Fourier Analysis

富氏分析

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Contents

Chapter 2

Fourier Series

讓我們回顧一下之前已經有的一些結論。在 §1.3 中我們學到了 Stone-Weierstrass 定理，它告訴我們定義在 $[0, 1]$ 上的連續函數 f 可以用多項式（例如 Bernstein 多項式）去逼近（在均勻收斂的意義下），而我們也注意到 Bernstein 多項式，在取不同次數 n 的多項式做逼近時，每一個單項式 x^k 前面的係數都跟 n 和 k 有關。但是從定理 1.16 中我們又發現，對某些擁有很好性質的函數 f（叫做解析函數 Analytic functions），即使取不同次數 n 的多項式做逼近時，每個單項式 x^k 前面的係數可以取成只跟函數 f 的 k 次導數有關（跟 n 無關）。這給了我們一個很粗略的概念，知道想用多項式去逼近連續函數時，多項式的係數有些時候會跟多項次的次數有關，有時則無關。

在這一章中，我們在前四節特別關注在週期為 2π 的連續函數。由定理 1.25 我們知道這樣的函數可用形如

$$p_n(x) = \frac{c_0^{(n)}}{2} + \sum_{k=1}^{n}(c_k^{(n)} \cos kx + s_k^{(n)} \sin kx)$$

的三角多項式 (trigonometric polynomials) 所逼近（在均勻收斂的意義下），其中上標 (n) 代表的是係數可能與用來逼近的三角多項式的次數 n 有關係。跟前一段所述的經驗類似，在數學理論上我們想知道下面問題的答案：

1. 什麼樣的函數，可以用係數與逼近次數無關的三角多項式去逼近。對這樣的函數，三角多項式要怎麼挑？
2. 對於實在沒辦法用係數與逼近次數無關的三角多項式去逼近的連續週期函數，有什麼好的方法逼近？而上面所挑出來的那個係數跟逼近次數無關的三角多項式，在次數接近無窮大時出了什麼問題？

上述的問題解決之後，我們用變數變換，也可以得到對於週期為 $2L$ 的函數的相關理論。

另外，由於在進行的過程中，我們發現我們所關心用來逼近連續函數的三角多項式（叫富氏級數），其係數的定法只要求函數可積分即可，因此，一個自然衍生的問題則是：對不連續（但可積分）的函數來說，有沒有什麼收斂理論可以說明？這個部份的研究則是第四、五節的主要重點。在第六節中，我們則提供了一個快速傅利葉變換 (FFT) 的演算法可供電腦去計算富氏級數（的係數）。

2.1 Basic properties of the Fourier series

Let $f \in \mathscr{C}(\mathbb{T})$ be given. We first assume that the trigonometric polynomials used to approximate f can be chosen in such a way that the coefficients does not depend on the degree of approximation; that is, $c_k^{(n)} = c_k$ and $s_k^{(n)} = s_k$. In this case, if $p_n \to f$ uniformy on $[-\pi, \pi]$, by Theorem 1.6 we must have

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} p_n(x)\cos kx\,dx = \int_{-\pi}^{\pi} f(x)\cos kx\,dx \qquad \forall\, k \in \{0, 1, \cdots, n\}$$

and

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} p_n(x)\sin kx\,dx = \int_{-\pi}^{\pi} f(x)\sin kx\,dx \qquad \forall\, k \in \{1, \cdots, n\}\,.$$

Since

$$\int_{-\pi}^{\pi} \cos kx \cos \ell x\,dx = \int_{-\pi}^{\pi} \sin kx \sin \ell x\,dx = \pi\delta_{k\ell} \qquad \forall\, k, \ell \in \mathbb{N}$$

and

$$\int_{-\pi}^{\pi} \sin kx \cos \ell x\,dx = 0 \qquad \forall\, k \in \mathbb{N}, \ell \in \mathbb{N} \cup \{0\}\,,$$

we find that

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \quad \text{and} \quad s_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx\,. \tag{2.1}$$

This induces the following

Definition 2.1. For a Riemann integrable function $f : [-\pi, \pi] \to \mathbb{R}$, the ***Fourier series representation*** of f, denoted by $s(f, \cdot)$, is given by

$$s(f, x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} (c_k \cos kx + s_k \sin kx)$$

whenever the sum makes sense, where sequences $\{c_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ given by (2.1) are called the ***Fourier coefficients*** associated with f. The n-th partial sum of the Fourier series representation to f, denoted by $s_n(f,\cdot)$, is given by

$$s_n(f,x) = \frac{c_0}{2} + \sum_{k=1}^{n}(c_k \cos kx + s_k \sin kx)\,.$$

We note that for the Fourier series $s(f,x)$ to be defined, f is not necessary continuous. Our goal is to establish the convergence of Fourier series in various senses.

Remark 2.2. Because of the Euler identity $e^{i\theta} = \cos\theta + i\sin\theta$, we can write

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)(e^{iky} + e^{-iky})dy \quad \text{and} \quad s_k = \frac{1}{2\pi i}\int_{-\pi}^{\pi} f(y)(e^{iky} - e^{-iky})dy$$

thus

$$\begin{aligned}
s_n(f,x) &= \frac{c_0}{2} + \sum_{k=1}^{n}\big(c_k\frac{e^{ikx}+e^{-ikx}}{2} + s_k\frac{e^{ikx}-e^{-ikx}}{2i}\big) \\
&= \frac{1}{2}\Big[c_0 + \sum_{k=1}^{n}\big((c_k - is_k)e^{ikx} + (c_k + is_k)e^{-ikx}\big)\Big] \\
&= \frac{1}{2}\Big[c_0 + \sum_{k=1}^{n}\big((c_k - is_k)e^{ikx} + \sum_{k=-n}^{-1}(c_{-k} + is_{-k})e^{ikx}\Big] \\
&= \frac{1}{2}\Big[c_0 + \frac{1}{\pi}\sum_{k=1}^{n}\int_{-\pi}^{\pi} f(y)e^{-iky}dye^{ikx}s + \frac{1}{\pi}\sum_{k=-n}^{-1}\int_{-\pi}^{\pi} f(y)e^{-iky}dye^{ikx}\Big]\,.
\end{aligned}$$

Define $\widehat{f}_k = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(y)e^{-iky}dy$. Then

$$s_n(f,x) = \sum_{k=-n}^{n}\widehat{f}_k e^{ikx}\,.$$

The sequence $\{\widehat{f}_k\}_{k=-\infty}^{\infty}$ is also called the Fourier coefficients associated with f, and one can write the Foruier series representation of f as $\sum\limits_{k=-\infty}^{\infty}\widehat{f}_k e^{ikx}$.

Remark 2.3. Given a continuous function g with period $2L$ (or a function g which is Riemann integrable on $[-L,L]$), let $f(x) = g\big(\frac{Lx}{\pi}\big)$. Then f is a continuous function with period 2π (or f is a Riemann integrable function on $[-\pi, \pi]$), and the Fourier series of f is given by

$$s(f, x) = \frac{c_0}{2} + \sum_{k=1}^{n} (c_k \cos kx + s_k \sin kx)\,,$$

where c_k and s_k are given by (2.1). Now, define the Fourier series of g by $s(g, x) = s\big(f, \frac{\pi x}{L}\big)$. Then the Fourier series of g is given by

$$s(g, x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} \Big(c_k \cos \frac{k\pi x}{L} + s_k \sin \frac{k\pi x}{L}\Big)\,,$$

where $\{c_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ is also called the Fourier coefficients associated with g and are given by

$$c_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} g\big(\frac{Lx}{\pi}\big) \cos kx\, dx = \frac{1}{L} \int_{-L}^{L} g(x) \cos \frac{k\pi x}{L}\, dx$$

and similarly, $s_k = \dfrac{1}{L} \displaystyle\int_{-L}^{L} g(x) \sin \frac{k\pi x}{L}\, dx$. Similar to Remark 2.2, the Fourier series of g can also be written as

$$\sum_{k=-\infty}^{\infty} \widehat{g}_k e^{\frac{i\pi kx}{L}}\,,$$

where $\widehat{g}_k = \dfrac{1}{2L} \displaystyle\int_{-L}^{L} g(y) e^{\frac{-i\pi ky}{L}}\, dy$.

Example 2.4. Consider the periodic function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \pi\,, \\ -x & \text{if } -\pi < x < 0\,, \end{cases}$$

and $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$. To find the Fourier representation of f, we compute the Fourier coefficients by

$$s_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx\, dx = \frac{1}{\pi}\Big(\int_0^{\pi} x \sin kx\, dx - \int_{-\pi}^{0} x \sin kx\, dx\Big) = 0$$

and

$$c_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, dx = \frac{1}{\pi}\Big(\int_0^{\pi} x \cos kx\, dx - \int_{-\pi}^{0} x \cos kx\, dx\Big) = \frac{2}{\pi} \int_0^{\pi} x \cos kx\, dx\,.$$

If $k = 0$, then $c_0 = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} x\, dx = \pi$, while if $k \in \mathbb{N}$,

$$c_k = \frac{2}{\pi}\Big(\frac{x \sin kx}{k}\Big|_0^{\pi} - \int_0^{\pi} \frac{\sin kx}{k}\, dx\Big) = \frac{2}{\pi} \frac{\cos kx}{k^2}\Big|_0^{\pi} = \frac{2((-1)^k - 1)}{\pi k^2}\,.$$

Therefore, $c_{2k} = 0$ and $c_{2k-1} = -\dfrac{4}{\pi(2k-1)^2}$ for all $k \in \mathbb{N}$. Therefore, the Fourier series of f is given by

$$s(f,x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}\,.$$

Example 2.5. Consider the periodic function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}\,, \\ 0 & \text{if } -\pi \leqslant x < -\frac{\pi}{2} \text{ or } \frac{\pi}{2} < x \leqslant \pi\,, \end{cases}$$

and $f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$. We compute the Fourier coefficients of f and find that $s_k = 0$ for all $k \in \mathbb{N}$ and $c_0 = 1$, as well as

$$c_k = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos kx\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos kx\,dx = \frac{2\sin\frac{k\pi}{2}}{\pi k}\,.$$

Therefore, $c_{2k} = 0$ and $c_{2k-1} = \dfrac{2(-1)^{k+1}}{\pi(2k-1)}$ for all $k \in \mathbb{N}$; thus the Fourier series of f is given by

$$s(f,x) = \frac{1}{2} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1}\cos(2k-1)x\,.$$

Example 2.6. Consider the periodic function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x \qquad \text{if} \quad -\pi < x \leqslant \pi$$

and $f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$. Then the Fourier coefficients of f are computed as follows: $c_k = 0$ for all $k \in \mathbb{N} \cup \{0\}$ since f is (more or less) an odd function, and

$$\begin{aligned} s_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin kx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin kx\,dx = \frac{2}{\pi}\Big(-\frac{x\cos kx}{k}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos kx}{k}\,dx\Big) \\ &= \frac{2(-1)^{k+1}}{k}\,. \end{aligned}$$

Therefore, the Fourier series of f is given by

$$s(f,x) = 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin kx\,.$$

2.2 Uniform Convergence of the Fourier Series

Before proceeding, we note that Remark 2.2 implies that

$$s_n(f,x) = \sum_{k=-n}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{ik(x-y)}\, dy = \int_{-\pi}^{\pi} f(y) \Big(\frac{1}{2\pi} \sum_{k=-n}^{n} e^{ik(x-y)} \Big)\, dy\,.$$

Define $D_n(x) = \frac{1}{2\pi} \sum\limits_{k=-n}^{n} e^{ikx}$. Then D_n is 2π-periodic, and

$$s_n(f,x) = \int_{-\pi}^{\pi} f(y) D_n(x-y)\, dy\,.$$

For 2π-periodic Riemann integrable functions f and g, we define the convolution of f and g on the circle by

$$(f \star g)(x) = \int_{-\pi}^{\pi} f(y) g(x-y)\, dy\,.$$

Then $s_n(f,x) = (D_n \star f)(x)$.

