x)\cos(\xi x)\, dx$ converges absolutely; thus by the fact that $x\cos(\xi x)$ are odd functions in x, we have

$$\int_{\mathbb{R}} xP_t(x)\cos(\xi x)\, dx = \lim_{R\to\infty} \int_{-R}^{R} xP_t(x)\cos(\xi x)\, dx = 0\,.$$

As a consequence,

$$\frac{d\widehat{P}_t}{d\xi}(\xi) = -\frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} xe^{-\frac{x^2}{2t}} \sin(x\xi)dx\,.$$

Similarly, $\widehat{P}_t(\xi) = \dfrac{1}{\sqrt{2\pi t}} \displaystyle\int_{\mathbb{R}} e^{-\frac{x^2}{2t}} \cos(x\xi)dx$, and the integration by parts formula implies that

$$\begin{aligned}
\frac{d\widehat{P}_t}{d\xi}(\xi) &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \frac{\partial}{\partial\xi}\Big(e^{-\frac{x^2}{2t}} \cos(x\xi)\Big) dx = -\frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} xe^{-\frac{x^2}{2t}} \sin(x\xi)dx \\
&= -\frac{1}{\sqrt{2\pi t}} \lim_{R\to\infty} \Big[-te^{-\frac{x^2}{2t}} \sin(x\xi)\Big|_{x=-R}^{x=R} + \int_{-R}^{R} \xi te^{-\frac{x^2}{2t}} \cos(x\xi)\, dx \Big] \\
&= -\frac{\xi t}{\sqrt{2\pi t}} \lim_{R\to\infty} \int_{-R}^{R} e^{-\frac{x^2}{2t}} \cos(x\xi)\, dx = -\frac{\xi t}{\sqrt{2\pi t}} \lim_{R\to\infty} \int_{-R}^{R} e^{-\frac{x^2}{2t}} \big[\cos(x\xi) - i\sin(x\xi)\big]\, dx \\
&= -\frac{\xi t}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{x^2}{2t}} e^{-ix\xi} dx = -\xi t\widehat{P}_t(\xi)\,;
\end{aligned}$$

thus $\widehat{P}_t(\xi) = Ce^{-\frac{t\xi^2}{2}}$. By the fact that $\widehat{P}_t(0) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\mathbb{R}} P_t(x)dx = 1$, we must have

$$\widehat{P}_t(\xi) = e^{-\frac{1}{2}t\xi^2}\,. \tag{3.6}$$

For $x \in \mathbb{R}^n$, if we define $\mathrm{P}_t(x) = \prod_{k=1}^n P_t(x_k) = \Big(\frac{1}{\sqrt{t}}\Big)^n e^{-\frac{|x|^2}{2t}}$, then (3.6) and the Fubini Theorem imply that $\widehat{\mathrm{P}}_t(\xi) = e^{-\frac{1}{2}t|\xi|^2}$. Therefore,

$$\widehat{\mathrm{P}}_t(\xi) = \Big(\frac{1}{\sqrt{t}}\Big)^n \mathrm{P}_{\frac{1}{t}}(\xi)$$

which, together with the fact that $\check{f}(x) = \widehat{f}(-x)$, further shows that

$$\check{\widehat{\mathrm{P}}}_t(x) = \Big(\frac{1}{\sqrt{t}}\Big)^n \widehat{\mathrm{P}_{\frac{1}{t}}}(-x) = \Big(\frac{1}{\sqrt{t}}\Big)^n \Big(\frac{1}{\sqrt{t^{-1}}}\Big)^n \mathrm{P}_t(-x) = \mathrm{P}_t(x)\,.$$

Similarly, $\widehat{\check{\mathrm{P}}}_t(\xi) = \mathrm{P}_t(\xi)$, so we establish that

$$\mathscr{F}^*\mathscr{F}(\mathrm{P}_t) = \mathscr{F}\mathscr{F}^*(\mathrm{P}_t) = \mathrm{P}_t\,. \tag{3.7}$$

The proof of the following lemma is similar to that of Theorem 2.20.

Lemma 3.19. *If $g \in \mathscr{S}(\mathbb{R}^n)$, then $\mathrm{P}_t * g \to g$ uniformly on $\mathbb{R}^n$ as $t \to 0^+$, where the convolution operator $*$ is given by*

$$(\mathrm{P}_t * g)(x) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} \mathrm{P}_t(x-y)g(y)\,dy = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} \mathrm{P}_t(y)g(x-y)\,dy\,. \tag{3.8}$$

Proof. Let $\varepsilon > 0$ be given. Since $g \in \mathscr{S}(\mathbb{R}^n)$, g is uniformly continuous; thus there exists $\delta > 0$ such that

$$\big|g(x) - g(y)\big| < \frac{\varepsilon}{2} \quad \text{whenever} \quad |x-y| < \delta\,.$$

Since $\displaystyle\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} \mathrm{P}_t(x)\,dx = 1$, for all $x \in \mathbb{R}^n$ we have

$$\begin{aligned}
\big|(\mathrm{P}_t * g)(x) - g(x)\big| &= \frac{1}{\sqrt{2\pi}^n}\Big|\int_{\mathbb{R}^n} g(x-y)\mathrm{P}_t(y)\,dy - \int_{\mathbb{R}^n} g(x)\mathrm{P}_t(y)\,dy\Big| \\
&= \frac{1}{\sqrt{2\pi}^n}\Big|\int_{\mathbb{R}^n} \big[(g(x-y) - g(x)\big]\mathrm{P}_t(y)\,dy\Big| \\
&\leqslant \frac{\varepsilon}{2}\frac{1}{\sqrt{2\pi}^n}\int_{|y|<\delta} \mathrm{P}_t(y)\,dy + \frac{2\|g\|_\infty}{\sqrt{2\pi}^n}\int_{|y|\geqslant\delta} \mathrm{P}_t(y)\,dy\,,
\end{aligned}$$

so we obtain that

$$\big\|(\mathrm{P}_t * g) - g\big\|_\infty \leqslant \frac{\varepsilon}{2} + \frac{2\|g\|_\infty}{\sqrt{2\pi}^n}\int_{|y|\geqslant\delta} \mathrm{P}_t(y)\,dy\,.$$

Note that

$$\int_{|y|>\delta} \mathrm{P}_t(y)\,dy = \frac{1}{\sqrt{t}^n}\int_{|y|>\delta} e^{-\frac{|y|^2}{2t}}\,dy = \int_{|z|>\frac{\delta}{\sqrt{t}}} e^{-\frac{|z|^2}{2}}\,dz$$

which approaches 0 as $t \to 0^+$; thus there exists $h > 0$ such that if $0 < |t| < h$,

$$\frac{2\|g\|_\infty}{\sqrt{2\pi}^n}\int_{|y|\geqslant\delta} \mathrm{P}_t(y)\,dy < \frac{\varepsilon}{2}\,.$$

Therefore, we conclude that

$$\big\|(P_t * g) - g\big\|_\infty < \varepsilon \quad \text{whenever} \quad 0 < t < h$$

which shows that $\mathrm{P}_t * g \to g$ uniformly as $t \to 0^+$. □

Theorem 3.20 (Fourier Inversion Formula)**.** *If $g \in \mathscr{S}(\mathbb{R}^n)$, then $\check{\hat{g}}(\xi) = \hat{\check{g}}(\xi) = g(\xi)$. In other words, $\mathscr{F}\mathscr{F}^* = \mathscr{F}^*\mathscr{F} = \mathrm{Id}$.*

Proof. Apply Lemma 3.17 with $f(\xi) = \widehat{\mathrm{P}}_t(\xi) = e^{-\frac{1}{2}t|\xi|^2}$, using (3.7) we find that

$$(\mathrm{P}_t * g)(x) = (\check{f} * g)(x) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|\xi|^2} e^{ix\cdot\xi}\hat{g}(\xi)\,d\xi\,.$$

Letting $t \to 0^+$, by Lemma 3.19 it suffices to show that

$$\lim_{t\to 0^+}\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|\xi|^2} e^{ix\cdot\xi}\hat{g}(\xi)\,d\xi = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{g}\,d\xi\,.$$

To see this, let $\varepsilon > 0$ be given. Since $\hat{g} \in \mathscr{S}(\mathbb{R}^n)$, there exists $R > 0$ such that

$$\int_{B(0,R)^\complement} \big|\hat{g}(\xi)\big|\,d\xi < \frac{\varepsilon}{2}\,.$$

For this particular R, there exists $\delta > 0$ such that if $0 < t < \delta$,

$$\frac{tR^2}{2}\|\hat{g}\|_{L^1(\mathbb{R}^n)} < \frac{\varepsilon}{2}\,.$$

Therefore, if $0 < t < \delta$, using the fact that $1 - e^{-x} \leqslant x$ for $x > 0$,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|\xi|^2} e^{ix\cdot\xi}\hat{g}(\xi)\,d\xi - \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{g}\,d\xi\Big| \\
&\qquad \leqslant \Big(\int_{B(0,R)} + \int_{B(0,R)^\complement}\Big)\big|e^{-\frac{1}{2}t|\xi|^2} - 1\big|\big|\hat{g}(\xi)\big|\,d\xi \\
&\qquad \leqslant \frac{1}{2}tR^2\int_{B(0,R)} \big|\hat{g}(\xi)\big|\,d\xi + \int_{B(0,R)^\complement}\big|\hat{g}(\xi)\big|\,d\xi < \varepsilon\,.
\end{aligned}$$

Therefore,

$$g(x) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} \hat{g}(\xi)e^{ix\cdot\xi}d\xi = \check{\hat{g}}(x)\,.$$

Let $\widetilde{\ }$ denote the reflection operator given by $\widetilde{f}(x) = f(-x)$. Then the change of variable formula implies that

$$\begin{aligned}\check{g}(\xi) &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} g(x)e^{ix\cdot\xi}dx = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} g(x)e^{-i(-x)\cdot\xi}dx \\ &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} g(-x)e^{-ix\cdot\xi}dx = \widehat{\widetilde{g}}(\xi)\,.\end{aligned}$$

On the other hand,

$$\check{g}(\xi) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} g(x)e^{-ix\cdot(-\xi)}dx = \widehat{g}(-\xi) = \widetilde{\widehat{g}}(\xi)\,;$$

thus $\widehat{\check{g}}(\xi) = \widehat{\widetilde{\widehat{g}}}(\xi) = \check{\widehat{g}}(\xi) = g(\xi)$. ◻

Corollary 3.21. *$\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is a bijection.*

Remark 3.22. In view of the Fourier Inversion Formula (Theorem 3.20), $\mathscr{F}^*$ sometimes is written as $\mathscr{F}^{-1}$, and is called the ***inverse Fourier transform***.