Note that $D_n(0) = \dfrac{2n+1}{2\pi}$, and if $e^{ix} \neq 1$,

$$D_n(x) = \frac{1}{2\pi} \frac{e^{-inx}\big[e^{i(2n+1)x} - 1\big]}{e^{ix} - 1} = \frac{1}{2\pi} \frac{e^{i(n+1/2)x} - e^{-i(n+1/2)x}}{e^{ix/2} - e^{-ix/2}} = \frac{\sin(n+\frac{1}{2})x}{2\pi \sin\frac{x}{2}}$$

so that we have the following

Definition 2.7. The function $D_n : \mathbb{R} \to \mathbb{R}$ defined by

$$D_n(x) = \begin{cases} \dfrac{\sin(n+\frac{1}{2})x}{2\pi \sin\frac{x}{2}} & \text{if } x \notin \{2k\pi \,|\, k \in \mathbb{Z}\}\,, \\ \dfrac{2n+1}{2\pi} & \text{if } x \in \{2k\pi \,|\, k \in \mathbb{Z}\}\,, \end{cases} \tag{2.2}$$

is called the ***Dirichlet kernel***.

By the fact that $D_n(x) = \frac{1}{2\pi} \sum\limits_{k=-n}^{n} e^{ikx}$, we immediately conclude the following

Lemma 2.8. *For each $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $\displaystyle\int_{-\pi}^{\pi} D_n(x-y)\, dy = 1$.*

In the following, we first consider the uniform convergence of the Fourier series of 2π-periodic continuously differentiable functions.

Definition 2.9. The normed vector space $\big(\mathscr{C}^1(\mathbb{T}), \|\cdot\|_{\mathscr{C}^1(\mathbb{T})}\big)$ is a vector space over $\mathbb{R}$ consisting of all 2π-periodic real-valued continuously differentiable functions and is equipped with a norm

$$\|f\|_{\mathscr{C}^1(\mathbb{T})} = \|f\|_\infty + \|f'\|_\infty = \max_{x\in\mathbb{R}} \big|f(x)\big| + \max_{x\in\mathbb{R}} \big|f'(x)\big| \qquad \forall\, f \in \mathscr{C}^1(\mathbb{T}) .$$

Theorem 2.10. *For any $f \in \mathscr{C}^1(\mathbb{T})$,the Fourier series of f converges uniformly to f on $\mathbb{R}$; that is, the sequence $\{s_n(f,\cdot)\}_{n=1}^\infty$ converges uniformly to f on $\mathbb{R}$.*

Proof. By Lemma 2.8, we find that for all $x \in \mathbb{R}$,

$$\begin{aligned} s_n(f,x) - f(x) = (D_n \star f - f)(x) &= \int_{-\pi}^{\pi} D_n(x-y)\big(f(y) - f(x)\big)\,dy \\ &= \int_{-\pi}^{\pi} D_n(y)\big(f(x-y) - f(x)\big)\,dy . \end{aligned}$$

We break the integral into two parts: one is the integral on $|y| \leqslant \delta$ and the other is the integral on $\delta < |y| \leqslant \pi$. Since $f \in \mathscr{C}^1(\mathbb{T})$,

$$|f(x-y) - f(x)| \leqslant \|f'\|_\infty |y| ;$$

thus by the fact that $\dfrac{x}{\sin x} \leqslant \dfrac{\pi}{2}$ for $0 < x < \dfrac{\pi}{2}$, we obtain that

$$\begin{aligned} &\Big|\int_{|y|\leqslant\delta} D_n(y)\big(f(x-y) - f(x)\big)\,dy\Big| \\ &\qquad \leqslant \int_{-\delta}^{\delta} \frac{|f(x-y) - f(x)|}{2\pi\big|\sin\frac{y}{2}\big|}\,dy \leqslant \frac{\|f'\|_\infty}{2\pi}\int_{-\delta}^{\delta} \frac{y}{\sin\frac{y}{2}}\,dy \leqslant \|f'\|_\infty \delta . \end{aligned} \tag{2.3}$$

Now we take care of the integral on $\delta < |y| \leqslant \pi$ by first looking at the integral on $\delta < y < \pi$. Integrating by parts,

$$\begin{aligned} \int_\delta^\pi D_n(y)\big(f(x-y) - f(y)\big)\,dy &= \frac{1}{2\pi}\int_\delta^\pi \sin\big(n+\frac{1}{2}\big)y\,\frac{f(x-y)-f(x)}{\sin\frac{y}{2}}\,dy \\ = -\frac{1}{2\pi}\frac{\cos\big(n+\frac{1}{2}\big)y}{n+\frac{1}{2}}\,\frac{f(x-y)-f(x)}{\sin\frac{y}{2}}\Big|_{y=\delta}^{y=\pi} &+ \frac{1}{2\pi}\int_\delta^\pi \frac{\cos\big(n+\frac{1}{2}\big)y}{n+\frac{1}{2}}\,\frac{d}{dy}\frac{f(x-y)-f(x)}{\sin\frac{y}{2}}\,dy . \end{aligned}$$

For the first term on the right-hand side,

$$\Big|\frac{1}{2\pi}\frac{\cos\big(n+\frac{1}{2}\big)y}{n+\frac{1}{2}}\,\frac{f(x-y)-f(x)}{\sin\frac{y}{2}}\Big|_{y=\delta}^{y=\pi}\Big| \leqslant \frac{2\|f\|_\infty}{2\pi n\sin\frac{\delta}{2}} \leqslant \frac{\|f\|_\infty}{n\sin\frac{\delta}{2}} \qquad \forall\, x \in \mathbb{R} .$$

For the second term on the right-hand side,

$$\begin{aligned}
&\Big|\frac{1}{2\pi}\int_\delta^\pi \frac{\cos\big(n+\frac{1}{2}\big)y}{n+\frac{1}{2}}\frac{d}{dy}\frac{f(x-y)-f(x)}{\sin\frac{y}{2}}dy\Big| \\
&\qquad \leqslant \frac{1}{2\pi}\Big[\Big|\int_\delta^\pi \frac{\cos\big(n+\frac{1}{2}\big)y}{n+\frac{1}{2}}\frac{f'(x-y)}{\sin\frac{y}{2}}dy\Big| + \Big|\int_\delta^\pi \frac{\cos\big(n+\frac{1}{2}\big)y}{n+\frac{1}{2}}\frac{\cos\frac{y}{2}\big(f(x-y)-f(x)\big)}{2\sin^2\frac{y}{2}}dy\Big|\Big] \\
&\qquad \leqslant \frac{1}{2\pi}\Big[\|f'\|_\infty\frac{\pi-\delta}{\big(n+\frac{1}{2}\big)\sin\frac{\delta}{2}} + \|f\|_\infty\frac{\pi-\delta}{\big(n+\frac{1}{2}\big)\sin^2\frac{\delta}{2}}\Big] \leqslant \frac{\|f\|_{\mathscr{C}^1(\mathbb{T})}}{n\sin^2\frac{\delta}{2}}\,.
\end{aligned}$$

Similarly,

$$\Big|\int_{-\pi}^{-\delta} D_n(y)\big(f(x-y)-f(x)\big)dy\Big| \leqslant \frac{\|f\|_\infty}{n\sin\frac{\delta}{2}} + \frac{\|f\|_{\mathscr{C}^1(\mathbb{T})}}{n\sin^2\frac{\delta}{2}}\,;$$

thus for all $x \in \mathbb{R}$,

$$\begin{aligned}
\big|s_n(f,x)-f(x)\big| &\leqslant \Big|\Big(\int_{-\delta}^\delta + \int_\delta^\pi + \int_{-\pi}^{-\delta}\Big)D_n(y)\big(f(x-y)-f(x)\big)\,dy\Big| \\
&\leqslant \|f'\|_\infty\delta + \frac{2\|f\|_\infty}{n\sin\frac{\delta}{2}} + \frac{2\|f\|_{\mathscr{C}^1(\mathbb{T})}}{n\sin^2\frac{\delta}{2}} \leqslant \|f'\|_\infty\delta + \frac{4\|f\|_{\mathscr{C}^1(\mathbb{T})}}{n\sin^2\frac{\delta}{2}}\,.
\end{aligned}$$

Let $\varepsilon > 0$ be given. Choose a fixed $\delta > 0$ such that $\|f'\|_\infty\delta < \dfrac{\varepsilon}{2}$. For this fixed δ, choose $N > 0$ such that

$$\frac{4\|f\|_{\mathscr{C}^1(\mathbb{T})}}{N\sin^2\frac{\delta}{2}} < \frac{\varepsilon}{2}\,.$$

Then if $n \geqslant N$ and $x \in \mathbb{R}$, we have

$$\big|s_n(f,x)-f(x)\big| < \frac{\varepsilon}{2} + \frac{4\|f\|_{\mathscr{C}^1(\mathbb{T})}}{n\sin^2\frac{\delta}{2}} \leqslant \frac{\varepsilon}{2} + \frac{4\|f\|_{\mathscr{C}^1(\mathbb{T})}}{N\sin^2\frac{\delta}{2}} < \varepsilon\,. \qquad \square$$

After showing the uniform convergence of the Fourier series of $\mathscr{C}^1$-functions, we next consider the convergence of the Fourier series of less regular functions. The functions of which we prove the convergence of the Fourier series representation belong to the so-called Hölder class continuous functions.

Definition 2.11. Let $I \subseteq \mathbb{R}$ be an interval, and $\alpha \in (0,1]$. A function f is said to be ***Hölder continuous with exponent*** α on I if $\displaystyle\sup_{x,y\in I, x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty$. The collection of all real-valued functions that are Hölder continuous with exponent α on I is denoted by $\mathscr{C}^{0,\alpha}(I;\mathbb{R})$, and $\mathscr{C}^{0,\alpha}(\mathbb{T})$ is the collection of all 2π-periodic functions that are Hölder continuous with exponent α on $\mathbb{R}$; that is,

$$\mathscr{C}^{0,\alpha}(\mathbb{T}) = \Big\{f \in \mathscr{C}(\mathbb{T})\,\Big|\, \sup_{x,y\in\mathbb{R},x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty\Big\}\,.$$

Let $\|\cdot\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}$ be defined by

$$\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})} = \sup_{x\in\mathbb{R}} \big|f(x)\big| + \sup_{\substack{x,y\in\mathbb{R}\\ x\neq y}} \frac{\big|f(x)-f(y)\big|}{|x-y|^\alpha}\,.$$

Then $\|\cdot\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}$ is a norm on $\mathscr{C}^{0,\alpha}(\mathbb{T})$, and

$$\mathscr{C}^{0,\alpha}(\mathbb{T}) = \big\{ f \in \mathscr{C}(\mathbb{T}) \,\big|\, \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})} < \infty \big\}\,.$$

In particular, when $\alpha = 1$, a function in $\mathscr{C}^{0,1}(\mathbb{T})$ is said to be Lipschitz continuous on $\mathbb{T}$; thus $\mathscr{C}^{0,1}(\mathbb{T})$ consists of Lipschitz continuous functions on $\mathbb{T}$.

The uniform convergence of $s_n(f,\cdot)$ to f for $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ with $\alpha \in (0,1)$ requires a lot more work. The idea is to estimate $\big\|f - s_n(f,\cdot)\big\|_{L^\infty(\mathbb{T})}$ in terms of the quantity $\inf\limits_{p\in\mathscr{P}_n(\mathbb{T})} \|f-p\|_{L^\infty(\mathbb{T})}$. Since $s_n(f,\cdot) \in \mathscr{P}_n(\mathbb{T})$, it is obvious that

$$\inf_{p\in\mathscr{P}_n(\mathbb{T})} \|f-p\|_{L^\infty(\mathbb{T})} \leqslant \big\|f - s_n(f,\cdot)\big\|_{L^\infty(\mathbb{T})}\,.$$

The goal is to show the inverse inequality

$$\big\|f - s_n(f,\cdot)\big\|_{L^\infty(\mathbb{T})} \leqslant C_n \inf_{p\in\mathscr{P}_n(\mathbb{T})} \|f-p\|_{L^\infty(\mathbb{T})} \tag{2.4}$$

for some constant C_n, and pick a suitable $p \in \mathscr{P}_n(\mathbb{T})$ which gives a good upper bound for $\big\|f - s_n(f,\cdot)\big\|_{L^\infty(\mathbb{T})}$. The inverse inequality is established via the following

Proposition 2.12. *The Dirichlet kernel D_n satisfies that for all $n \in \mathbb{N}$,*

$$\int_{-\pi}^{\pi} \big|D_n(x)\big|dx \leqslant 2 + \log n\,. \tag{2.5}$$

Proof. The validity of (2.5) for the case $n = 1$ is left to the reader, and we provide the proof for the case $n \geqslant 2$ here. Recall that $D_n(x) = \sum\limits_{k=-n}^{n} \dfrac{e^{ikx}}{2\pi} = \dfrac{\sin(n+\frac{1}{2})x}{2\pi\sin\frac{x}{2}}$. Therefore,

$$\int_{-\pi}^{\pi} \big|D_n(x)\big|dx = 2\int_0^{\pi} \big|D_n(x)\big|dx = \int_0^{\frac{1}{n}} 2\big|D_n(x)\big|dx + \int_{\frac{1}{n}}^{\pi} \Big|\frac{\sin(n+\frac{1}{2})x}{\pi\sin\frac{x}{2}}\Big|dx\,.$$

Since $|D_n(x)| \leqslant \lim\limits_{t\to 0^+} |D_n(t)| = \dfrac{2n+1}{2\pi}$ for all $0 < x \leqslant \dfrac{1}{n}$, the first integral can be estimated by

$$\int_0^{\frac{1}{n}} 2\big|D_n(x)\big|dx \leqslant \frac{1}{\pi}\frac{2n+1}{n}\,. \tag{2.6}$$

Since $\dfrac{2x}{\pi} \leqslant \sin x$ for $0 \leqslant x \leqslant \dfrac{\pi}{2}$, the second integral can be estimated by

$$\int_{\frac{1}{n}}^{\pi} \Big|\frac{\sin(n+\frac{1}{2})x}{\pi \sin\frac{x}{2}}\Big| dx \leqslant \int_{\frac{1}{n}}^{\pi} \frac{1}{x}\, dx = \log \pi + \log n\,. \tag{2.7}$$

We then conclude (2.5) from (2.6) and (2.7) by noting that $\log \pi + \dfrac{2n+1}{n\pi} \leqslant 2$ for all $n \geqslant 2$. □

Remark 2.13. A more subtle estimate can be done to show that

$$\int_{-\pi}^{\pi} \big|D_n(x)\big| dx \geqslant c_1 + c_2 \log n \qquad \forall\, n \in \mathbb{N}$$

for some positive constants c_1 and c_2. Therefore, the integral of $|D_n|$ over $[-\pi, \pi]$ blows up as $n \to \infty$.

With the help of Proposition 2.12, we are able to prove the inverse inequality (2.4). The following theorem is a direct consequence of Proposition 2.12.

Theorem 2.14. *Let $f \in \mathscr{C}(\mathbb{T})$; that is, f is a continuous function with period 2π. Then*

$$\big\|f - s_n(f,\cdot)\big\|_\infty \leqslant (3 + \log n) \inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty\,. \tag{2.8}$$

Proof. For $n \in \mathbb{N}$ and $x \in \mathbb{T}$,

$$\big|s_n(f,x)\big| \leqslant \int_{-\pi}^{\pi} \big|D_n(y)\big|\big|f(x-y)\big| dy \leqslant (2 + \log n)\|f\|_\infty\,.$$

Given $\varepsilon > 0$, let $p \in \mathscr{P}_n(\mathbb{T})$ such that

$$\|f - p\|_\infty \leqslant \inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty + \varepsilon.$$

Then by the fact that $s_n(p,x) = p(x)$ if $p \in \mathscr{P}_n(\mathbb{T})$, we obtain that

$$\begin{aligned}
\big\|f - s_n(f,\cdot)\big\|_\infty &\leqslant \big\|f - p\big\|_\infty + \big\|p - s_n(f,\cdot)\big\|_\infty \leqslant \big\|f - p\big\|_\infty + \big\|s_n(f-p,\cdot)\big\|_\infty \\
&\leqslant \big\|f - p\big\|_\infty + (2 + \log n)\|f - p\|_\infty \\
&\leqslant (3 + \log n)\Big[\inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty + \varepsilon\Big],
\end{aligned}$$

and (2.8) is obtained by passing to the limit as $\varepsilon \to 0$. □

Having established Theorem 2.14, the study of the uniform convergence of $s_n(f,\cdot)$ to f then amounts to the study of the quantity $\inf\limits_{p\in\mathscr{P}_n(\mathbb{T})}\|f-p\|_\infty$. The estimate of $\inf\limits_{p\in\mathscr{P}_n(\mathbb{T})}\|f-p\|_\infty$ for $f\in\mathscr{C}^{0,\alpha}(\mathbb{T})$, where $\alpha\in(0,1)$, is more difficult, and requires a clever choice of p. We begin with the following

Lemma 2.15. *If f is a continuous function on $[a,b]$, then for all $\delta_1,\delta_2>0$,*

$$\sup_{|x-y|\leqslant\delta_1}\big|f(x)-f(y)\big|\leqslant\Big(1+\frac{\delta_1}{\delta_2}\Big)\sup_{|x-y|\leqslant\delta_2}\big|f(x)-f(y)\big|\,.$$

The proof of Lemma 2.15 is not very difficult, and is left to the readers.

Now we are in position to prove the theorem due to D. Jackson.

Theorem 2.16 (Jackson)**.** *There exists a constant $C>0$ such that*

$$\inf_{p\in\mathscr{P}_n(\mathbb{T})}\|f-p\|_{L^\infty(\mathbb{T})}\leqslant C\sup_{|x-y|\leqslant\frac{1}{n}}|f(x)-f(y)|\qquad\forall\,f\in\mathscr{C}(\mathbb{T})\,.$$

Proof. Let $p(x)=1+c_1\cos x+\cdots+c_n\cos nx$ be a positive trigonometric function of degree n with coefficients $\{c_i\}_{i=1}^n$ determined later. Define an operator K on $\mathscr{C}(\mathbb{T})$ by

$$\mathrm{K}f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}p(y)f(x-y)\,dy\,.$$

Then $\mathrm{K}f\in\mathscr{P}_n(\mathbb{T})$. Lemma 2.15 then implies

$$\begin{aligned}\big|\mathrm{K}f(x)-f(x)\big|&\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}p(y)\big|f(x-y)-f(x)\big|dy\\&\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}p(y)\big(1+n|y|\big)\sup_{|x-y|\leqslant\frac{1}{n}}\big|f(x)-f(y)\big|dy\\&=\Big[1+\frac{n}{2\pi}\int_{-\pi}^{\pi}|y|p(y)\,dy\Big]\sup_{|x-y|\leqslant\frac{1}{n}}\big|f(x)-f(y)\big|\,.\end{aligned}$$

Since $y^2\leqslant\dfrac{\pi^2}{2}(1-\cos y)$ for $y\in[-\pi,\pi]$, by Hölder's inequality we find that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|y|p(y)\,dy&\leqslant\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi}y^2p(y)\,dy\Big]^{\frac12}\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi}p(y)\,dy\Big]^{\frac12}\\&\leqslant\Big[\frac{\pi}{4}\int_{-\pi}^{\pi}(1-\cos y)p(y)\,dy\Big]^{\frac12}=\frac{\pi}{2}\sqrt{2-c_1}\,.\end{aligned}$$

Therefore,

$$\|\mathrm{K}f - f\|_\infty \leqslant \Big(1 + \frac{n\pi}{2}\sqrt{2-c_1}\Big) \sup_{|x-y|\leqslant \frac{1}{n}} \big|f(x)-f(y)\big| .$$

To conclude the theorem, we need to show that the number $n\sqrt{2-c_1}$ can be made bounded by choosing p properly. Nevertheless, let

$$\begin{aligned} p(x) &= c\Big|\sum_{k=0}^n \sin\frac{(k+1)\pi}{n+2}e^{ikx}\Big|^2 = c\sum_{k=0}^n\sum_{\ell=0}^n \sin\frac{(k+1)\pi}{n+2}\sin\frac{(\ell+1)\pi}{n+2}e^{i(k-\ell)x} \\ &= c\sum_{k=0}^n \sin^2\frac{(k+1)\pi}{n+2} + 2c\sum_{\substack{k,\ell=0\\k>\ell}}^n \sin\frac{(k+1)\pi}{n+2}\sin\frac{(\ell+1)\pi}{n+2}\cos(k-\ell)x \end{aligned}$$

and choose c so that $p(x) = 1 + c_1\cos x + \cdots + c_n\cos nx$. Then

$$\begin{aligned} c^{-1} &= \sum_{k=0}^n \sin^2\frac{(k+1)\pi}{n+2} = \frac{1}{2}\sum_{k=0}^n\Big[1-\cos\frac{2(k+1)\pi}{n+2}\Big] \\ &= \frac{n+1}{2} - \frac{\sin\frac{(2n+3)\pi}{n+2} - \sin\frac{\pi}{n+2}}{4\sin\frac{\pi}{n+2}} = \frac{n+2}{2}, \end{aligned}$$

and

$$\begin{aligned} c_1 &= 2c\sum_{k=1}^n \sin\frac{(k+1)\pi}{n+2}\sin\frac{k\pi}{n+2} = c\sum_{k=1}^n\Big[\cos\frac{\pi}{n+2} - \cos\frac{(2k+1)\pi}{n+2}\Big] \\ &= c\Big[n\cos\frac{\pi}{n+2} - \frac{\sin\frac{(2n+2)\pi}{n+2} - \sin\frac{2\pi}{n+2}}{2\sin\frac{\pi}{n+2}}\Big] \\ &= c\Big[n\cos\frac{\pi}{n+2} + \frac{\sin\frac{2\pi}{n+2}}{\sin\frac{\pi}{n+2}}\Big] \\ &= c(n+2)\cos\frac{\pi}{n+2} = 2\cos\frac{\pi}{n+2}. \end{aligned}$$

As a consequence,

$$\begin{aligned} n\sqrt{2-c_1} &= n\Big(2-2\cos\frac{\pi}{n+2}\Big)^{\frac{1}{2}} = 2n\sin\frac{\pi}{2(n+2)} \\ &= 2(n+2)\sin\frac{\pi}{2(n+2)} - 4\sin\frac{\pi}{2(n+2)} \\ &= \pi\frac{2(n+2)}{\pi}\sin\frac{\pi}{2(n+2)} - 4\sin\frac{\pi}{2(n+2)} \end{aligned}$$

which is bounded by π; thus

$$\inf_{p\in\mathscr{P}_n(\mathbb{T})}\|f-p\|_{L^\infty(\mathbb{T})} \leqslant \|\mathrm{K}f-f\|_{L^\infty(\mathbb{T})} \leqslant \big(1+\frac{\pi^2}{2}\big)\sup_{|x-y|\leqslant\frac{1}{n}}\big|f(x)-f(y)\big| . \qquad \square$$

Finally, since $\lim\limits_{n\to\infty} n^{-\alpha}\log n = 0$ for all $\alpha \in (0,1]$, we conclude the following

Theorem 2.17. *For all $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ with $\alpha \in (0,1]$, $s_n(f,\cdot) = D_n \star f$ converges to f uniformly as $n \to \infty$.*

Remark 2.18. The converse of Theorem 2.16 is the Bernstein theorem which states that if f is a 2π-periodic function such that for some constant C (independent of n) and $\alpha \in (0,1)$,

$$\inf_{p\in\mathscr{P}_n(\mathbb{T})} \|f-p\|_\infty \leqslant Cn^{-\alpha} \tag{2.9}$$

for all $n \in \mathbb{N}$, then $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$. In other words, (2.9) is an equivalent condition to the Hölder continuity with exponent α of 2π-periodic continuous functions.