Theorem 3.23 (Plancherel formula for $\mathscr{S}(\mathbb{R}^n)$)**.** *If $f,\ g \in \mathscr{S}(\mathbb{R}^n)$, then*

$$\langle f, g\rangle_{L^2(\mathbb{R}^n)} = \langle \widehat{f}, \widehat{g}\rangle_{L^2(\mathbb{R}^n)}.$$

Proof. Recall that $(f, g\rangle_{L^2(\mathbb{R}^n)} = \displaystyle\int_{\mathbb{R}^n} f(x)\overline{g(x)}dx$. By Fubini's theorem,

$$\begin{aligned}\langle \check{f}, g\rangle_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \check{f}(x)\overline{g(x)}\,dx = \int_{\mathbb{R}^n}\Big[\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}d\xi\Big]\overline{g(x)}\,dx \\ &= \int_{\mathbb{R}^n} f(\xi)\Big[\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\overline{g(x)e^{-ix\cdot\xi}dx}\Big]d\xi = \langle f, \widehat{g}\rangle_{L^2(\mathbb{R}^n)}\,.\end{aligned}$$

Therefore, $\langle f, g\rangle_{L^2(\mathbb{R}^n)} = \langle \check{\widehat{f}}, g\rangle_{L^2(\mathbb{R}^n)} = \langle \widehat{f}, \widehat{g}\rangle_{L^2(\mathbb{R}^n)}$. ◻

Remark 3.24. The Plancherel formula is a "generalization" of the Parseval identity in the following sense. Define the ℓ^2 space as the collection of all square summable (complex) sequences; that is,

$$\ell^2 = \Big\{\{a_k\}_{k=-\infty}^{\infty} \subseteq \mathbb{C}\,\Big|\,\sum_{k=-\infty}^{\infty}|a_k|^2 < \infty\Big\}$$

with inner product

$$\big\langle\{a_k\}_{k=-\infty}^{\infty}, \{b_k\}_{k=-\infty}^{\infty}\big\rangle_{\ell^2} = \sum_{k=-\infty}^{\infty} a_k\overline{b_k}\,.$$

Here we treat $\{a_k\}_{k=-\infty}^{\infty}$ and $\{a_{k+1}\}_{k=-\infty}^{\infty}$ as different sequences. With $\|\cdot\|_{\ell^2}$ denoting the norm induced by the inner product above, the Parseval identity then implies that

$$\|f\|_{L^2(\mathbb{T})} = \big\|\{\widehat{f}_k\}_{k=-\infty}^{\infty}\big\|_{\ell^2},$$

thus by the identities

$$\|f+g\|^2_{L^2(\mathbb{T})} = \|f\|^2_{L^2(\mathbb{T})} + 2\mathrm{Re}\big(\langle f, g\rangle_{L^2(\mathbb{T})}\big) + \|g\|^2_{L^2(\mathbb{T})},$$
$$\|f-g\|^2_{L^2(\mathbb{T})} = \|f\|^2_{L^2(\mathbb{T})} - 2\mathrm{Re}\big(\langle f, g\rangle_{L^2(\mathbb{T})}\big) + \|g\|^2_{L^2(\mathbb{T})},$$

we find that

$$\begin{aligned}\mathrm{Re}\big(\langle f, g\rangle_{L^2(\mathbb{T})}\big) &= \frac{1}{4}\Big(\|f+g\|^2_{L^2(\mathbb{T})} + \|f-g\|^2_{L^2(\mathbb{T})}\Big) = \frac{1}{4}\Big(\sum_{k=-\infty}^{\infty} \big|\widehat{f}_k + \widehat{g}_k\big|^2 + \sum_{k=-\infty}^{\infty} \big|\widehat{f}_g - \widehat{g}_k\big|^2\Big) \\ &= \sum_{k=-\infty}^{\infty} \mathrm{Re}\big(\widehat{f}_k \overline{\widehat{g}_k}\big).\end{aligned}$$

Replacing g by ig in the identities above shows that $\mathrm{Im}\big(\langle f, g\rangle_{L^2(\mathbb{T})}\big) = \sum\limits_{k=-\infty}^{\infty} \mathrm{Im}\big(\widehat{f}_k\overline{\widehat{g}_k}\big)$; thus

$$\langle f, g\rangle_{L^2(\mathbb{T})} = \mathrm{Re}\big(\langle f, g\rangle_{L^2(\mathbb{T})}\big) + i\mathrm{Im}\big(\langle f, g\rangle_{L^2(\mathbb{T})}\big) = \sum_{k=-\infty}^{\infty} \widehat{f}_k\overline{\widehat{g}_k} = \big\langle \{\widehat{f}_k\}_{k=-\infty}^{\infty}, \{\widehat{g}_k\}_{k=-\infty}^{\infty}\big\rangle_{\ell^2}.$$

Define $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2$ by $F(f) = \{\widehat{f}_k\}_{k=-\infty}^{\infty}$. Then the identity above shows that

$$\langle f, g\rangle_{L^2(\mathbb{T})} = \langle \mathcal{F}(f), \mathcal{F}(g)\rangle_{\ell^2} \qquad \forall\, f, g \in L^2(\mathbb{T})$$

so that we obtain an identity similar to the Plancherel formula.

Remark 3.25. Even though in general an square integrable function might not be integrable, using the Plancherel formula the Fourier transform of L^2-functions can still be defined. Note that the Plancherel formula provides that

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\widehat{f}\|_{L^2(\mathbb{R}^n)} \qquad \forall\, f \in \mathscr{S}(\mathbb{R}^n). \tag{3.9}$$

If $f \in L^2(\mathbb{R}^n)$; that is, $|f|$ is square integrable, by the fact that $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, there exists a sequence $\{f_k\}_{k=1}^{\infty} \subseteq \mathscr{S}(\mathbb{R}^n)$ such that $\lim\limits_{k\to\infty} \|f_k - f\|_{L^2(\mathbb{R}^n)} = 0$. Then $\{f_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$; thus (3.9) implies that $\{\widehat{f}_k\}_{k=1}^{\infty}$ is also a Cauchy sequence in $L^2(\mathbb{R}^n)$. By the completeness of $L^2(\mathbb{R}^n)$ (which we did not cover in this lecture), there exists $g \in L^2(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty} \|\widehat{f}_k - g\|_{L^2(\mathbb{R}^n)} = 0.$$

We note that such a limit g is independent of the choice of sequence $\{f_k\}_{k=1}^{\infty}$ used to approximate f; thus we can denote this limit g as $\widehat{f}$. In other words, $\mathscr{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Moreover, by that $f_k \to f$ and $\widehat{f}_k \to \widehat{f}$ in $L^2(\mathbb{R}^n)$ as $k \to \infty$, we find that

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\widehat{f}\|_{L^2(\mathbb{R}^n)} \qquad \forall\, f \in L^2(\mathbb{R}^n)\,,$$

and the parallelogram law further implies that $\langle f, g\rangle_{L^2(\mathbb{R}^n)} = \langle \widehat{f}, \widehat{g}\rangle_{L^2(\mathbb{R}^n)}$ for all $f, g \in L^2(\mathbb{R}^n)$. Similar argument applies to the case of inverse transform of L^2-functions; thus we conclude that

$$\langle f, g\rangle_{L^2(\mathbb{R}^n)} = \langle \widehat{f}, \widehat{g}\rangle_{L^2(\mathbb{R}^n)} = \langle \check{f}, \check{g}\rangle_{L^2(\mathbb{R}^n)} \qquad \forall\, f, g \in L^2(\mathbb{R}^n)\,. \tag{3.10}$$

We have established the Fourier inversion formula for Schwartz class functions. Our goal next is to show that the Fourier inversion formula holds (in certain sense) for absolutely integrable function whose Fourier transform is also absolutely integrable. Motivated by the Fourier inversion formula, we would like to show, if possible, that

$$\hat{\check{f}} = \check{\hat{f}} = f \qquad \forall\, f \in L^1(\mathbb{R}^n) \text{ such that } \widehat{f} \in L^1(\mathbb{R}^n).$$

The above assertion cannot be true since $\hat{\check{f}}$ and $\check{\hat{f}}$ are both continuous (by Proposition 3.3) while $f \in L^1(\mathbb{R}^n)$ which is not necessary continuous. However, we will prove that the identity above holds for points x at which f is continuous.

Before proceeding, let us discuss some properties concerning the Fourier transform the product and the convolution of two Schwartz class functions.