2.3 Cesàro Mean of Fourier Series

While Theorem 1.25 shows that the collection of trigonometric polynomials

$$\Big\{\frac{c_0}{2} + \sum_{k=1}^n (c_k\cos kx + s_k \sin kx)\,\Big|\,\{c_k\}_{k=0}^n, \{s_k\}_{k=1}^n \subseteq \mathbb{R}\Big\}$$

is dense in $\mathscr{C}(\mathbb{T})$, Theorem 2.17 only implies the uniform convergence of the Fourier series of Hölder continuous functions. Since the Fourier coefficients $\{c_k\}_{k=0}^n$ and $\{s_k\}_{k=1}^n$ are independent of the order of approximation n, as we discussed in the beginning of this chapter we do not expect that $s_n(f,\cdot)$ uniformly to f on $[-\pi,\pi]$ for general $f \in \mathscr{C}(\mathbb{T})$. To approximate continuous functions uniformly, the coefficients of the trigonometric polynomials should depend on the order of approximation.

The motivation of the discussion below is due to the following observation. Let $\{a_k\}_{k=1}^\infty$ be a sequence. Define a new sequence $\{b_n\}_{n=1}^\infty$, called the ***Cesàro mean*** of the sequence $\{a_k\}_{k=1}^\infty$, by

$$b_n = \frac{a_1+\cdots+a_n}{n} = \frac{1}{n}\sum_{k=1}^n a_k\,.$$

If $\{a_k\}_{k=1}^\infty$ converges to a, then $\{b_n\}_{n=1}^\infty$ converges to a as well. Even though the convergence of a sequence cannot be guaranteed by the convergence of its Cesàro mean, it is worthwhile investigating the convergence behavior of the Cesàro mean.

Let $\sigma_n(f,\cdot)$ denote the Cesàro mean of the Fourier series of f given by

$$\sigma_n(f,\cdot) \equiv \frac{1}{n+1}\sum_{k=0}^n s_k(f,\cdot) = \frac{1}{n+1}\sum_{k=0}^n (D_k\star f) = \Big(\frac{1}{n+1}\sum_{k=0}^n D_k\Big)\star f\,.$$

We note that the coefficients of the Cesàro mean $\sigma_n(f,\cdot)$ depend on the order of approximation n since

$$\sigma_n(f,x) = \frac{c_0}{2} + \sum_{k=1}^{n} \Big(\underbrace{\frac{n+1-k}{n+1} c_k}_{\equiv c_k^{(n)}} \cos kx + \underbrace{\frac{n+1-k}{n+1} s_k}_{\equiv s_k^{(n)}} \sin kx \Big) .$$

Recall that $D_k(x) = \dfrac{\sin(k+\frac{1}{2})x}{2\pi \sin\frac{x}{2}}$. By the product-to-sum formula, we find that if $x \in (0,\pi)$,

$$\begin{aligned}\sum_{k=0}^{n} D_k(x) &= \frac{1}{2\pi\sin\frac{x}{2}} \sum_{k=0}^{n} \sin(k+\frac{1}{2})x = \frac{1}{4\pi\sin^2\frac{x}{2}} \sum_{k=0}^{n} 2\sin\frac{x}{2}\sin(k+\frac{1}{2})x \\ &= \frac{1}{4\pi\sin^2\frac{x}{2}} \sum_{k=0}^{n} \big(\cos kx - \cos(k+1)x\big) \\ &= \frac{1}{4\pi\sin^2\frac{x}{2}} \big(1-\cos(n+1)x\big) = \frac{\sin^2\frac{n+1}{2}x}{2\pi\sin^2\frac{x}{2}} .\end{aligned}$$

This induces the following

Definition 2.19. The ***Fejér kernel*** is the Cesàro mean of the Dirichlet kernel given by

$$F_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(x) = \frac{1}{2\pi(n+1)} \frac{\sin^2\frac{(n+1)x}{2}}{\sin^2\frac{x}{2}} .$$

We note that $\sigma_n(f,\cdot) = F_n \star f$, where $F_n \geqslant 0$ and has the property that $\displaystyle\int_{-\pi}^{\pi} F_n(x)\,dx = 1$ (since the integral of the Dirichlet kernel is 1). Moreover, for any $\delta > 0$,

$$\lim_{n\to\infty} \int_{\delta\leqslant|x|\leqslant\pi} F_n(x)\,dx = 0 \tag{2.10}$$

since $\big|F_n(x)\big| \leqslant \dfrac{1}{2\pi(n+1)\sin^2\frac{\delta}{2}}$ if $\delta \leqslant |x| \leqslant \pi$. Inequality (2.10) allows us to show that $\big\{\sigma_n(f,\cdot)\big\}_{n=1}^{\infty}$ converges uniformly to f.

Theorem 2.20. *For any $f \in \mathscr{C}(\mathbb{T})$, the Cesàro mean $\big\{\sigma_n(f,\cdot)\big\}_{n=1}^{\infty}$ of the Fourier series of f converges uniformly to f.*

Proof. Let $\varepsilon > 0$ be given. Since $f \in \mathscr{C}(\mathbb{T})$, f is uniformly continuous on $\mathbb{R}$; thus there exists $\delta > 0$ such that

$$\big|f(x) - f(y)\big| < \frac{\varepsilon}{2} \quad \text{whenever} \quad |x-y| < \delta .$$

Therefore, by the fact that $\int_{-\pi}^{\pi} F_n(x)dx = 1$ and $F_n \geqslant 0$,

$$\begin{aligned}
\big|\sigma_n(f,x) - f(x)\big| &= \Big|\int_{-\pi}^{\pi} F_n(y)f(x-y)\,dy - \int_{-\pi}^{\pi} F_n(y)f(x)\,dy\Big| \\
&\leqslant \int_{-\pi}^{\pi} F_n(y)\big|f(x-y) - f(x)\big|\,dy \\
&= \int_{|y|<\delta} F_n(y)\big|f(x-y) - f(x)\big|\,dy + \int_{\delta\leqslant|y|\leqslant\pi} F_n(y)\big|f(x-y) - f(x)\big|\,dy \\
&\leqslant \varepsilon \int_{|y|<\delta} F_n(y)\,dy + 2\|f\|_\infty \int_{\delta\leqslant|y|\leqslant\pi} F_n(y)\,dy \\
&\leqslant \frac{\varepsilon}{2} + 2\|f\|_\infty \int_{\delta\leqslant|y|\leqslant\pi} F_n(y)\,dy\,.
\end{aligned}$$

Using (2.10), there exists $N > 0$ such that

$$2\|f\|_\infty \int_{\delta\leqslant|y|\leqslant\pi} F_n(y)\,dy < \frac{\varepsilon}{2} \quad \textit{whenever} \quad n \geqslant N\,.$$

Therefore, $\big|\sigma_n(f,x) - f(x)\big| < \varepsilon$ whenever $n \geqslant N$ and $x \in \mathbb{R}$; thus we conclude that the Cesàro mean $\big\{\sigma_n(f,\cdot)\big\}_{n=1}^{\infty}$ converges uniformly to f. ▫

2.4 Convergence of Fourier Series for Functions with Jump Discontinuity

In previous sections we discussed the convergence of the Fourier series representation of continuous functions. However, since the Fourier series can be defined for bounded Riemann integrable functions, it is natural to ask what happen if the function under consideration is not continuous. In this section, we focus on the convergence behavior of Fourier series representation of functions with jump discontinuities.

Definition 2.21. A function $f : [-\pi,\pi] \to \mathbb{R}$ is said to have jump discontinuity at $a \in (-\pi,\pi)$ if

1. $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ both exist.
2. $\lim\limits_{x\to a^+} f(x) \neq \lim\limits_{x\to a^-} f(x)$.

Now suppose that $f : [-\pi, \pi] \to \mathbb{R}$ is piecewise Hölder continuous with exponent $\alpha \in (0, 1]$; that is, there exists $\{a_1, \cdots, a_m\} \subseteq (-\pi, \pi)$ such that $f \in \mathscr{C}^{0,\alpha}((a_j, a_{j+1}); \mathbb{R})$ for all $j \in \{0, \cdots, m\}$, where $a_0 = -\pi$ and $a_{m+1} = \pi$. Then for all $a \in (-\pi, \pi)$, the limits $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ exist since if $\{x_k\}_{k=1}^{\infty}$ is a sequence in $(-\pi, \pi)$ which approaches to a from the right/left, then for some $0 \leqslant j \leqslant m$ we must have $x_k \in (a_j, a_{j+1})$ for all large k so that the Hölder continuity implies that

$$\big|f(x_k) - f(x_\ell)\big| \leqslant M|x_k - x_\ell|^\alpha \qquad \forall\, k, \ell \text{ large}$$

which shows that $\{f(x_k\}_{k=1}^{\infty}$ is a Cauchy sequence $\big($converging to $\lim\limits_{x\to a^\pm} f(x)\big)$. In other words, if $f : [-\pi, \pi] \to \mathbb{R}$ is piecewise Hölder continuous and $a \in (-\pi, \pi)$ is a discontinuity of f, then f has either removable discontinuity at a $\big($which means $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) \neq f(a)\big)$ or jump discontinuity at a. In the following, we always assume that f is piecewise Hölder continuous with exponent $\alpha \in (0, 1]$ and has only jump discontinuities at $\{a_1, \cdots, a_m\}$ in $(-\pi, \pi)$.

Let $f(a_j^+) = \lim\limits_{x\to a_j^+} f(x)$, $f(a_j^-) = \lim\limits_{x\to a_j^-} f(x)$, and define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = \frac{1}{2\pi}(x - \pi) \qquad \forall\, x \in [0, 2\pi) \tag{2.11}$$

and $\phi(x + 2\pi) = \phi(x)$ for all $x \in \mathbb{R}$. Since f has jump discontinuities at $\{a_1, \cdots, a_m\}$, with a_0^- denoting a_{m+1}^- the function $g : [-\pi, \pi] \to \mathbb{R}$ defined by

$$g(x) \equiv \begin{cases} f(x) + \sum\limits_{j=0}^{m} \big(f(a_j^+) - f(a_j^-)\big)\phi(x - a_j) & \text{if } x \neq a_k \text{ for all } k, \\ \dfrac{f(a_k^+) + f(a_k^-)}{2} + \sum\limits_{\substack{0\leqslant j\leqslant m \\ j\neq k}} \big(f(a_j^+) - f(a_j^-)\big)\phi(a_k - a_j) & \text{if } x = a_k \text{ for some } k, \end{cases} \tag{2.12}$$

is Hölder continuous with exponent α and $g(a_0^+) = g(a_0^-) = g(-\pi)$. Let G be the 2π-periodic extension of g; that is, $G = g$ on $[-\pi, \pi]$ and $G(x + 2\pi) = G(x)$ for all $x \in \mathbb{R}$. Then $G \in \mathscr{C}^{0,\alpha}(\mathbb{T})$; thus Theorem 2.17 implies that $s_n(G, \cdot) \to G$ uniformly on $\mathbb{R}$. In particular, $s_n(g, \cdot) \to g$ uniformly on $[-\pi, \pi]$.

Using the identity

$$\int_{-\pi}^{\pi} \phi(x - a)e^{-ikx}\,dx = e^{-ika}\int_{-\pi}^{\pi} \phi(x)e^{-ikx}\,dx = \widehat{\phi}_k e^{-ika}\,,$$

we obtain that

$$s_n(\phi(\cdot - a), x) = \sum_{k=-n}^{n} \widehat{\phi}_k e^{ik(x-a)} = s_n(\phi, x - a)\,; \tag{2.13}$$

thus (2.12) implies that the Fourier series representation of f is given by

$$\begin{aligned} s_n(f,x) &= s_n(g,x) - \sum_{j=0}^{m} \big(f(a_j^+) - f(a_j^-)\big) s_n(\phi(\cdot - a_j), x) \\ &= s_n(g,x) - \sum_{j=0}^{m} \big(f(a_j^+) - f(a_j^-)\big) s_n(\phi, x - a_j)\,. \end{aligned} \tag{2.14}$$

Therefore, to understand the convergence of the Fourier series representation of f, without loss of generality it suffices to consider the convergence of $s_n(\phi,\cdot)$.

2.4.1 Uniform convergence on compact subsets

In this sub-section, we show that the Fourier series of a piecewise Hölder continuous function whose discontinuities are all jump discontinuities converges uniformly on each compact subset containing no jump discontinuities.

Based on the discussion above, we first study the convergence of $s_n(\phi,\cdot)$. Since ϕ is an odd function, for $k \in \mathbb{N}$,

$$\begin{aligned} s_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\sin kx\,dx = \frac{1}{\pi^2}\int_0^{\pi} (x-\pi)\sin kx\,dx \\ &= \frac{1}{\pi^2}\Big[\frac{-(x-\pi)\cos kx}{k}\Big|_{x=0}^{x=\pi} + \int_0^{\pi}\frac{\cos kx}{k}\,dx\Big] = -\frac{1}{\pi k}\,. \end{aligned}$$

Therefore, the Fourier series of ϕ is given by

$$s_n(\phi,x) = -\frac{1}{\pi}\sum_{k=1}^{n}\frac{\sin kx}{k}\,. \tag{2.15}$$

Lemma 2.22. *The series* $\sum\limits_{k=1}^{\infty}\frac{\sin kx}{k}$ *converges uniformly on* $[-\pi,-\delta]\cup[\delta,\pi]$ *for all* $0<\delta<\pi$.

Proof. Let $0<\delta<\pi$ be given, and $S_n(x)$ denote the sum $\sum\limits_{k=1}^{n}\sin kx$. Using the identity

$$\sum_{k=1}^{n}\sin kx = \frac{\cos(n+\frac{1}{2})x - \cos\frac{x}{2}}{2\sin\frac{x}{2}} \qquad \forall\, x\in[-\pi,-\delta]\cup[\delta,\pi]\,,$$

we find that $|S_n| \leqslant M < \infty$ for some fixed constant M. For $m>n$,

$$\begin{aligned} \sum_{k=n+1}^{m}\frac{1}{k}\sin kx &= \frac{1}{m}(S_m - S_{m-1}) + \frac{1}{m-1}(S_{m-1}-S_{m-2}) + \cdots + \frac{1}{n+1}(S_{n+1}-S_n) \\ &= \frac{S_m}{m} - \frac{S_n}{n+1} + \frac{1}{m(m-1)}S_{m-1} + \frac{1}{(m-1)(m-2)}S_{m-2} + \cdots \frac{1}{(n+1)n}S_{n+1}\,; \end{aligned}$$

thus

$$\Big|\sum_{k=n+1}^{m}\frac{1}{k}\sin kx\Big| \leqslant M\Big(\frac{1}{m}+\frac{1}{n+1}+\sum_{k=n+1}^{m}\frac{1}{k(k-1)}\Big) \leqslant 2M\Big(\frac{1}{m}+\frac{1}{n}\Big)\,.$$

Since the right-hand side converges to 0 as $n, m \to \infty$, the Cauchy criterion (for the convergence of series of functions) implies that the series

$$\sum_{k=1}^{\infty}\frac{\sin kx}{k}$$

converges uniformly on $[-\pi, -\delta] \cup [\delta, \pi]$. ▫

Lemma 2.22 provides the uniform convergence of $s_n(\phi, \cdot)$ in $[-\pi, -\delta] \cup [\delta, \pi]$. To see the limit is exactly ϕ, we consider an anti-derivative Φ of ϕ and establish that $\Phi' = s(\phi, \cdot)$.

Let $\Psi : \mathbb{R} \to \mathbb{R}$ be 2π-periodic and $\Psi(x) = \dfrac{x^2}{4\pi}$ for $x \in [-\pi, \pi]$. Then $\Psi \in \mathscr{C}^{0,1}(\mathbb{T})$ is an even function and the Fourier coefficients of Ψ is

$$\widehat{\Psi}_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{x^2}{4\pi}\,dx = \frac{\pi}{12}$$

and for $k \neq 0$,

$$\widehat{\Psi}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{x^2}{4\pi}e^{-ikx}\,dx = \frac{1}{8\pi^2}\int_{-\pi}^{\pi}x^2(\cos kx + i\sin kx)\,dx = \frac{(-1)^k}{2k^2\pi}\,.$$

Therefore, using (2.13) we find that the Fourier series of $\Phi \equiv \Psi(\cdot - \pi)$ is

$$\begin{aligned} s(\Phi, x) = s(\Psi, x-\pi) &= \frac{\pi}{12} + \sum_{k\in\mathbb{Z}, k\neq 0}\widehat{\Psi}_k e^{ik(x-\pi)} = \frac{\pi}{12} + \frac{1}{2\pi}\sum_{k\in\mathbb{Z}, k\neq 0}\frac{e^{ikx}}{k^2} \\ &= \frac{\pi}{12} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\cos kx}{k^2}\,. \end{aligned}$$

Since $\Phi \in \mathscr{C}^{0,1}(\mathbb{T})$, $s_n(\Phi, \cdot)$ converges uniformly to Φ on $\mathbb{R}$. Moreover, $s_n(\Phi, \cdot)' = s_n(\phi, \cdot)$ which converges uniformly on $[-\pi, -\delta] \cup [\delta, \pi]$. Therefore, Theorem 1.5 implies that $s(\phi, \cdot)$, the uniform limit of $s_n(\phi, \cdot)$, must equal Φ' on $[-\pi, -\delta] \cup [\delta, \pi]$. Finally, we note that $\phi = \Phi'$ on $[-\pi, -\delta] \cup [\delta, \pi]$, so we establish that $s_n(\phi, \cdot) \to \phi$ uniformly on $[-\pi, -\delta] \cup [\delta, \pi]$.

Since a discontinuity of a piecewise Hölder continuous function f is either removable or a jump discontinuity, and the value of the function at removable discontinuities does not change the value of the Fourier series of f, the uniform convergence of $s_n(\phi, \cdot)$ to ϕ on $[-\pi, -\delta] \cup [\delta, \pi]$ for all $0 < \delta < \pi$ implies the following

Theorem 2.23. *Let $f : (-\pi,\pi) \to \mathbb{R}$ be piecewise Hölder continuous with exponent $\alpha \in (0,1]$. If f is continuous on (a,b), then the Fourier series of f converges uniformly to f on any compact subsets of (a,b).*

By Remark 2.3, we can also conclude the following

Corollary 2.24. *Let $f : (-L,L) \to \mathbb{R}$ be piecewise Hölder continuous with exponent $\alpha \in (0,1]$. If f is continuous on (a,b), then the Fourier series of f converges uniformly to f on any compact subsets of (a,b) (where the Fourier series of f is given in Remark 2.3). In particular, $\lim\limits_{n\to\infty} s_n(f,x_0) = f(x_0)$ if f is continuous at x_0. In other words, the Fourier series of f converges pointwise to f except the discontinuities.*

2.4.2 Jump discontinuity and Gibbs phenomenon

In this sub-section, we show that the Fourier series evaluated at the jump discontinuity converges to the average of the limits from the left and the right. Moreover, the convergence of the Fourier series is never uniform in the domain including these jump discontinuities due to the famous Gibbs phenomenon: near the jump discontinuity the maximum difference between the limit of the Fourier series and the function itself is at least 8% of the jump. To be more precise, we have the following

Theorem 2.25. *Let $f : \mathbb{R} \to \mathbb{R}$ be $2L$-periodic piecewise Hölder continuous with exponent $\alpha \in (0,1]$. Then*

$$\lim_{n\to\infty} s_n(f,x_0) = \frac{f(x_0^+) + f(x_0^-)}{2} \qquad \forall\, x_0 \in \mathbb{R}\,. \tag{2.16}$$

Moreover, if x_0 is a jump discontinuity of f so that

$$f(x_0^+) - f(x_0^-) = a \neq 0\,,$$

then there exists a constant $c > 0$, independent of f, x_0 and L (in fact, $c = \dfrac{1}{\pi}\displaystyle\int_0^\pi \frac{\sin x}{x}\,dx - \frac{1}{2} \approx 0.089490$), such that

$$\lim_{n\to\infty} s_n\big(f, x_0 + \frac{L}{n}\big) = f(x_0^+) + ca\,, \tag{2.17a}$$

$$\lim_{n\to\infty} s_n\big(f, x_0 - \frac{L}{n}\big) = f(x_0^-) - ca\,. \tag{2.17b}$$

Proof. By Remark 2.3, W.L.O.G. we can assume that $L = \pi$. Let $\{a_1,\cdots,a_m\} \subseteq (-\pi,\pi)$ be the collection of jump discontinuities of f in $(-\pi,\pi)$, $a_0 = -\pi$, $a_{m+1} = \pi$ (so by periodicity

$f(a_0^-) = f(a_{m+1}^-)$ automatically), and define g by (2.12). Then $g \in \mathscr{C}^{0,\alpha}(\mathbb{T})$. Suppose that x_0 is a jump discontinuity of f in $[-\pi, \pi)$ (so a_0 could be a possible jump discontinuity of f). Then $x_0 = a_k$ for some $k \in \{0, 1, \cdots, m\}$. Therefore, by the fact that ϕ is continuous at $x_0 - a_j$ if $j \neq k$ and $s_n(\phi, 0) = 0$ for all $n \in \mathbb{N}$, Corollary 2.24 implies that

$$\begin{aligned}
&\sum_{j=0}^{m} \big(f(a_j^+) - f(a_j^-)\big) \lim_{n\to\infty} s_n(\phi, x_0 - a_j) \\
&\quad = \sum_{\substack{0\leqslant j\leqslant m \\ j\neq k}} \big(f(a_j^+) - f(a_j^-)\big) \lim_{n\to\infty} s_n(\phi, x_0 - a_j) = \sum_{\substack{0\leqslant j\leqslant m \\ j\neq k}} \big(f(a_j^+) - f(a_j^-)\big)\phi(x_0 - a_j)\,.
\end{aligned}$$

On the other hand,

$$\lim_{n\to\infty} s_n(g, x_0) = g(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2} + \sum_{\substack{0\leqslant j\leqslant m \\ j\neq k}} \big(f(a_j^+) - f(a_j^-)\big)\phi(x_0 - a_j)\,.$$

Identity (2.16) is then concluded using (2.14).

Now we focus on (2.17a). Since $g \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, $s_n(g, \cdot) \to g$ uniformly on $\mathbb{R}$; thus

$$\lim_{n\to\infty} s_n\Big(g, x_0 + \frac{\pi}{n}\Big) = g(x_0)\,.$$

Similarly, since $s_n(\phi, \cdot) \to \phi$ uniformly on $[-\pi, -\delta] \cup [\delta, \pi]$ for all $\delta > 0$, if $j \neq k$,

$$\lim_{n\to\infty} s_n\Big(\phi, x_0 + \frac{\pi}{n} - a_j\Big) = \phi(x_0 - a_j)\,.$$

On the other hand,

$$s_n\Big(\phi, \frac{\pi}{n}\Big) = -\sum_{k=1}^{n} \frac{1}{\pi k} \sin\frac{k\pi}{n} = -\frac{1}{\pi} \sum_{k=1}^{n} \frac{n}{k\pi} \sin\frac{k\pi}{n}\frac{\pi}{n} \to -\frac{1}{\pi}\int_0^{\pi} \frac{\sin x}{x} dx \equiv -\Big(c + \frac{1}{2}\Big)\,.$$

As a consequence,

$$\begin{aligned}
\lim_{n\to\infty} s_n\Big(f, x_0 + \frac{\pi}{n}\Big) &= \lim_{n\to\infty} \Big[s_n\Big(g, x_0 + \frac{\pi}{n}\Big) - \sum_{j=0}^{m} \big(f(a_j^+) - f(a_j^-)\big) s_n\Big(\phi, x_0 + \frac{\pi}{n} - a_j\Big)\Big] \\
&= g(x_0) - \sum_{\substack{0\leqslant j\leqslant m \\ j\neq k}} \big(f(a_j^+) - f(a_j^-)\big)\phi(x_0 - a_j) + \Big(c + \frac{1}{2}\Big)\big(f(x_0^+) - f(x_0^-)\big) \\
&= f(x_0^+) + c\big(f(x_0^+) - f(x_0^-)\big)\,.
\end{aligned}$$