Theorem 3.26. *If $f, g \in \mathscr{S}(\mathbb{R}^n)$, then $\mathscr{F}(f * g) = \widehat{f}\,\widehat{g}$. In particular, $f * g \in \mathscr{S}(\mathbb{R}^n)$ if $f, g \in \mathscr{S}(\mathbb{R}^n)$.*

Proof. By the definition of the Fourier transform and the convolution,

$$\begin{aligned}
\widehat{f * g}(\xi) &= \frac{1}{\sqrt{2\pi}^{\,n}} \mathscr{F}\Big(\int_{\mathbb{R}^n} f(\cdot - y) g(y)\, dy\Big)(\xi) \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^n} f(x-y) g(y)\, dy\Big] e^{-ix\cdot\xi} dx \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) \Big(\int_{\mathbb{R}^n} g(y) e^{-i(x+y)\cdot\xi} dx\Big) dy \\
&= \Big(\frac{1}{\sqrt{2\pi}^{\,n}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx\Big)\Big(\frac{1}{\sqrt{2\pi}^{\,n}} \int_{\mathbb{R}^n} g(y) e^{-iy\cdot\xi} dy\Big)
\end{aligned}$$

which concludes the theorem. □

Corollary 3.27. *$\mathscr{F}^*(f * g) = \check{f}\,\check{g}$, $\widehat{fg} = \widehat{f} * \widehat{g}$ and $\widecheck{fg} = \check{f} * \check{g}$ for all $f, g \in \mathscr{S}(\mathbb{R}^n)$.*

Lemma 3.28. *Let $f \in L^1(\mathbb{R}^n)$ and $g \in \mathscr{S}(\mathbb{R}^n)$. Then $\langle \widehat{f}, g\rangle = \langle f, \widehat{g}\rangle$ and $\langle \check{f}, g\rangle = \langle f, \check{g}\rangle$, where $\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)g(x)\,dx$.*

Proof. We only prove $\langle \widehat{f}, g\rangle = \langle f, \widehat{g}\rangle$ if $f \in L^1(\mathbb{R}^n)$ and $g \in \mathscr{S}(\mathbb{R}^n)$. By Proposition 3.4, $\widehat{f}$ is bounded and continuous on $\mathbb{R}^n$; thus $\widehat{f}g$ is an absolutely integrable continuous function. By the Fubini Theorem (Proposition 3.16),

$$\begin{aligned}\langle \widehat{f}, g\rangle &= \int_{\mathbb{R}^n}\Big(\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx\Big)g(\xi)d\xi = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(x)g(\xi)e^{-ix\cdot\xi}dx\Big)d\xi\\ &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(x)g(\xi)e^{-ix\cdot\xi}d\xi\Big)dx = \int_{\mathbb{R}^n} f(x)\Big(\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} g(\xi)e^{-ix\cdot\xi}d\xi\Big)dx\end{aligned}$$

which is exactly $\langle f, \widehat{g}\rangle$. □

Next, we shall establish some useful tools in analysis that can be applied in a wide range of applications. Those tools are fundamental in real analysis; however, we assume only knowledge of elementary analysis again to derive those results. We first define the class of locally integrable functions.

Definition 3.29. The space $L^1_{\rm loc}(\mathbb{R}^n)$ consists of all functions (defined on $\mathbb{R}^n$) that are absolutely integrable on all bounded open subsets of $\mathbb{R}^n$ and whose integrals are absolute convergent. In other words,

$$L^1_{\rm loc}(\mathbb{R}^n) = \Big\{ f : \mathbb{R}^n \to \mathbb{C} \,\Big|\, \int_{\mathcal{U}} f(x)\,dx \text{ is absolutely convergent for all bounded open } \mathcal{U} \subseteq \mathbb{R}^n \Big\}.$$

Again, we emphasize that we ***abuse*** the notation $L^1_{\rm loc}(\mathbb{R}^n)$ which in fact stands for a larger class of functions. We also note that $L^1(\mathbb{R}^n) \subseteq L^1_{\rm loc}(\mathbb{R}^n)$.

Lemma 3.30. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support (that is, the collection $\{x \in \mathbb{R}^n \,|\, \phi(x) \neq 0\}$ is bounded), and $f \in L^1_{loc}(\mathbb{R}^n)$. Then $\int_{\mathbb{R}^n} \phi(x-y)f(y)\,dy$ is smooth.*

Proof. It suffices to show that

$$\frac{\partial}{\partial x_j}\int_{\mathbb{R}^n} \phi(x-y)f(y)\,dy = \int_{\mathbb{R}^n} \phi_{x_j}(x-y)f(y)\,dy\,.$$

Let $x \in \mathbb{R}^n$ be given, and suppose that $\big\{y \in \mathbb{R}^n \,\big|\, \phi(y) \neq 0\big\} \subseteq B(0,R)$. Since ϕ has compact support, ϕ_{x_j} is uniformly continuous on $\mathbb{R}$; thus there exists $0 < \delta < 1$ such that

$$\big|\phi_{x_j}(z_1) - \phi_{x_j}(z_2)\big| < \frac{\varepsilon}{1 + \int_{B(x,R+1)} |f(y)|\, dy} \quad \text{whenever} \quad |z_1 - z_2| < \delta\,.$$

Define $g(x) = \int_{\mathbb{R}^n} \phi(x-y) f(y)\, dy$. Then for some function $\vartheta : \mathbb{R} \to (0,1)$,

$$\phi(x + he_j - y) - \phi(x-y) = h\phi_{x_j}(x - y + \vartheta(h)he_j)\,;$$

thus if $0 < |h| < \delta$,

$$\begin{aligned}
&\Big|\frac{g(x+he_j) - g(x)}{h} - \int_{\mathbb{R}^n} \phi_{x_j}(x-y) f(y)\, dy\Big| \\
&\quad \leqslant \int_{\mathbb{R}^n} \Big|\frac{\phi(x+he_j-y) - \phi(x-y)}{h} - \phi_{x_j}(x-y)\Big|\big|f(y)\big|\, dy \\
&\quad = \int_{B(x,R+1)} \big|\phi_{x_j}(x-y+\vartheta(h)he_j) - \phi_{x_j}(x-y)\big|\big|f(y)\big|\, dy < \varepsilon\,.
\end{aligned}$$

This implies that $g_{x_j}(x) = \int_{\mathbb{R}^n} \phi_{x_j}(x-y) f(y)\, dy$. □

A special class of functions will be used as the role of ϕ in Lemma 3.30. Let $\zeta : \mathbb{R} \to \mathbb{R}$ be a smooth function defined by

$$\zeta(x) = \begin{cases} \exp\Big(\dfrac{1}{x^2 - 1}\Big) & \text{if } |x| < 1\,, \\ 0 & \text{if } |x| \geqslant 1\,. \end{cases}$$

For $x \in \mathbb{R}^n$, define $\eta_1(x) = C\zeta(|x|)$, where C is chosen so that $\int_{\mathbb{R}^n} \eta_1(x)\, d = 1$. The change of variables formula then implies that $\eta_\varepsilon(x) \equiv \varepsilon^{-n}\eta_1(x/\varepsilon)$ has integral 1.

Definition 3.31. The sequence $\{\eta_\varepsilon\}_{\varepsilon > 0}$ is called the ***standard mollifiers***.

Example 3.32. Let $f = \mathbf{1}_{[a,b]}$, the characteristic/indicator function of the closed interval $[a,b]$. Then for $\varepsilon \ll 1$, the function $\eta_\varepsilon * f = \sqrt{2\pi}\eta_\varepsilon * f$ is smooth and has the property that

$$(\eta_\varepsilon * f)(x) = \begin{cases} 1 & \text{if } x \in [a+\varepsilon, b-\varepsilon]\,, \\ 0 & \text{if } x \in [a-\varepsilon, b+\varepsilon]^{\complement}, \end{cases}$$

and $0 \leqslant f \leqslant 1$. Therefore, $\eta_\varepsilon * f$ converges pointwise to f on $\mathbb{R} \backslash \{a,b\}$.

Since η_ε is supported in the closure of $B(0,\varepsilon)$, Lemma 3.30 implies that for any $f \in L^1_{\rm loc}(\mathbb{R}^n)$, $\eta_\varepsilon * f$ is smooth function. The following lemma shows that $\eta_\varepsilon * f$ converges to f at points of continuity of f.

Lemma 3.33. *Let $f \in L^1(\mathbb{R}^n)$ and x_0 be a continuity of f. Then*

$$(\eta_\varepsilon * f)(x_0) = \sqrt{2\pi}^n(\eta_\varepsilon \ast f)(x_0) \to f(x_0) \quad as \quad \varepsilon \to 0\,.$$

Proof. Let $\epsilon > 0$ be given. Since f is continuous at x_0, there exists $\delta > 0$ such that

$$\big|f(y) - f(x_0)\big| < \frac{\epsilon}{2} \qquad \text{whenever} \quad |y - x_0| < \delta\,.$$

Therefore, by the fact that $\displaystyle\int_{\mathbb{R}^n} \eta_\varepsilon(x_0 - y)\,dy = 1$, if $0 < \varepsilon < \delta$,

$$\begin{aligned}\big|(\eta_\varepsilon * f)(x_0) - f(x_0)\big| &= \Big|\int_{\mathbb{R}^n} \eta_\varepsilon(x_0 - y) f(y)\,dy - \int_{\mathbb{R}^n} \eta_\varepsilon(x_0 - y) f(x_0)\,dy\Big| \\ &\leqslant \int_{B(x_0,\varepsilon)} \eta_\varepsilon(x_0 - y)\big|f(y) - f(x_0)\big|\,dy \leqslant \frac{\epsilon}{2}\int_{B(x_0,\varepsilon)} \eta_\varepsilon(x_0 - y)\,dy < \epsilon\end{aligned}$$

which implies $(\eta_\varepsilon * f)(x_0) \to f(x_0)$ as $\varepsilon \to 0$. □

Lemma 3.34. *Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$. If $\langle f, g\rangle = 0$ for all $g \in \mathscr{S}(\mathbb{R}^n)$, then $f(x_0) = 0$ whenever f is continuous at x_0.*

Proof. W.L.O.G. we can assume that f is real-valued. Let $\{\eta_\varepsilon\}_{\varepsilon>0}$ be the standard mollifiers, x_0 be a point of continuity of f, and $f_\varepsilon \equiv \eta_\varepsilon * f = \sqrt{2\pi}^n(\eta_\varepsilon \star f)$. Then Lemma 3.30 shows that f_ε are smooth for all $\varepsilon > 0$.

Define $g(x) \equiv \eta_1(x - x_0) f_\varepsilon(x)$. Then $g \in \mathscr{S}(\mathbb{R}^n)$ since f_ε, η_1 are smooth and $\eta_1(\cdot - x_0)$ vanishes outside $B(x_0, 1)$. Since $\eta_\varepsilon, g \in \mathscr{S}(\mathbb{R}^n)$, Theorem 3.26 implies that $\eta_\varepsilon * g \equiv \sqrt{2\pi}^n(\eta_\varepsilon \star g) \in \mathscr{S}(\mathbb{R}^n)$; thus

$$\langle f, \eta_\varepsilon * g\rangle = 0 \qquad \forall\,\varepsilon > 0\,.$$

Since $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and g has compact support, Tonelli's Theorem implies that the function $F(x, y) = f(x)g(y)$ is absolutely integrable on $\mathbb{R}^n \times \mathbb{R}^n$. Moreover, by the boundedness and continuity of η_ε, the comparison test implies that the function $G(x, y) = F(x, y)\eta_\varepsilon(x - y)$ is also absolutely integrable on $\mathbb{R}^n \times \mathbb{R}^n$. Fubini's theorem then implies that

$$\langle f, \eta_\varepsilon * g\rangle = \int_{\mathbb{R}^n} f(x)\Big(\int_{\mathbb{R}^n} \eta_\varepsilon(x - y) g(y)\,dy\Big)dx = \int_{\mathbb{R}^n} g(y)\Big(\int_{\mathbb{R}^n} \eta_\varepsilon(x - y) f(x)\,dx\Big)dy\,;$$

thus by the fact that $\eta_\varepsilon(x-y) = \eta_\varepsilon(y-x)$ we conclude that $\langle f, \eta_\varepsilon * g\rangle = \langle \eta_\varepsilon * f, g\rangle$. As a consequence,

$$0 = \langle f, \eta_\varepsilon * g\rangle = \langle \eta_\varepsilon * f, \eta_1(\cdot - x_0)(\eta_\varepsilon * f)\rangle = \int_{\mathbb{R}^n} \eta_1(x-x_0)\big|(\eta_\varepsilon * f)(x)\big|^2\, dx$$

which implies that $\eta_\varepsilon * f = 0$ on $B(x_0, 1)$. We then conclude from Lemma 3.33 that $(\eta_\varepsilon * f)(x_0) \to f(x_0)$ as $\varepsilon \to 0$. ▫

Now we state the Fourier inversion formula for functions of more general class.

Theorem 3.35 (Fourier Inversion Formula)**.** *Let $f \in L^1(\mathbb{R}^n)$ such that $\hat{f} \in L^1(\mathbb{R}^n)$. Then*

$$\check{\hat{f}}(x) = \hat{\check{f}}(x) = f(x) \qquad \text{whenever } f \text{ is continuous at } x.$$

Proof. Let $f : \mathbb{R}^n \to \mathbb{C}$ be such that $f, \hat{f} \in L^1(\mathbb{R}^n)$. By the fact that $\check{f} = \tilde{\hat{f}}$ (where $\tilde{}$ is the reflection operator), we also have $\check{f} \in L^1(\mathbb{R}^n)$. By Lemma 3.28 and the Fourier inversion formula for Schwartz class functions (Theorem 3.20),

$$\langle \check{\hat{f}}, g\rangle = \langle \hat{f}, \check{g}\rangle = \langle f, \hat{\check{g}}\rangle = \langle f, g\rangle \;\text{ and }\; \langle \hat{\check{f}}, g\rangle = \langle \check{f}, \hat{g}\rangle = \langle f, \check{\hat{g}}\rangle = \langle f, g\rangle \quad \forall\, g \in \mathscr{S}(\mathbb{R}^n)\,.$$

In other words, if $f, \hat{f} \in L^1(\mathbb{R}^n)$,

$$\langle \check{\hat{f}} - f, g\rangle = \langle \hat{\check{f}} - f, g\rangle = 0 \qquad \forall\, g \in \mathscr{S}(\mathbb{R}^n)\,.$$

By Proposition 3.4, $\check{\hat{f}}, \hat{\check{f}} \in L^1_{\rm loc}(\mathbb{R}^n)$; thus the theorem is concluded by Lemma 3.34 and the fact that $\check{\hat{f}}$ and $\hat{\check{f}}$ are continuous (which is guaranteed by Proposition 3.4). ▫

Remark 3.36. Since an integrable function $f : \mathbb{R}^n \to \mathbb{R}$ must be continuous **almost everywhere** on $\mathbb{R}^n$, Theorem 3.35 implies that if $f : \mathbb{R}^n \to \mathbb{R}$ is a function such that f, $\hat{f} \in L^1(\mathbb{R}^n)$, then $\check{\hat{f}} = \hat{\check{f}} = f$ almost everywhere.

Remark 3.37. In some occasions (especially in engineering applications), the Fourier transform and inverse Fourier transform of a (Schwartz) function f are defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i2\pi x\cdot\xi} dx \qquad \text{and} \qquad \check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{i2\pi x\cdot\xi} d\xi\,. \tag{3.11}$$

Using this definition, we still have

1. $\check{\hat{f}} = \hat{\check{f}} = f$ for all $f \in \mathscr{S}(\mathbb{R}^n)$;
2. if $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$, then $\check{\hat{f}}(x) = \hat{\check{f}}(x) = f(x)$ for all x at which f is continuous.

3.4 The Fourier Transform of Generalized Functions

It is often required to consider the Fourier transform of functions which do not belong to $L^1(\mathbb{R}^n)$. For example, the **normalized sinc function** $\text{sinc}: \mathbb{R} \to \mathbb{R}$ defined by

$$\text{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x} & \text{if } x \neq 0\,, \\ 1 & \text{if } x = 0\,, \end{cases} \tag{3.12}$$

does not belong to $L^1(\mathbb{R})$ but it is a very important function in the study of signal processing.

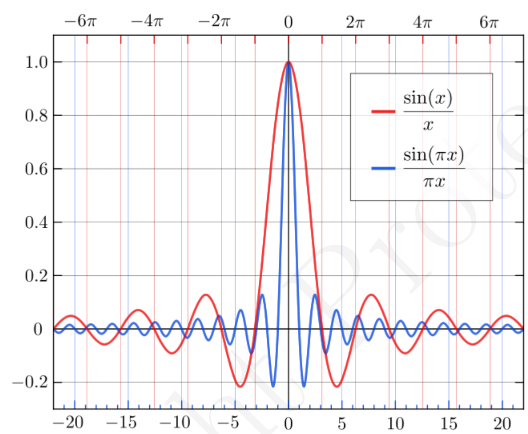

Figure 3.1: The graphs of unnormalized and normalized sinc functions (from wiki)

Moreover, there are "functions" that are not even functions in the traditional sense. For example, in physics and engineering applications the Dirac delta "function" δ is defined as the "function" which validates the relation

$$\int_{\mathbb{R}^n} \delta(x)\phi(x)\,dx = \phi(0) \qquad \forall\, \phi \in \mathscr{C}(\mathbb{R}^n)$$

In fact, there is no function (in the traditional sense) satisfying the property given above. Can we take the Fourier transform of those "functions" as well? To understand this topic better, it is required to study the theory of distributions.

The fundamental idea of the theory of distributions (generalized functions) is to identify a function v defined on $\mathbb{R}^n$ with the family of its integral averages

$$v \approx \int_{\mathbb{R}^n} v(x)\phi(x)\,dx \qquad \forall\, \phi \in \mathscr{C}_c^\infty(\mathbb{R}^n)\,,$$

where $\mathscr{C}_c^\infty(\mathbb{R}^n)$ denotes the collection of $\mathscr{C}^\infty$-functions with compact support, and is often denoted by $\mathcal{D}(\mathbb{R}^n)$ in the theory of distributions. Note that this makes sense for any locally integrable function v, and $\mathcal{D}(\mathbb{R}^n) \subseteq \mathscr{S}(\mathbb{R}^n)$.

To understand the meaning of distributions, let us turn to a situation in physics: measuring the temperature. To measure the temperature T at a point a, instead of outputting the exact value of $T(a)$ the thermometer instead outputs the **overall value** of the temperature near a point. In other words, the reading of the temperature is determined by a pairing of the temperature distribution with the thermometer. The role of the test function ϕ is like the thermometer used to measure the temperature.

The Fourier transform can be defined on the space of tempered distributions, a smaller class of generalized functions. A tempered distribution on $\mathbb{R}^n$ is a continuous linear functional on $\mathscr{S}(\mathbb{R}^n)$. In other words, T is a tempered distribution if

$$T : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C},\ T(c\phi + \psi) = cT(\phi) + T(\psi) \text{ for all } c \in \mathbb{C} \text{ and } \phi, \psi \in \mathscr{S}(\mathbb{R}^n),$$
$$\text{and } \lim_{j\to\infty} T(\phi_j) = T(\phi) \text{ if } \{\phi_j\}_{j=1}^\infty \subseteq \mathscr{S}(\mathbb{R}^n) \text{ and } \phi_j \to \phi \text{ in } \mathscr{S}(\mathbb{R}^n).$$

The convergence in $\mathscr{S}(\mathbb{R}^n)$ is described by semi-norms, and is given in the following

Definition 3.38 (Convergence in $\mathscr{S}(\mathbb{R}^n)$). For each $k \in \mathbb{N}$, define the semi-norm

$$p_k(u) = \sup_{x\in\mathbb{R}^n, |\alpha|\leqslant k} \langle x\rangle^k |D^\alpha u(x)|,$$

where $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$. A sequence $\{u_j\}_{j=1}^\infty \subseteq \mathscr{S}(\mathbb{R}^n)$ is said to converge to u in $\mathscr{S}(\mathbb{R}^n)$ if $p_k(u_j - u) \to 0$ as $j \to \infty$ for all $k \in \mathbb{N}$.

We note that $p_k(u) \leqslant p_{k+1}(u)$, so $\{u_j\}_{j=1}^\infty \subseteq \mathscr{S}(\mathbb{R}^n)$ converges to u in $\mathscr{S}(\mathbb{R}^n)$ if $p_k(u_j - u) \to 0$ as $j \to \infty$ for $k \gg 1$. We also note that if $\{u_j\}_{j=1}^\infty$ converge to u in $\mathscr{S}(\mathbb{R}^n)$, then $\{u_j\}_{j=1}^\infty$ converges uniformly to u on $\mathbb{R}^n$.

Definition 3.39 (Tempered Distributions). A linear map $T : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ is continuous if there exists $N \in \mathbb{N}$ such that for each $k \geqslant N$, there exists a constant C_k such that

$$\big|\langle T, u\rangle\big| \leqslant C_k p_k(u) \quad \forall\, u \in \mathscr{S}(\mathbb{R}^n),$$

where $\langle T, u\rangle \equiv T(u)$ is the usual notation for the value of T at u. The collection of continuous linear functionals on $\mathscr{S}(\mathbb{R}^n)$ is denoted by $\mathscr{S}(\mathbb{R}^n)'$. Elements of $\mathscr{S}(\mathbb{R}^n)'$ are called ***tempered distributions***.