Identity (2.17b) can be proved in the same fashion, and is left as an exercise. □

Remark 2.26. Let f be a function given in Theorem 2.25, x_0 be a jump discontinuity of f, and $I = (x_0, x_0 + r)$ for some $r > 0$ so that f is continuous on I. By the definition of the right limit, there exists $0 < \delta < r$ such that

$$\big|f(x) - f(x_0^+)\big| < \frac{c|a|}{2} \qquad \forall\, x \in (x_0, x_0 + \delta)\,.$$

Choose $N > 0$ such that $\dfrac{L}{N} < \delta$. Then $x_0 + \dfrac{L}{N} \in (x_0, x_0 + \delta)$ for all $n \geqslant N$; thus if $n \geqslant N$,

$$\begin{aligned}\sup_{x\in I}\big|s_n(f,x) - f(x)\big| &\geqslant \big|s_n(f, x_0 + \frac{L}{N}) - f(x_0 + \frac{L}{N})\big| \\ &\geqslant \big|s_n(f, x_0 + \frac{L}{N}) - f(x_0^+)\big| - \big|f(x_0 + \frac{L}{N}) - f(x_0^+)\big| \\ &\geqslant \big|s_n(f, x_0 + \frac{L}{N}) - f(x_0^+)\big| - \frac{c|a|}{2}\end{aligned}$$

which implies that

$$\liminf_{n\to\infty}\sup_{x\in I}\big|s_n(f,x) - f(x)\big| \geqslant c|a| - \frac{c|a|}{2} = \frac{c|a|}{2}\,.$$

Therefore, $\big\{s_n(f,\cdot)\big\}_{n=1}^\infty$ does not converge uniformly (to f) on I, while Corollary 2.24 shows that $\big\{s_n(f,\cdot)\big\}_{n=1}^\infty$ converges pointwise to f on I. Similarly, if x_0 is a jump discontinuity of f and f is continuous on $(x_0 - r, x_0)$ for some $r > 0$, then $\big\{s_n(f,\cdot)\big\}_{n=1}^\infty$ converge pointwise but not uniformly on $(x_0 - r, x_0)$.

For a function f given in Theorem 2.25, let $\widetilde{f}$ be defined by

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x\,, \\ \dfrac{f(x^+) + f(x^-)}{2} & \text{if } x \text{ is a discontinuity of } f\,. \end{cases}$$

Then $s_n(\widetilde{f},\cdot) = s_n(f,\cdot)$ for all $n \in \mathbb{N}$, and Corollary 2.24 and Theorem 2.25 together imply that $\big\{s_n(f,\cdot)\big\}_{n=1}^\infty$ converges pointwise to $\widetilde{f}$. However, the discussion above shows that $\big\{s_n(f,\cdot)\big\}_{n=1}^\infty$ cannot converge uniformly on I as long as I contains jump discontinuities of f.

2.5 The Inner-Product Point of View

除了逐點收斂或均勻收斂的觀點之外，還有一個更自然（就數學而言）的觀點可以用來看 Fourier series。我們可以把定義在 $[-\pi, \pi]$ 的所有 Riemann integrable 函數所形成的集合

看成一個向量空間，然後在上面定義一個內積的結構。一個可積分函數（也可視為一個向量）的 Fourier series representation 可以看成這個向量在一組正交基底向量的線性組合。

Let $L^2(\mathbb{T})$ denote the collection of Riemann measurable, square integrable function over $[-\pi, \pi]$ modulo the relation that $f \sim g$ if $f - g = 0$ except on a set of measure zero (or $f = g$ almost everywhere). In other words,

$$L^2(\mathbb{T}) = \Big\{ f : [-\pi, \pi) \to \mathbb{C} \,\Big|\, \int_{[-\pi,\pi)} |f(x)|^2\, dx < \infty \Big\} / \sim \,.$$

Here again we **abuse** the use of notation $L^2(\mathbb{T})$ for that it indeed denotes a more general space. We also note that the domain $[-\pi, \pi)$ can be replaced by any intervals with $-\pi$, π as end-points for we can easily modify functions defined on those domains to functions defined on $[-\pi, \pi)$ without changing the Riemann measurability and the square integrability.

Define a bilinear function $\langle \cdot, \cdot \rangle$ on $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx \,.$$

Then $\langle \cdot, \cdot \rangle$ is an inner product on $L^2(\mathbb{T})$. Indeed, if f, g belong to $L^2(\mathbb{T})$, then the product $f\overline{g}$ is also Riemann measurable, and the Cauchy-Schwartz inequality as well as the monotone convergence theorem imply that

$$\begin{aligned}
\big|\langle f, g \rangle\big| &= \lim_{k\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \big|(f \wedge k)(x)\big|\big|(g \wedge k)(x)\big|\, dx \\
&\leqslant \lim_{k\to\infty} \frac{1}{2\pi} \Big(\int_{-\pi}^{\pi} \big|(f \wedge k)(x)\big|^2 dx \Big)^{\frac{1}{2}} \Big(\int_{-\pi}^{\pi} \big|(g \wedge k)(x)\big|^2 dx \Big)^{\frac{1}{2}} \\
&= \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} \big|f(x)\big|^2 dx \Big)^{\frac{1}{2}} \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} \big|g(x)\big|^2 dx \Big)^{\frac{1}{2}} = \|f\|_{L^2(\mathbb{T})} \|g\|_{L^2(\mathbb{T})} < \infty \,;
\end{aligned}$$

thus the definition of the inner product $\langle \cdot, \cdot \rangle$ given above is well-defined. The norm induced by the inner product above is denoted by $\|\cdot\|_{L^2(\mathbb{T})}$.

For $k \in \mathbb{Z}$, define $\mathbf{e}_k : [-\pi, \pi] \to \mathbb{C}$ by $\mathbf{e}_k(x) = e^{ikx}$. Then $\{\mathbf{e}_k\}_{k=-\infty}^{\infty}$ is an orthonormal set in $L^2(\mathbb{T})$ since

$$\langle \mathbf{e}_k, \mathbf{e}_\ell \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-i\ell x}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-\ell)x}\, dx = \begin{cases} 1 & \text{if } k = \ell \,, \\ 0 & \text{if } k \neq \ell \,. \end{cases}$$

Let $\mathcal{V}_n = \text{span}(\mathbf{e}_{-n}, \mathbf{e}_{-n+1}, \cdots, \mathbf{e}_0, \mathbf{e}_1, \cdots, \mathbf{e}_n) = \Big\{ \sum\limits_{k=-n}^{n} a_k \mathbf{e}_k \,\Big|\, \{a_k\}_{k=-n}^{n} \subseteq \mathbb{C} \Big\}$. For each vector $f \in L^2(\mathbb{T})$, the orthogonal projection of f onto $\mathcal{V}_n$ is, conceptually, given by

$$\sum_{k=-n}^{n} \langle f, \mathbf{e}_k \rangle \mathbf{e}_k = \sum_{k=-n}^{n} \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx \Big) \mathbf{e}_k = \sum_{k=-n}^{n} \widehat{f}_k \mathbf{e}_k \,.$$

By the definition of $\mathbf{e}_k$, we obtain that the orthogonal projection of f on $\mathcal{V}_n$ is exactly $s_n(f, \cdot)$. We also note that $\mathcal{V}_n = \mathscr{P}_n(\mathbb{T})$.

Now we prove that $s_n(f, \cdot)$ is exactly the orthogonal projection of f onto $\mathcal{V}_n = \mathscr{P}_n(\mathbb{T})$.

Proposition 2.27. *Let* $f \in L^2(\mathbb{T})$. *Then*

$$\langle f - s_n(f, \cdot), p \rangle = 0 \qquad \forall\, p \in \mathscr{P}_n(\mathbb{T})\,.$$

Proof. Let $p \in \mathscr{P}_n(\mathbb{T})$. Then $p = s_n(p, \cdot)$; thus

$$\begin{aligned} \langle f - s_n(f, \cdot), p \rangle &= \langle f, p \rangle - \langle s_n(f, \cdot), p \rangle = \Big\langle f, \sum_{k=-n}^{n} \widehat{p}_k \mathbf{e}_k \Big\rangle - \Big\langle \sum_{k=-n}^{n} \widehat{f}_k \mathbf{e}_k, p \Big\rangle \\ &= \sum_{k=-n}^{n} \overline{\widehat{p}_k} \langle f, \mathbf{e}_k \rangle - \sum_{k=-n}^{n} \widehat{f}_k \overline{\langle p, \mathbf{e}_k \rangle} = \sum_{k=-n}^{n} \overline{\widehat{p}_k} \widehat{f}_k - \sum_{k=-n}^{n} \widehat{f}_k \overline{\widehat{p}_k} = 0\,. \end{aligned}$$

□

Theorem 2.28. *Let* $f \in L^2(\mathbb{T})$. *Then*

$$\|f - p\|^2_{L^2(\mathbb{T})} = \|f - s_n(f, \cdot)\|^2_{L^2(\mathbb{T})} + \|s_n(f, \cdot) - p\|^2_{L^2(\mathbb{T})} \qquad \forall\, p \in \mathscr{P}_n(\mathbb{T})\,. \tag{2.18}$$

Proof. By Proposition 2.27, if $p \in \mathscr{P}_n(\mathbb{T})$, $s_n(f, \cdot) - p = s_n(f - p, \cdot) \in \mathscr{P}_n(\mathbb{T})$; thus

$$\begin{aligned} \|f - p\|^2_{L^2(\mathbb{T})} &= \langle f - p, f - p \rangle = \langle f - s_n(f, \cdot) + s_n(f, \cdot) - p, f - s_n(f, \cdot) + s_n(f, \cdot) - p \rangle \\ &= \big\|f - s_n(f, \cdot)\big\|^2_{L^2(\mathbb{T})} + 2\mathrm{Re}\big(\langle f - s_n(f, \cdot), s_n(f, \cdot) - p \rangle\big) + \big\|s_n(f, \cdot) - p\big\|^2_{L^2(\mathbb{T})} \\ &= \big\|f - s_n(f, \cdot)\big\|^2_{L^2(\mathbb{T})} + \big\|s_n(f, \cdot) - p\big\|^2_{L^2(\mathbb{T})} \end{aligned}$$

which concludes the proposition. □

We note that (2.18) implies that

$$\|f - s_n(f, \cdot)\|_{L^2(\mathbb{T})} \leqslant \|f - p\|_{L^2(\mathbb{T})} \qquad \forall\, p \in \mathscr{P}_n(\mathbb{T})\,. \tag{2.19}$$

Since $s_n(f, \cdot) \in \mathscr{P}_n(\mathbb{T})$, we conclude that

$$\|f - s_n(f, \cdot)\|_{L^2(\mathbb{T})} = \inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_{L^2(\mathbb{T})}\,.$$

Moreover, letting $p = 0$ in (2.18) we establish the famous Bessel's inequality.

Corollary 2.29. *Let $f \in L^2(\mathbb{T})$. Then for all $n \in \mathbb{N}$,*

$$\|s_n(f,\cdot)\|_{L^2(\mathbb{T})} \leqslant \|f\|_{L^2(\mathbb{T})} . \tag{2.20}$$

In particular,

$$\sum_{k=-\infty}^{\infty} |\widehat{f}_k|^2 \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|f(x)\big|^2\,dx . \qquad \text{(Bessel's inequality)}$$

Remark 2.30. When $f \in L^2(\mathbb{T})$ and f is real-valued, then

$$\sum_{k=-\infty}^{\infty} |\widehat{f}_k|^2 = \frac{c_0^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}(c_k^2 + s_k^2) ;$$

thus in this case the Bessel inequality can also be written as

$$\frac{c_0^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}(c_k^2 + s_k^2) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|f(x)\big|^2\,dx .$$

Next, we prove that the Bessel inequality is in fact an equality, called the Parseval identity. Using (2.18), it is equivalent to that $\big\{s_n(f,\cdot)\big\}_{n=1}^{\infty}$ converges to f in the sense of L^2-norm; that is,

$$\lim_{n\to\infty} \big\|s_n(f,\cdot) - f\big\|_{L^2(\mathbb{T})} = 0 \qquad \forall\, f \in L^2(\mathbb{T}) .$$

Before proceeding, we first prove that every $f \in L^2(\mathbb{T})$ can be approximated by a sequence $\{g_n\}_{n=1}^{\infty} \subseteq \mathscr{C}(\mathbb{T})$ in the sense of L^2-norm.

Proposition 2.31. *Let $f \in L^2(\mathbb{T})$. Then for all $\varepsilon > 0$ there exists $g \in \mathscr{C}(\mathbb{T})$ such that*

$$\|f - g\|_{L^2(\mathbb{T})} < \varepsilon .$$

In other words, $\mathscr{C}(\mathbb{T})$ is dense in $\big(L^2(\mathbb{T}), \|\cdot\|_{L^2(\mathbb{T})}\big)$.

Proof. W.L.O.G., we can assume that f is real-valued and non-zero. Let $\varepsilon > 0$ be given. Since $f \in L^2(\mathbb{T})$, the monotone convergence theorem implies that

$$\lim_{k\to\infty} \|f - (-k) \vee (f \wedge k)\|_{L^2(\mathbb{T})}^2 = \lim_{k\to\infty} \int_{-\pi}^{\pi} \mathbf{1}_{\{|f(x)|>k\}}(x)|f(x)|^2\,dx = 0 ;$$

thus there exists $K > 0$ such that

$$\|f - (-k) \vee (f \wedge k)\|_{L^2(\mathbb{T})} < \frac{\varepsilon}{2} \qquad \forall\, k \geqslant K .$$

Let $h = (-K) \vee (f \wedge K)$. Then h is bounded and Riemann measurable; thus h is Riemann integrable on $[-\pi, \pi]$. Therefore, there exists a partition $\mathcal{P} = \{-\pi = x_0 < x_1 < \cdots < x_n = \pi\}$ of $[-\pi, \pi]$ such that $U(h, \mathcal{P}) - L(h, \mathcal{P}) < \frac{\pi\varepsilon^2}{8K}$. Define

$$S(x) = \sum_{k=0}^{n-1} \sup_{\xi \in [x_k, x_{k+1}]} h(\xi) \mathbf{1}_{[x_k, x_{k+1}]}(x) \quad \text{and} \quad s(x) = \sum_{k=0}^{n-1} \inf_{\xi \in [x_k, x_{k+1}]} h(\xi) \mathbf{1}_{[x_k, x_{k+1}]}(x) \,,$$

where $\mathbf{1}_A$ denotes the characteristic/indicator function of set A. Then

1. $-K \leqslant s \leqslant h \leqslant S \leqslant K$ on $[-\pi, \pi] \backslash \{x_1, x_2, \cdots, x_{n-1}\}$;
2. $\displaystyle\int_{-\pi}^{\pi} S(x)\, dx = U(h, \mathcal{P})$;
3. $\displaystyle\int_{-\pi}^{\pi} s(x)\, dx = L(h, \mathcal{P})$.

The properties above show that

$$\begin{aligned} \int_{-\pi}^{\pi} \big|h(x) - s(x)\big|\, dx &= \int_{-\pi}^{\pi} h(x) - s(x)\, dx \leqslant \int_{-\pi}^{\pi} \big(S(x) - s(x)\big)\, dx \\ &= U(h, \mathcal{P}) - L(h, \mathcal{P}) < \frac{\pi\varepsilon^2}{8K} \,. \end{aligned}$$

Now, for the step function s defined on $[-\pi, \pi]$, we can always find a continuous function $g \in \mathscr{C}(\mathbb{T})$ (for example, g can be a trapezoidal function) such that

1. $\|g\|_{L^\infty(\mathbb{T})} \leqslant K$.
2. $\displaystyle\int_{-\pi}^{\pi} \big|s(x) - g(x)\big|\, dx < \frac{\varepsilon^2}{16K}$.

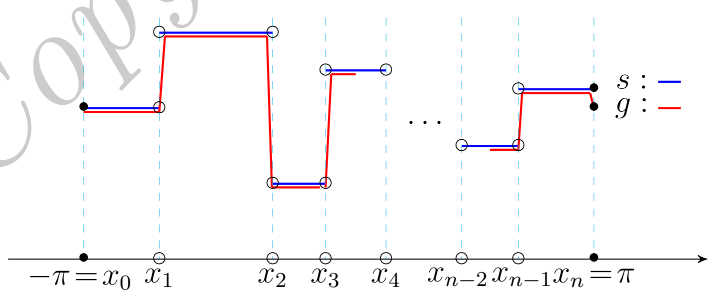

Figure 2.1: One way of constructing $g \in \mathscr{C}(\mathbb{T})$ given step function s

Therefore,

$$\int_{-\pi}^{\pi} \big|h(x) - g(x)\big|\, dx \leqslant \int_{-\pi}^{\pi} \big|h(x) - s(x)\big|\, dx + \int_{-\pi}^{\pi} \big|s(x) - g(x)\big|\, dx < \frac{\pi\varepsilon^2}{4K}$$

which implies that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|h(x)-g(x)\big|^2\,dx \leqslant \frac{K}{\pi}\int_{[-\pi,\pi]}\big|h(x)-g(x)\big|\,dx < \frac{\varepsilon^2}{4}\,;$$

thus $\|h-g\|_{L^2(\mathbb{T})} < \frac{\varepsilon}{2}$. The proposition is then concluded by the triangle inequality. □

Theorem 2.32. *Let $f \in L^2(\mathbb{T})$. Then*

$$\lim_{n\to\infty}\big\|f - s_n(f,\cdot)\big\|_{L^2(\mathbb{T})} = 0 \tag{2.21}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|f(x)\big|^2\,dx = \sum_{k=-\infty}^{\infty}|\widehat{f}(k)|^2\,. \qquad \text{(Parseval's identity)}$$

Proof. Let $\varepsilon > 0$ be given. By Proposition 2.31 there exists $g \in \mathscr{C}(\mathbb{T})$ such that

$$\|f-g\|_{L^2(\mathbb{T})} < \frac{\varepsilon}{3}\,.$$

By the denseness of the trigonometric polynomials in $\mathscr{C}(\mathbb{T})$, there exists $h \in \mathscr{P}(\mathbb{T})$ such that $\sup\limits_{x\in\mathbb{R}}\big|g(x)-h(x)\big| < \frac{\varepsilon}{3}$. Suppose that $h \in \mathscr{P}_N(\mathbb{T})$. Using (2.19),

$$\big\|g - s_N(g,\cdot)\big\|^2_{L^2(\mathbb{T})} \leqslant \|g-h\|^2_{L^2(\mathbb{T})} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\big|g(x)-h(x)\big|^2\,dx \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\varepsilon^2}{9}\,dx = \frac{\varepsilon^2}{9}\,.$$

Since $s_N(g,\cdot) \in \mathscr{P}_n(\mathbb{T})$ if $n \geqslant N$, using (2.19) again we must have

$$\big\|g - s_n(g,\cdot)\big\|_{L^2(\mathbb{T})} \leqslant \big\|g - s_N(g,\cdot)\big\|_{L^2(\mathbb{T})} \leqslant \frac{\varepsilon}{3} \qquad \forall\, n \geqslant N\,.$$

Therefore, for $n \geqslant N$, inequality (2.20) and the triangle inequality yield that

$$\begin{aligned}\big\|f - s_n(f,\cdot)\big\|_{L^2(\mathbb{T})} &\leqslant \|f-g\|_{L^2(\mathbb{T})} + \|g - s_n(g,\cdot)\|_{L^2(\mathbb{T})} + \|s_n(g-f,\cdot)\|_{L^2(\mathbb{T})} \\ &\leqslant 2\|f-g\|_{L^2(\mathbb{T})} + \|g - s_n(g,\cdot)\|_{L^2(\mathbb{T})} < \varepsilon\,;\end{aligned}$$

thus (2.21) is concluded. Finally, using (2.18) with $p = 0$ we obtain that

$$\int_{-\pi}^{\pi}\big|f(x)\big|^2\,dx = \int_{-\pi}^{\pi}\big|s_n(f,x)\big|^2\,dx + \int_{-\pi}^{\pi}\big|f(x)-s_n(f,x)\big|^2\,dx\,.$$

Using the fact that $\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|s_n(f,x)\big|^2\,dx = \sum\limits_{k=-n}^{n}\big|\widehat{f}_k\big|^2$ and passing to the limit as $n \to \infty$, we conclude the Parseval identity. □

Example 2.33. Example 2.6 provides that $\int_{-\pi}^{\pi} x^2\,dx = \pi\sum\limits_{k=1}^{\infty}\frac{4}{k^2}$; thus $\sum\limits_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$.

2.6 The Discrete Fourier "Transform" and the Fast Fourier "Transform"

Let $f:\mathbb{R}\to\mathbb{R}$ be a periodic function with period L and f is bounded Riemann integrable on $[0,L)$. Similar to Remark 2.2, the Fourier series of f, defined in Remark 2.3, can be written as

$$s(f,x)=\sum_{k=-\infty}^{\infty}\widehat{f}_k e^{\frac{2\pi ikx}{L}},$$

where $\widehat{f}_k=\dfrac{1}{L}\displaystyle\int_0^L f(y)e^{\frac{-2\pi iky}{L}}\,dy$; thus $\widehat{f}_k$ can be approximated by the Riemann sum

$$\frac{1}{L}\sum_{\ell=0}^{N-1}f\big(\frac{L\ell}{N}\big)e^{\frac{-2\pi ik\ell}{N}}\frac{L}{N}=\frac{1}{N}\sum_{\ell=0}^{N-1}f\big(\frac{L\ell}{N}\big)e^{\frac{-2\pi ik\ell}{N}}.$$

In other words, the values of f at N evenly distributed points can be used to determine an approximation of the Fourier coefficients of f.

There is another point of view of finding the sum $\dfrac{1}{N}\displaystyle\sum_{\ell=0}^{N-1}f\big(\frac{L\ell}{N}\big)e^{\frac{-2\pi ik\ell}{N}}$. Even though $s_n(f,x)$ will be a good approximation of $s(f,x)$ for large n, the computation of the exact Fourier coefficients will be expensive (and probably impossible). Therefore, instead of compute the exact Fourier coefficients, we look for a Fourier-like series of the form

$$\frac{1}{N}\sum_{k=0}^{N-1}X_k e^{\frac{2\pi ikx}{L}}.$$

so that it agrees with the value of f at points $\left\{\dfrac{Lj}{N}\right\}_{j=0}^{N-1}$. Therefore, we look for $\{X_k\}_{k=0}^{N-1}$ satisfying that

$$\frac{1}{N}\begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ 1 & e^{\frac{2\pi i}{N}} & e^{\frac{4\pi i}{N}} & \cdots & e^{\frac{2\pi(N-1)i}{N}}\\ 1 & e^{\frac{4\pi i}{N}} & e^{\frac{8\pi i}{N}} & \cdots & e^{\frac{4\pi(N-1)i}{N}}\\ \vdots & & & \ddots & \vdots\\ 1 & e^{\frac{2\pi(N-1)i}{N}} & e^{\frac{4\pi(N-1)i}{N}} & \cdots & e^{\frac{2\pi(N-1)^2 i}{N}}\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\\ \vdots\\ X_{N-1}\end{bmatrix}=\begin{bmatrix}f(0)\\ f\big(\frac{L}{N}\big)\\ \vdots\\ f\big(\frac{(N-1)L}{N}\big)\end{bmatrix}.$$