Example 3.40. Let $L^p(\mathbb{R}^n)$ denote the collection of Riemann measurable functions whose p-th power is integrable; that is,

$$L^p(\mathbb{R}^n) = \Big\{ f : \mathbb{R}^n \to \mathbb{C} \,\Big|\, f \text{ is Riemann measurable and } \int_{\mathbb{R}^n} \big|f(x)\big|^p \, dx < \infty \Big\} .$$

Every L^p-function $f : \mathbb{R}^n \to \mathbb{C}$ can be viewed as a tempered distribution for all $p \in [1, \infty]$. In fact, the tempered distribution T_f associated with f is defined by

$$T_f(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x) \, dx \qquad \forall \, \phi \in \mathscr{S}(\mathbb{R}^n) . \tag{3.13}$$

Since we have use $\langle \cdot, \cdot \rangle$ for the integral of product of functions, the value of the tempered distribution of f at ϕ is exactly $\langle f, \phi \rangle$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$. This should explain the use of the notation $\langle T, \phi \rangle$.

Now we show that T_f given by (3.13) is indeed a tempered distribution. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ be given. Then $\|\phi\|_{L^\infty(\mathbb{R}^n)} \leqslant p_k(\phi)$ for all $k \in \mathbb{N}$, while for $1 \leqslant q < \infty$ and $k > \dfrac{n}{q}$,

$$\begin{aligned} \|\phi\|_{L^q(\mathbb{R}^n)} &\equiv \int_{\mathbb{R}^n} \big|\phi(x)\big|^q \, dx \Big)^{\frac{1}{q}} = \Big(\int_{\mathbb{R}^n} \langle x \rangle^{-kq} \big[\langle x \rangle^k |\phi(x)| \big]^q \, dx \Big)^{\frac{1}{q}} \leqslant \Big(\int_{\mathbb{R}^n} \langle x \rangle^{-kq} \, dx \Big)^{\frac{1}{q}} p_k(\phi) \\ &\leqslant \Big(\omega_{n-1} \int_0^\infty (1 + r^2)^{-\frac{kq}{2}} r^{n-1} dr \Big)^{\frac{1}{q}} p_k(\phi) . \end{aligned}$$

Note that $\displaystyle\int_0^\infty (1 + r^2)^{-\frac{kq}{2}} r^{n-1} dr < \infty$ if $k > \dfrac{n}{q}$; thus for all $q \in [1, \infty]$, there exists $C_{k,q,n} > 0$ such that

$$\|\phi\|_{L^q(\mathbb{R}^n)} \leqslant C_{k,q,n} p_k(\phi) \qquad \forall \, k \gg 1 . \tag{3.14}$$

Therefore, if $f \in L^p(\mathbb{R}^n)$, by the Hölder inequality we have

$$\big|\langle f, \phi \rangle\big| \leqslant \|f\|_{L^p(\mathbb{R}^n)} \|\phi\|_{L^{p'}(\mathbb{R}^n)} \leqslant C_{k,p',n} \|f\|_{L^p(\mathbb{R}^n)} p_k(\phi) \qquad \forall \, k \gg 1 ,$$

where $p' \in [1, \infty]$ is the Hölder conjugate of p satisfying $\dfrac{1}{p} + \dfrac{1}{p'} = 1$; thus $T_f \in \mathscr{S}(\mathbb{R}^n)'$ if $f \in L^p(\mathbb{R}^n)$. Note that the sinc function belongs to $L^2(\mathbb{R})$ so that $T_{\text{sinc}} \in \mathscr{S}(\mathbb{R})'$.

Example 3.41. Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic, Riemann measurable function such that $\displaystyle\int_{-\pi}^{\pi} |f(x)| \, dx < \infty$, and $\phi \in \mathscr{S}(\mathbb{R})$. Lemma 3.11 (or Corollary 3.13) and Proposition 3.4

imply that

$$\begin{aligned}
|\xi|^2|\phi(\xi)| &= \big|\mathscr{F}_x\big[(\check{\phi})''(x)\big](\xi)\big| \leqslant \big\|(\check{\phi})''\big\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} \big|(\check{\phi})''(x)\big|\,dx = \int_{\mathbb{R}} \big|(\widehat{\phi})''(x)\big|\,dx \\
&= \int_{\mathbb{R}} \langle x\rangle^{-2}\big|\langle x\rangle^2(\widehat{\phi})''(x)\big|\,dx \leqslant \Big(\sup_{x\in\mathbb{R}}\big|\langle x\rangle^2(\widehat{\phi})''(x)\big|\Big)\int_{\mathbb{R}}\langle x\rangle^{-2}\,dx \\
&= \pi\sup_{x\in\mathbb{R}}\Big|\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Big(1-\frac{d^2}{d\xi^2}\Big)\big[\xi^2\phi(\xi)\big]e^{-ix\xi}\,d\xi\Big| \\
&\leqslant \sqrt{\frac{\pi}{2}}\int_{\mathbb{R}} 2\sum_{|\alpha|\leqslant 2}\langle\xi\rangle^2\big|D^\alpha\phi(\xi)\big|\,d\xi \leqslant \sqrt{2\pi}\int_{\mathbb{R}}\langle\xi\rangle^{-2}p_4(\phi)\,d\xi \leqslant \pi^2 p_4(\phi)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\big|\langle f,\phi\rangle\big| &= \Big|\sum_{k=-\infty}^{\infty}\int_{-\pi+2k\pi}^{\pi+2k\pi} f(x)\phi(x)\,dx\Big| \leqslant \sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi}\big|f(x)\big|\big|\phi(x-2k\pi)\big|\,dx \\
&= \int_{-\pi}^{\pi}\big|f(x)\big|\big|\phi(x)\big|\,dx + \sum_{|k|\geqslant 1}\int_{-\pi}^{\pi}\big|f(x)\big|\big|\phi(x-2k\pi)\big|\,dx \\
&\leqslant p_0(\phi)\int_{-\pi}^{\pi}\big|f(x)\big|\,dx + \sum_{|k|\geqslant 1}\int_{-\pi}^{\pi}\big|f(x)\big|\frac{\pi^2}{|x-2k\pi|^2}p_4(\phi)\,dx \\
&\leqslant \Big(\int_{-\pi}^{\pi}\big|f(x)\big|\,dx\Big)\Big(1+2\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\Big)p_4(\phi)
\end{aligned}$$

which implies that T_f is a tempered distribution. In particular, $T_c \in \mathscr{S}(\mathbb{R})'$ for all constant $c \in \mathbb{R}$.

From now on, we identify f with the tempered distribution T_f if $f \in L^p(\mathbb{R}^n)$. For example, if $T \in \mathscr{S}(\mathbb{R}^n)'$ and $f : \mathbb{R}^n \to \mathbb{C}$ is bounded or integrable, we say that $T = f$ in $\mathscr{S}(\mathbb{R}^n)'$ if $T = T_f$, where T_f is the tempered distribution associated with the function f.

Remark 3.42. Let $f(x) = e^{x^4} \in L^1_{\rm loc}(\mathbb{R}^n)$. Then $\langle T_f, e^{-x^2}\rangle = \infty$. Therefore, being in $L^1_{\rm loc}(\mathbb{R}^n)$ is not good enough to generate elements in $\mathscr{S}(\mathbb{R}^n)'$, and it requires that $|f(x)| \leqslant C(1+|x|^N)$ for any N. In such a case, $T_f \in \mathscr{S}(\mathbb{R}^n)'$ is well-defined.

Example 3.43 (Dirac delta function)**.** Consider the map $\delta : \mathscr{C}(\mathbb{R}^n) \to \mathbb{R}$ defined by $\delta(\phi) = \phi(0)$. Then $\big|\langle\delta,\phi\rangle\big| \leqslant p_0(\phi) \leqslant p_k(\phi)$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$; thus $\delta \in \mathscr{S}(\mathbb{R}^n)'$. Similarly, the Dirac delta function at a point ω defined by $\langle\delta_\omega,\phi\rangle = \phi(\omega)$ is also a tempered distribution.

As shown in the example above, a tempered distribution might not be defined in the pointwise sense. Therefore, how to define usual operations such as translation, dilation, and reflection on generalized functions should be answered prior to define the Fourier transform of tempered distributions. For completeness, let us start from providing the definitions of translation, dilation and reflection operators.

Definition 3.44 (Translation, dilation, and reflection)**.** Let $f : \mathbb{R}^n \to \mathbb{C}$ be a function.

1. For $h \in \mathbb{R}^n$, the translation operator τ_h maps f to $\tau_h f$ given by $(\tau_h f)(x) = f(x - h)$.
2. For $\lambda > 0$, the dilation operator $d_\lambda : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ maps f to $d_\lambda f$ given by $(d_\lambda f)(x) = f(\lambda^{-1}x)$.
3. The Reflection operator $\widetilde{\ }$ maps f to $\widetilde{f}$ given by $\widetilde{f}(x) = f(-x)$.

Now suppose that $T \in \mathscr{S}(\mathbb{R}^n)'$. We expect that $\tau_h T$, $d_\lambda T$ and $\widetilde{T}$ are also tempered distributions, so we need to provide the values of $\langle \tau_h T, \phi\rangle$, $\langle d_\lambda T, \phi\rangle$ and $\langle \widetilde{T}, \phi\rangle$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$. If $T = T_f$ is the tempered distribution associated with $f \in L^1(\mathbb{R}^n)$, then for $\phi \in \mathscr{S}(\mathbb{R}^n)$, the change of variable formula implies that

$$\begin{aligned}
\langle \tau_h f, g\rangle &= \int_{\mathbb{R}^n} f(x-h)g(x)\,dx = \int_{\mathbb{R}^n} f(x)g(x+h)\,dx = \langle f, \tau_{-h} g\rangle\,,\\
\langle d_\lambda f, g\rangle &= \int_{\mathbb{R}^n} f(\lambda^{-1}x)g(x)\,dx = \int_{\mathbb{R}^n} f(x)g(\lambda x)\lambda^n\,dx = \langle f, \lambda^n d_{\lambda^{-1}} g\rangle\,,\\
\langle \widetilde{f}, g\rangle &= \int_{\mathbb{R}^n} f(-x)g(x)\,dx = \int_{\mathbb{R}^n} f(x)g(-x)\,dx = \langle f, \widetilde{g}\rangle\,.
\end{aligned}$$

The computations above motivate the following

Definition 3.45. Let $h \in \mathbb{R}^n$, $\lambda > 0$, and τ_h and d_λ be the translation and dilation operator given in Definition 3.44. For $T \in \mathscr{S}(\mathbb{R}^n)'$, $\tau_h T$, $d_\lambda T$ and $\widetilde{T}$ are the tempered distributions defined by

$$\langle \tau_h T, \phi\rangle = \langle T, \tau_{-h}\phi\rangle\,,\quad \langle d_\lambda T, \phi\rangle = \langle T, \lambda^n d_{\lambda^{-1}}\phi\rangle \text{ and } \langle \widetilde{T}, \phi\rangle = \langle T, \widetilde{\phi}\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

We note that $\tau_h T$, $d_\lambda T$ and $\widetilde{T}$ are tempered distributions since

$$\begin{aligned}
p_k(\tau_{-h}\phi) &\leqslant \sup_{x\in\mathbb{R}^n, |\alpha|\leqslant k} \langle x\rangle^k \big|D^\alpha\phi(x-h)\big| \leqslant (2|h|^2+1)^{\frac{k}{2}} p_k(\phi)\,,\\
p_k(\lambda^n d_{\lambda^{-1}}\phi) &\leqslant \lambda^n \sup_{x\in\mathbb{R}^n, |\alpha|\leqslant k} \langle x\rangle^k \lambda^{|\alpha|}\big|(D^\alpha\phi)(\lambda x)\big| \leqslant \lambda^n \max\{\lambda^k, \lambda^{-k}\} p_k(\phi)\,,\\
p_k(\widetilde{\phi}) &= p_k(\phi)
\end{aligned}$$

so that for $k \gg 1$,

$$
\begin{aligned}
\big|\langle \tau_h T, \phi\rangle\big| &= \big|\langle T, \tau_{-h}\phi\rangle\big| \leqslant C_k(2|h|^2+1)^{\frac{k}{2}} p_k(\phi) = \widetilde{C}_k p_k(\phi)\,,\\
\big|\langle d_\lambda T, \phi\rangle\big| &= \big|\langle T, \lambda^n d_{\lambda^{-1}}\phi\rangle\big| \leqslant C_k \lambda^n \max\{\lambda^k, \lambda^{-k}\} p_k(\phi) = \widetilde{C}_k p_k(\phi)\,,\\
\big|\langle \widetilde{T}, \phi\rangle\big| &= \big|\langle T, \widetilde{\phi}\rangle\big| \leqslant C_k p_k(\phi)\,.
\end{aligned}
$$

Example 3.46. Let $\omega, h \in \mathbb{R}^n$ and $\lambda > 0$.

1. $\tau_h \delta_\omega = \delta_{\omega-h}$ since if $\phi \in \mathscr{S}(\mathbb{R}^n)$, $\langle \tau_h\delta_\omega, \phi\rangle = \langle \delta_\omega, \tau_{-h}\phi\rangle = \phi(\omega - h) = \langle \delta_{\omega-h}, \phi\rangle$.
2. $d_\lambda \delta_\omega = \lambda^n \delta_{\lambda\omega}$ since if $\phi \in \mathscr{S}(\mathbb{R}^n)$, $\langle d_\lambda \delta_\omega, \phi\rangle = \langle \delta_\omega, \lambda^n d_{1/\lambda}\phi\rangle = \lambda^n \phi(\lambda\omega) = \langle \lambda^n \delta_{\lambda\omega}, \phi\rangle$.
3. $\widetilde{\delta_\omega} = \delta_{-\omega}$ since if $\phi \in \mathscr{S}(\mathbb{R}^n)$, $\langle \widetilde{\delta_\omega}, \phi\rangle = \langle \delta_\omega, \widetilde{\phi}\rangle = \phi(-\omega) = \langle \delta_{-\omega}, \phi\rangle$.

From the experience of defining the translation, dilation and reflection of tempered distribution, now we can talk about how to defined Fourier transform of tempered distributions. Recall that in Lemma 3.28 we have established that

$$
\langle \widehat{f}, g\rangle = \langle f, \widehat{g}\rangle \quad \text{and} \quad \langle \check{f}, g\rangle = \langle f, \check{g}\rangle \qquad \forall\, f \in L^1(\mathbb{R}^n), g \in \mathscr{S}(\mathbb{R}^n)\,.
$$

Since the identities above hold for all L^1-functions f (and L^1-functions corresponds to tempered distributions T_f through (3.13)), we expect that the Fourier transform of tempered distributions has to satisfy the identities above as well. Let $T \in \mathscr{S}(\mathbb{R}^n)'$ be given, and define $\widehat{T} : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ by

$$
\widehat{T}(\phi) = \langle \widehat{T}, \phi\rangle \equiv \langle T, \widehat{\phi}\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,. \tag{3.15}
$$

Note that if $k \geqslant 2$,

$$
\begin{aligned}
p_k(\widehat{\phi}) &= \sup_{\xi\in\mathbb{R}^n, |\alpha|\leqslant k} \langle \xi\rangle^k \big|D^\alpha \widehat{\phi}(\xi)\big| = \sup_{\xi\in\mathbb{R}^n, |\alpha|\leqslant k} \langle \xi\rangle^k \Big|\mathscr{F}_x\big[x^\alpha \phi(x)\big](\xi)\Big|\\
&\leqslant \sup_{\xi\in\mathbb{R}^n, |\alpha|\leqslant k} (n+1)^{\frac{k}{2}-1}\big(1 + |\xi_1|^k + \cdots + |\xi_n|^k\big)\Big|\mathscr{F}_x\big[x^\alpha\phi(x)\big](\xi)\Big|\\
&\leqslant (n+1)^{\frac{k}{2}-1} \sup_{\xi\in\mathbb{R}^n, |\alpha|\leqslant k} \Big|\mathscr{F}_x\big[(1 + \partial_{x_1}^k + \cdots + \partial_{x_n}^k)\big(x^\alpha\phi(x)\big)\big](\xi)\Big|\,.
\end{aligned}
$$

Since

$$
\begin{aligned}
\sup_{\xi\in\mathbb{R}^n, |\alpha|\leqslant k} \Big|\mathscr{F}_x\big[x^\alpha\phi(x)\big](\xi)\Big| &\leqslant \sup_{|\alpha|\leqslant k} \big\|x^\alpha \phi(x)\big\|_{L^1(\mathbb{R}^n)} \leqslant \big\|\langle x\rangle^k \phi(x)\big\|_{L^1(\mathbb{R}^n)}\\
&\leqslant \big\|\langle x\rangle^{-n-1}\big\|_{L^1(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n} \langle x\rangle^{n+k+1}\big|\phi(x)\big| \leqslant \big\|\langle x\rangle^{-n-1}\big\|_{L^1(\mathbb{R}^n)}\, p_{n+k+1}(\phi)
\end{aligned}
$$

and for $1 \leqslant j \leqslant n$,

$$\begin{aligned}
\sup_{\xi\in\mathbb{R}^n,|\alpha|\leqslant k}\Big|\mathscr{F}_x\big[\partial^k_{x_j}(x^\alpha\phi(x)\big](\xi)\Big| &\leqslant \sum_{\ell=0}^k C^k_\ell \sup_{\xi\in\mathbb{R}^n,|\alpha|\leqslant k}\Big|\mathscr{F}_x\big[\partial^{k-\ell}_{x_j}x^\alpha\partial^\ell_{x_j}\phi(x)\big](\xi)\Big| \\
&\leqslant \sum_{\ell=0}^k C^k_\ell \sup_{|\alpha|\leqslant k}\big\|\partial^{k-\ell}_{x_j}x^\alpha\partial^\ell_{x_j}\phi(x)\big\|_{L^1(\mathbb{R}^n)} \leqslant \sum_{\ell=0}^k C^k_\ell \sup_{|\alpha|\leqslant k}|\alpha|!\big\|\langle x\rangle^{|\alpha|-k+\ell}\partial^\ell_{x_j}\phi(x)\big\|_{L^1(\mathbb{R}^n)} \\
&\leqslant \sum_{\ell=0}^k C^k_\ell k! \sup_{|\beta|=\ell}\big\|\langle x\rangle^\ell D^\beta\phi(x)\big\|_{L^1(\mathbb{R}^n)} \leqslant k!\sum_{\ell=0}^k C^k_\ell\big\|\langle x\rangle^{-n-1}\big\|_{L^1(\mathbb{R}^n)} p_{n+\ell+1}(\phi) \\
&\leqslant k!\big\|\langle x\rangle^{-n-1}\big\|_{L^1(\mathbb{R}^n)} p_{n+k+1}(\phi)\sum_{\ell=0}^k C^k_\ell = k!2^k\big\|\langle x\rangle^{-n-1}\big\|_{L^1(\mathbb{R}^n)} p_{n+k+1}(\phi)\,,
\end{aligned}$$

we conclude that

$$p_k(\widehat{\phi}) \leqslant (n+1)^{\frac{k}{2}-1}(1+nk!2^k)\big\|\langle x\rangle^{-n-1}\big\|_{L^1(\mathbb{R}^n)} p_{n+k+1}(\phi) = \bar{C}(n,k)p_{n+k+1}(\phi)\,. \tag{3.16}$$

Therefore,

$$\big|\langle\widehat{T},\phi\rangle\big| = \big|\langle T,\widehat{\phi}\rangle\big| \leqslant C_k p_k(\widehat{\phi}) \leqslant C_k\bar{C}(n,k)p_{k+n+1}(\phi) \qquad \forall\, k \gg 1 \tag{3.17}$$

which shows that $\widehat{T}$ defined by (3.15) is a tempered distribution. Similarly, $\check{T} : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ defined by $\langle\check{T},\phi\rangle = \langle T,\check{\phi}\rangle$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$ is also a tempered distribution. The discussion above leads to the following

Definition 3.47. Let $T \in \mathscr{S}(\mathbb{R}^n)'$. The Fourier transform of T and the inverse Fourier transform of T, denoted by $\widehat{T}$ and $\check{T}$ respectively, are tempered distributions satisfying

$$\langle\widehat{T},\phi\rangle = \langle T,\widehat{\phi}\rangle \qquad \text{and} \qquad \langle\check{T},\phi\rangle = \langle T,\check{\phi}\rangle \qquad \forall\,\phi\in\mathscr{S}(\mathbb{R}^n)\,.$$

In other words, if $T \in \mathscr{S}(\mathbb{R}^n)'$, then $\widehat{T},\check{T} \in \mathscr{S}(\mathbb{R}^n)'$ as well and the actions of $\widehat{T}$, $\check{T}$ on $\phi \in \mathscr{S}(\mathbb{R}^n)$ are given in the relations above.

Example 3.48 (The Fourier transform of the Dirac delta function)**.** Consider the Dirac delta function $\delta : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ defined in Example 3.43. Then for $\phi \in \mathscr{S}(\mathbb{R}^n)$,

$$\langle\delta,\widehat{\phi}\rangle = \widehat{\phi}(0) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\phi(x)e^{-ix\cdot 0}\,dx = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\phi(x)\,dx = \langle\frac{1}{\sqrt{2\pi}^n},\phi\rangle\,;$$

thus the Fourier transform of the Dirac delta function is a constant function and $\widehat{\delta}(\xi) = \dfrac{1}{\sqrt{2\pi}^n}$. Similarly, $\check{\delta}(\xi) = \dfrac{1}{\sqrt{2\pi}^n}$, so $\widehat{\delta} = \check{\delta}$.

Next we consider the Fourier transform of δ_ω, the Dirac delta function at point $\omega \in \mathbb{R}^n$. Note that for $\phi \in \mathscr{S}(\mathbb{R}^n)$,

$$\langle \delta_\omega, \widehat{\phi} \rangle = \widehat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \phi(x) e^{-ix\cdot\omega}\, dx = \Big\langle \frac{e^{-ix\cdot\omega}}{\sqrt{2\pi}^n}, \phi \Big\rangle \equiv \langle \widehat{\delta_\omega}, \phi \rangle\,;$$

thus the Fourier transform of the Dirac delta function at point ω is the function $\widehat{\delta_\omega}(\xi) = \dfrac{e^{-i\xi\cdot\omega}}{\sqrt{2\pi}^n}$. The inverse Fourier transform of δ_ω can be computed in the same fashion and we have $\widecheck{\delta_\omega}(\xi) = \dfrac{e^{i\xi\cdot\omega}}{\sqrt{2\pi}^n}$. We note that $\widecheck{\delta_\omega} = \widetilde{\widehat{\delta_\omega}} = \widehat{\widetilde{\delta_\omega}}$.

Symbolically, "assuming" that $\delta_\omega(\phi) = \phi(\omega)$ for all continuous function ϕ,

$$\widehat{\delta_\omega}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \delta_\omega(x) e^{-ix\cdot\xi}\, dx = \frac{1}{\sqrt{2\pi}^n} e^{-ix\cdot\xi}\Big|_{x=\omega} = \frac{e^{-i\xi\cdot\omega}}{\sqrt{2\pi}^n}$$

and

$$\widecheck{\delta_\omega}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \delta_\omega(x) e^{ix\cdot\xi}\, dx = \frac{1}{\sqrt{2\pi}^n} e^{ix\cdot\xi}\Big|_{x=\omega} = \frac{e^{i\xi\cdot\omega}}{\sqrt{2\pi}^n}\,.$$

Example 3.49 (The Fourier transform of $e^{ix\cdot\omega}$)**.** By "definition" and the Fourier inversion formula, for $\phi \in \mathscr{S}(\mathbb{R}^n)$ we have

$$\langle e^{ix\cdot\omega}, \widehat{\phi} \rangle = \int_{\mathbb{R}^n} e^{ix\cdot\omega} \widehat{\phi}(x)\, dx = \sqrt{2\pi}^n \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \widehat{\phi}(x) e^{ix\cdot\omega}\, dx = \sqrt{2\pi}^n \widecheck{\widehat{\phi}}(\omega) = \sqrt{2\pi}^n \phi(\omega)\,;$$

thus

$$\langle e^{ix\cdot\omega}, \widehat{\phi} \rangle = \sqrt{2\pi}^n \phi(\omega) = \langle \sqrt{2\pi}^n \delta_\omega, \phi \rangle\,.$$

Therefore, the Fourier transform of the function $s(x) = e^{ix\cdot\omega}$ is $\sqrt{2\pi}^n \delta_\omega$, where δ_ω is the Dirac delta function at point ω introduced in Example 3.48. We note that this result also implies that

$$\widecheck{\widehat{\delta_\omega}} = \delta_\omega \qquad \forall\, \omega \in \mathbb{R}^n\,.$$

Similarly, $\widehat{\widecheck{\delta_\omega}} = \delta_\omega$ for all $\omega \in \mathbb{R}^n$; thus the Fourier inversion formula is also valid for the Dirac δ function.

Example 3.50 (The Fourier Transform of the Sine function)**.** Let $s(x) = \sin\omega x$, where ω denotes the frequency of this sine wave. Since $\sin\omega x = \dfrac{e^{i\omega x} - e^{-i\omega x}}{2i}$, we conclude that the Fourier transform of $s(x) = \sin\omega x$ is

$$\frac{\sqrt{2\pi}}{2i}\big(\delta_\omega - \delta_{-\omega}\big)$$

since if T_1, T_2 are tempered distributions, then $T = T_1 + T_2$ satisfies

$$\langle \widehat{T}, \phi \rangle = \langle T_1 + T_2, \widehat{\phi} \rangle = \langle T_1, \widehat{\phi} \rangle + \langle T_2, \widehat{\phi} \rangle = \langle \widehat{T}_1, \phi \rangle + \langle \widehat{T}_2, \phi \rangle = \langle \widehat{T}_1 + \widehat{T}_2, \phi \rangle \quad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)$$

which shows that $\widehat{T} = \widehat{T}_1 + \widehat{T}_2$.

Theorem 3.51. *Let $T \in \mathscr{S}(\mathbb{R}^n)'$. Then $\check{\widehat{T}} = \widehat{\check{T}} = T$.*

Proof. To see that $\check{\widehat{T}}$ and T are the same tempered distribution, we need to show that $\langle \check{\widehat{T}}, \phi \rangle = \langle T, \phi \rangle$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$. Nevertheless, by the defintion of the Fourier transform and the inverse Fourier transform of tempered distributions,

$$\langle \check{\widehat{T}}, \phi \rangle = \langle \widehat{T}, \check{\phi} \rangle = \langle T, \widehat{\check{\phi}} \rangle = \langle T, \phi \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n) .$$

That $\widehat{\check{T}} = T$ can be proved in the same fashion. □

Example 3.52 (The Fourier Transform of the sinc function)**.** The rect/rectangle function, also called the gate function or windows function, is a function $\Pi : \mathbb{R} \to \mathbb{R}$ defined by

$$\Pi(x) = \begin{cases} 1 & \text{if } |x| < 1 , \\ 0 & \text{if } |x| \geqslant 1 . \end{cases}$$

Since $\Pi \in L^1(\mathbb{R})$, we can compute its (inverse) Fourier transform in the usual way, and we have

$$\widehat{\Pi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Pi(x) e^{-ix\xi}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ix\xi}\, dx = \frac{1}{\sqrt{2\pi}} \frac{e^{-ix\xi}}{-i\xi} \Big|_{x=-1}^{x=1} = \sqrt{\frac{2}{\pi}} \frac{\sin \xi}{\xi} \qquad \forall\, \xi \neq 0$$

and $\widehat{\Pi}(0) = \sqrt{\dfrac{2}{\pi}}$. Define the ***unnormalized sinc function*** $\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 . \end{cases}$

Then $\widehat{\Pi}(\xi) = \sqrt{\dfrac{2}{\pi}} \operatorname{sinc}(\xi)$. Similar computation shows that $\check{\Pi}(\xi) = \widehat{\Pi}(\xi) = \sqrt{\dfrac{2}{\pi}} \operatorname{sinc}(\xi)$.

Even though the sinc function is not integrable, we can apply Theorem 3.51 and see that

$$\widehat{\operatorname{sinc}}(\xi) = \widecheck{\operatorname{sinc}}(\xi) = \sqrt{\frac{\pi}{2}}\, \Pi(\xi) \qquad \forall\, \xi \in \mathbb{R} .$$

Theorem 3.53. *Let $T \in \mathscr{S}(\mathbb{R}^n)'$. Then*

$$\langle \widehat{\tau_h T}, \phi \rangle = \langle \widehat{T}(\xi), \phi(\xi) e^{-i\xi\cdot h} \rangle , \quad \langle \widehat{d_\lambda T}, \phi \rangle = \langle \widehat{T}, d_\lambda \phi \rangle \ \ \text{and} \ \ \langle \widehat{\widehat{T}}, \phi \rangle = \langle \check{T}, \phi \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n) .$$

A short-hand notation for identities above are $\widehat{\tau_h T}(\xi) = \widehat{T}(\xi) e^{-i\xi\cdot h}$, $\widehat{d_\lambda T}(\xi) = \lambda^n \widehat{T}(\lambda \xi)$, and $\widehat{\widehat{T}}(\xi) = \check{T}(\xi)$.

Proof. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$. For $h \in \mathbb{R}^n$, define $\phi_h(x) = \phi(x)e^{-ix\cdot h}$. Then

$$(\tau_{-h}\widehat{\phi})(\xi) = \widehat{\phi}(\xi + h) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\phi(x)e^{-ix\cdot(\xi+h)}\,dx = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\phi(x)e^{-ix\cdot h}e^{-ix\cdot\xi}\,dx = \widehat{\phi_h}(\xi)\,.$$

By the definition of the Fourier transform of tempered distribution and the translation operator,

$$\langle\widehat{\tau_h T}, \phi\rangle = \langle T, \tau_{-h}\widehat{\phi}\rangle = \langle T, \widehat{\phi_h}\rangle = \langle\widehat{T}(x), \phi(x)e^{-ix\cdot h}\rangle = \langle\widehat{T}(\xi), \phi(\xi)e^{-i\xi\cdot h}\rangle\,.$$

On the other hand, for $\lambda > 0$,

$$(d_{\lambda^{-1}}\widehat{\phi})(\xi) = \widehat{\phi}(\lambda\xi) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\phi(x)e^{-ix\cdot(\lambda\xi)}\,dx = \lambda^{-n}\frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\phi\big(\frac{x}{\lambda}\big)e^{-ix\cdot\xi}\,dx = \lambda^{-n}\widehat{d_\lambda\phi}(\xi)\,.$$

Therefore,

$$\langle\widehat{d_\lambda T}, \phi\rangle = \langle T, \lambda^n d_{\lambda^{-1}}\widehat{\phi}\rangle = \langle T, \widehat{d_\lambda\phi}\rangle = \langle\widehat{T}, d_\lambda\phi\rangle = \langle\lambda^n d_{\lambda^{-1}}\widehat{T}, \phi\rangle\,.$$

The identity $\langle\widehat{\widehat{T}}, \phi\rangle = \langle\check{T}, \phi\rangle$ follows from that $\widehat{\widehat{\phi}} = \check{\phi}$, and the detail proof is left to the readers. □

Remark 3.54. One can check (using the change of variable formula) that $\widehat{\tau_h f}(\xi) = \widehat{f}(\xi)e^{-i\xi\cdot h}$ and $\widehat{d_\lambda f}(\xi) = \lambda^n\widehat{f}(\lambda\xi)$ if $f \in L^1(\mathbb{R}^n)$.

Next we define the convolution of a tempered distribution and a Schwartz function. Before proceeding, we note that if $f, g \in \mathscr{S}(\mathbb{R}^n)$, then

$$\begin{aligned}\langle f \star g, \phi\rangle &= \int_{\mathbb{R}^n}(f \star g)(x)\phi(x)\,dx = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}f(y)g(x-y)\,dy\Big)\phi(x)\,dx\\ &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}g(x-y)\phi(x)\,dx\Big)f(y)\,dy\\ &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\widetilde{g}(y-x)\phi(x)\,dx\Big)f(y)\,dy = \langle f, \widetilde{g} \star \phi\rangle\,.\end{aligned}$$

The change of variable formula implies that

$$\begin{aligned}(\widetilde{g} \star \phi)(y) &= \frac{1}{\sqrt{2\pi}^n}\Big(\int_{\mathbb{R}^n}\widetilde{g}(x)\phi(y-x)\,dx = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}\widetilde{g}(-x)\phi(y+x)\,dx\\ &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}g(x)\widetilde{\phi}(-y-x)\,dx = (g \star \widetilde{\phi})(-y) = \widetilde{g \star \widetilde{\phi}}(y)\,;\end{aligned}$$

thus

$$\langle f \star g, \phi\rangle = \langle f, \widetilde{g} \star \phi\rangle = \langle f, \widetilde{g \star \widetilde{\phi}}\rangle = \langle\widetilde{f}, g \star \widetilde{\phi}\rangle\,.$$

The identity above serves as the origin of the convolution of a tempered distribution and a Schwartz function.

Definition 3.55 (Convolution)**.** Let $T \in \mathscr{S}(\mathbb{R}^n)'$ and $f \in \mathscr{S}(\mathbb{R}^n)$. The convolution of T and f, denoted by $T \star f$, is the tempered distribution given by

$$\langle T \star f, \phi \rangle = \langle T, \widetilde{f} \star \phi \rangle = \langle \widetilde{T}, f \star \widetilde{\phi} \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,,$$

where $\widetilde{T}$ is the tempered distribution given in Definition 3.45.

Example 3.56. Let δ_ω be the Dirac delta function at point $\omega \in \mathbb{R}^n$, and $f \in \mathscr{S}(\mathbb{R}^n)$. Then $\delta_\omega \star f = \dfrac{\tau_\omega f}{\sqrt{2\pi}^n}$ since if $\phi \in \mathscr{S}(\mathbb{R}^n)$,

$$\begin{aligned}\langle \delta_\omega, \widetilde{f} \star \phi \rangle = (\widetilde{f} \star \phi)(\omega) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \widetilde{f}(y)\phi(\omega - y)\, dy = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} f(z-\omega)\phi(z)\, dz \\ = \Big\langle \frac{\tau_\omega f}{\sqrt{2\pi}^n}, \phi \Big\rangle\end{aligned}$$

In symbol,

$$(\delta_\omega \star f)(x) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \delta_\omega(y) f(x-y)\, dy = f(x-\omega) = \frac{1}{\sqrt{2\pi}^n} (\tau_\omega f)(x)\,. \tag{3.18}$$

Remark 3.57. If $S \in \mathscr{S}(\mathbb{R}^n)'$ satisfies that $S \star \phi \in \mathscr{S}(\mathbb{R}^n)$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$, we can also define the convolution of T and S by

$$\langle T \star S, \phi \rangle = \langle \widetilde{T}, S \star \widetilde{\phi} \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

In other words, it is possible to define the convolution of two tempered distributions.

For example, from Example 3.56 we find that $\delta_\omega \star \phi = \dfrac{\tau_\omega \phi}{\sqrt{2\pi}^{\mathrm{n}}}$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$; thus $\delta_\omega \star \phi \in \mathscr{S}(\mathbb{R}^n)$ for all $\mathscr{S}(\mathbb{R}^n)$ (and $\omega \in \mathbb{R}^n$). Therefore, if T is a tempered distribution, $T \star \delta_\omega$ is also a tempered distribution and is given by

$$\langle T \star \delta_\omega, \phi \rangle = \Big\langle \widetilde{T}, \frac{1}{\sqrt{2\pi}^n} \tau_\omega \widetilde{\phi} \Big\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

Further computation shows that

$$\langle T \star \delta_\omega, \phi \rangle = \Big\langle \widetilde{T}, \frac{1}{\sqrt{2\pi}^n} \widetilde{\tau_{-\omega}\phi} \Big\rangle = \Big\langle T, \frac{1}{\sqrt{2\pi}^n} \tau_{-\omega}\phi \Big\rangle = \Big\langle \frac{1}{\sqrt{2\pi}^n} \tau_\omega T, \phi \Big\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

The identity above shows that $T \star \delta_\omega = \dfrac{\tau_\omega T}{\sqrt{2\pi}^n}$ for all $T \in \mathscr{S}(\mathbb{R}^n)'$. This formula agrees with (3.18).

Similar to Theorem 3.26 and Corollary 3.27, the product and the convolutions of functions are related under Fourier transform.

Theorem 3.58. *Let $T \in \mathscr{S}(\mathbb{R}^n)'$ and $f \in \mathscr{S}(\mathbb{R}^n)$. Then*

$$\langle \widehat{T \star f}, \phi \rangle = \langle \widehat{T}, \widehat{f}\phi \rangle \quad \textit{and} \quad \langle \widecheck{T \star f}, \phi \rangle = \langle \check{T}, \check{f}\phi \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,,$$

and

$$\langle \widehat{f\,T}, \phi \rangle = \langle \widehat{T} \star \widehat{f}, \phi \rangle \quad \textit{and} \quad \langle \widecheck{f\,T}, \phi \rangle = \langle \check{T} \star \check{f}, \phi \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,,$$

where $fT \in \mathscr{S}(\mathbb{R}^n)'$ is defined by $\langle fT, \phi \rangle = \langle T, f\phi \rangle$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$. A short-hand notation for the identities above are $\widehat{T \star f} = \widehat{f}\widehat{T}$, $\widecheck{T \star f} = \check{f}\check{T}$, $\widehat{f\,T} = \widehat{T} \star \widehat{f}$ and $\widecheck{f\,T} = \check{T} \star \check{f}$ in $\mathscr{S}(\mathbb{R}^n)'$.

Proof. By Theorem 3.26,

$$\langle \widehat{T \star f}, \phi \rangle = \langle T \star f, \widehat{\phi} \rangle = \langle \widetilde{T}, f \star \widetilde{\widehat{\phi}} \rangle = \langle \widetilde{T}, f \star \check{\phi} \rangle = \langle \check{\widetilde{T}}, \mathscr{F}(f \star \check{\phi}) \rangle = \langle \widehat{T}, \widehat{f}\phi \rangle$$

and by the definition of the convolution of tempered distributions and Schwartz functions,

$$\langle \widehat{f\,T}, \phi \rangle = \langle T, f\widehat{\phi} \rangle = \langle \widehat{T}, \mathscr{F}^*(f\widehat{\phi}) \rangle = \langle \widehat{T}, \check{f} \star \phi \rangle = \langle \widehat{T}, \widetilde{\widehat{f}} \star \phi \rangle = \langle \widehat{T} \star \widehat{f}, \phi \rangle\,.$$

The counterpart for the inverse Fourier transform can be proved similarly. □

Remark 3.59. Let $f, \phi \in \mathscr{S}(\mathbb{R}^n)$, and $T \in \mathscr{S}(\mathbb{R}^n)'$ satisfy $\big|\langle T, u \rangle\big| \leqslant C_k p_k(u)$ for all $u \in \mathscr{S}(\mathbb{R}^n)$ and $k \gg 1$. By Theorem 3.58, we find that

$$\langle T \star f, \phi \rangle = \big\langle T \star f, \widehat{\check{\phi}} \big\rangle = \big\langle \widehat{T \star f}, \check{\phi} \big\rangle = \langle \widehat{T}, \widehat{f}\check{\phi} \rangle\,.$$

By the fact that

$$\begin{aligned} p_k(gh) &= \sup_{x \in \mathbb{R}^n, |\alpha| \leqslant k} \langle x \rangle^k \big|D^\alpha(gh)(x)\big| \leqslant \sum_{\substack{0 \leqslant \beta \leqslant \alpha \\ |\alpha| \leqslant k}} C^k_\beta \langle x \rangle^k \big|D^{\alpha-\beta} g(x) D^\beta h(x)\big| \\ &\leqslant \sum_{\substack{0 \leqslant \beta \leqslant \alpha \\ |\alpha| \leqslant k}} C^k_\beta p_k(g) p_k(h) = \Big(\sum_{|\beta| \leqslant k} C^k_\beta\Big) p_k(g) p_k(h) \qquad \forall\, g, h \in \mathscr{S}(\mathbb{R}^n)\,, \end{aligned}$$

we conclude from (3.16) and (3.17) that for $k \gg 1$,

$$\begin{aligned} \big|\langle T \star f, \phi \rangle\big| &\leqslant C_k \bar{C}(n,k) p_{k+n+1}\big(\widehat{f}\check{\phi}\big) \leqslant C_k \bar{C}(n,k) \Big(\sum_{|\beta| \leqslant k} C^k_\beta\Big) p_k\big(\widehat{f}\big) p_k\big(\widehat{\widetilde{\phi}}\big) \\ &\leqslant C_k \Big(\sum_{|\beta| \leqslant k} C^k_\beta\Big) \bar{C}(n,k)^3 p_{n+k+1}(f) p_{n+k+1}\big(\widetilde{\phi}\big) = \widetilde{C}(n,k) p_{n+k+1}(f) p_{n+k+1}(\phi)\,. \end{aligned}$$

Therefore, $T \star f$ is a tempered distribution.

Index