Let $\boldsymbol{v}_k=\big[v_k^{(1)},v_k^{(2)},\cdots,v_k^{(N)}\big]^{\mathrm{T}}$ denote the k-th column of the $N\times N$ matrix F on the left-hand side of the equation above. Then $v_k^{(j)} = e^{\frac{2\pi(k-1)(j-1)i}{N}}$ so that

$$\begin{aligned}\boldsymbol{v}_\ell \cdot \boldsymbol{v}_k = \boldsymbol{v}_k^* \boldsymbol{v}_\ell &= \sum_{j=1}^{N} e^{\frac{-2\pi(j-1)(k-1)i}{N}} e^{\frac{2\pi(j-1)(\ell-1)i}{N}} = \sum_{j=1}^{N} e^{\frac{2\pi(j-1)(\ell-k)i}{N}} \\ &= \sum_{j=0}^{N-1} \cos\frac{2\pi j(\ell-k)}{N} + i\sum_{j=0}^{N-1} \sin\frac{2\pi j(\ell-k)i}{N}\end{aligned}$$

which shows that

$$\boldsymbol{v}_\ell \cdot \boldsymbol{v}_k = \begin{cases} N & \text{if } k = \ell\,, \\ 0 & \text{if } k \neq \ell\,. \end{cases}$$

Therefore, $F^*F = N\mathrm{I}_{N\times N}$; thus

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-\frac{2\pi i}{N}} & e^{-\frac{4\pi i}{N}} & \cdots & e^{-\frac{2\pi(N-1)i}{N}} \\ 1 & e^{-\frac{4\pi i}{N}} & e^{-\frac{8\pi i}{N}} & \cdots & e^{-\frac{4\pi(N-1)i}{N}} \\ \vdots & & & \ddots & \vdots \\ 1 & e^{-\frac{2\pi(N-1)i}{N}} & e^{-\frac{4\pi(N-1)i}{N}} & \cdots & e^{-\frac{2\pi(N-1)^2 i}{N}} \end{bmatrix} \begin{bmatrix} f(0) \\ f\big(\frac{L}{N}\big) \\ \vdots \\ f\big(\frac{(N-1)L}{N}\big) \end{bmatrix}.$$

The discussions above induce the following

Definition 2.34. The ***discrete Fourier transform***, symbolized by DFT, of a sequence of N complex numbers $\{x_0, x_1, \cdots, x_{N-1}\}$ is a sequence $\{X_k\}_{k\in\mathbb{Z}}$ defined by

$$X_k = \sum_{\ell=0}^{N-1} x_\ell e^{\frac{-2\pi i k\ell}{N}} \qquad \forall\, k \in \mathbb{Z}\,.$$

We note that the sequence $\{X_k\}_{k\in\mathbb{Z}}$ is N-periodic; that is, $X_{k+N} = X_k$ for all $k \in \mathbb{Z}$. Therefore, often time we only focus on one of the following N consecutive terms $\{X_0, X_1, \cdots, X_{N-1}\}$ of the DFT.

Example 2.35. The DFT of the sequence $\{x_0, x_1\}$ is $\{x_0 + x_1, x_0 - x_1\}$.

2.6.1 The inversion formula

Let $\{X_k\}_{k=0}^{N-1}$ be the discrete Fourier transform of the sequence $\{x_\ell\}_{\ell=0}^{N-1}$. Then $\{x_\ell\}_{\ell=0}^{N-1}$ can be recovered given $\{X_k\}_{k=0}^{N-1}$ by the inversion formula

$$x_\ell = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{\frac{2\pi i k\ell}{N}}\,. \tag{2.22}$$

To see this, we compute $\sum_{k=0}^{N-1}\Big(\sum_{j=0}^{N-1} x_j e^{\frac{-2\pi ikj}{N}}\Big)e^{\frac{2\pi ik\ell}{N}}$ and obtain that

$$\sum_{k=0}^{N-1}\Big(\sum_{j=0}^{N-1} x_j e^{\frac{-2\pi ikj}{N}}\Big)e^{\frac{2\pi ik\ell}{N}} = \sum_{j=0}^{N-1}\Big(x_j\sum_{k=0}^{N-1} e^{\frac{2\pi ik(\ell-j)}{N}}\Big) = Nx_\ell + \sum_{\substack{j=0\\ j\neq\ell}}^{N-1}\Big(x_j\sum_{k=0}^{N-1} e^{\frac{2\pi ik(\ell-j)}{N}}\Big)$$
$$= Nx_\ell + \sum_{\substack{j=0\\ j\neq\ell}}^{N-1}\Big(x_j\frac{e^{2\pi i(\ell-j)}-1}{e^{\frac{2\pi i(\ell-j)}{N}}-1}\Big) = Nx_\ell\,.$$

The map from $\{X_k\}_{k=0}^{N-1}$ to $\{x_\ell\}_{\ell=0}^{N-1}$ is called the ***discrete inverse Fourier transform***.

We note that the inversion formula (2.22) is an analogy of

$$f(x) = \sum_{k=-\infty}^{\infty} \widehat{f}_k e^{ikx}$$

for all piecewise constant function f and $x \in \mathbb{R}$ at which f is continuous.

Remark 2.36. Given a sample data $[x_0, x_1, \cdots, x_{N-1}]$ which is the values of a function f on N evenly distributed points on $[0, L)$ (for some unknown $L > 0$), the DFT $[X_0, X_1, \cdots, X_{N-1}]$ can be thought as Fourier coefficients which provides the approximation

$$f(x) \text{ ``$=$'' } \sum_{k=0}^{N-1} X_k e^{\frac{2\pi ikx}{L}} = \sum_{k=-[\frac{N}{2}]}^{-1} X_{k+N} e^{\frac{2\pi ikx}{L}} + \sum_{k=0}^{[\frac{N-1}{2}]} X_k e^{\frac{2\pi ikx}{L}}\,,$$

where the first equality " = " holds only for $x = \frac{L\ell}{N}$, $0 \leqslant \ell \leqslant N-1$. Therefore, for $0 \leqslant k \leqslant \big[\frac{N-1}{2}\big]$ each X_k is the coefficient associated with the wave with frequency $\frac{k}{L}$. To determine L, we introduce the ***sampling frequency*** F_s which is the number of samples per unit time/length. Then $F_s = \frac{N}{L}$ so that X_k is the coefficient associated with the wave with frequency $\frac{F_s}{N}k$.

2.6.2 The fast Fourier transform

Let $M = [m_{k\ell}]$ be an $N \times N$ matrix with entry $m_{k\ell}$ defined by

$$m_{k\ell} = e^{\frac{-2\pi ik\ell}{N}} \qquad 0 \leqslant k, \ell \leqslant N-1\,,$$

and write $\boldsymbol{x} = (x_0, x_1, \cdots, x_{N-1})^{\rm T}$ and $\boldsymbol{X} = (X_0, \cdots, X_{N-1})^{\rm T}$. Then $\boldsymbol{X} = M\boldsymbol{x}$ and it requires N^2 multiplications to compute $\boldsymbol{X}$. The ***fast Fourier transform***, symbolized by

FFT, is a much faster way to compute $\boldsymbol{X}$. In the following, we show that when $N = 2^\gamma$ for some $\gamma \in \mathbb{N}$, then there is a way to compute the DFT with at most $N \log_2 N$ multiplications.

With $N = 2^\gamma$, suppose that $(x_0, \cdots, x_{N-1})$ is a given sequence, and $\{X_k\}_{k=0}^{N-1}$ is the DFT of $\{x_k\}_{k=0}^{N-1}$. Let $\omega = e^{-\frac{2\pi i}{N}}$, and

$$\boldsymbol{x}_{\text{even}} = \begin{bmatrix} x_0 & x_2 & x_4 & \cdots & x_{N-2} \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}_{\text{odd}} = \begin{bmatrix} x_1 & x_3 & x_5 & \cdots & x_{N-1} \end{bmatrix}$$

Then

$$\begin{aligned} X_j &= \sum_{\ell=0}^{N-1} x_\ell \omega^{j\ell} = \sum_{\substack{0 \leqslant \ell \leqslant N-1 \\ \ell \text{ is even}}} x_\ell \omega^{j\ell} + \omega^j \sum_{\substack{0 \leqslant \ell \leqslant N-1 \\ \ell \text{ is odd}}} x_\ell \omega^{j(\ell-1)} \\ &= \boldsymbol{x}_{\text{even}} \cdot \begin{bmatrix} \omega^0 & \omega^{2j} & \omega^{4j} & \cdots & \omega^{j(N-2)} \end{bmatrix} + \omega^j \boldsymbol{x}_{\text{odd}} \cdot \begin{bmatrix} \omega^0 & \omega^{2j} & \omega^{4j} & \cdots & \omega^{j(N-2)} \end{bmatrix}. \end{aligned}$$

In particular, for $0 \leqslant j \leqslant \frac{N}{2} - 1$,

$$\begin{aligned} X_{\frac{N}{2}+j} &= \boldsymbol{x}_{\text{even}} \cdot \begin{bmatrix} \omega^0 & \omega^{2(\frac{N}{2}+j)} & \omega^{4(\frac{N}{2}+j)} & \cdots & \omega^{(\frac{N}{2}+j)(N-2)} \end{bmatrix} \\ &\quad + \omega^{\frac{N}{2}+j} \boldsymbol{x}_{\text{odd}} \cdot \begin{bmatrix} \omega^0 & \omega^{2(\frac{N}{2}+j)} & \omega^{4(\frac{N}{2}+j)} & \cdots & \omega^{(\frac{N}{2}+j)(N-2)} \end{bmatrix} \\ &= \boldsymbol{x}_{\text{even}} \cdot \begin{bmatrix} \omega^0 & \omega^{2j} & \omega^{4j} & \cdots & \omega^{j(N-2)} \end{bmatrix} - \omega^j \boldsymbol{x}_{\text{odd}} \cdot \begin{bmatrix} \omega^0 & \omega^{2j} & \omega^{4j} & \cdots & \omega^{j(N-2)} \end{bmatrix}, \end{aligned}$$

where we have used the fact that $\omega^{\frac{N}{2}} = -1$. We note that

$$\left\{ \boldsymbol{x}_{\text{even}} \cdot \begin{bmatrix} \omega^0 & \omega^{2j} & \omega^{4j} & \cdots & \omega^{j(N-2)} \end{bmatrix} \right\}_{j=0}^{N/2}$$

is exactly the DFT of the sequence $\{x_0, x_2, \cdots, x_{N-2}\}$ and

$$\left\{ \boldsymbol{x}_{\text{odd}} \cdot \begin{bmatrix} \omega^0 & \omega^{2j} & \omega^{4j} & \cdots & \omega^{j(N-1)} \end{bmatrix} \right\}_{j=0}^{N/2}$$

is exactly the DFT of the sequence $\{x_1, x_3, \cdots, x_{N-1}\}$. In other words, to compute the DFT of $\{x_0, \cdots, x_{N-1}\}$, where $N = 2^\gamma$, it suffices to compute the DFTs of the sequence $\{x_0, x_2, \cdots, x_{N-2}\}$ and $\{x_1, x_3, \cdots, x_{N-1}\}$. As long as the DFTs of the sequences $\{x_0, x_2, \cdots, x_{N-2}\}$ and $\{x_1, x_3, \cdots, x_{N-1}\}$ are known, it requires another $\frac{N}{2}$ multiplications to compute the DFT of $\{x_0, x_1, \cdots, x_{N-1}\}$.

Now we compute the total multiplications it requires to compute the DFT of the sequence $\{x_k\}_{k=0}^{2^\gamma-1}$ using the procedure above. Suppose that to compute the DFT of $\{x_k\}_{k=0}^{2^\gamma-1}$ requires $f(\gamma)$ multiplications. Then

$$f(\gamma) = 2f(\gamma-1) + 2^{\gamma-1}.$$

It is easy to see that it requires no multiplication to compute the DFT of $\{x_0, x_1\}$ since it is simply $\{x_0+x_1, x_0-x_1\}$; thus $f(1) = 0$. Solving the iteration relation above, we obtain that $f(\gamma) = 2^{\gamma-1}(\gamma-1)$ which implies the total multiplications requires to compute the DFT of $\{x_k\}_{k=0}^{N-1}$, where $N = 2^\gamma$, is $\dfrac{N}{2}(\log_2 N - 1)$.

Example 2.37. To compute the DFT of $\{x_0, x_1, \cdots, x_7\}$, we first compute the DFT of $\{x_0, x_2, x_4, x_6\}$ and $\{x_1, x_3, x_5, x_7\}$, and it requires another 4 multiplications (to compute the multiplication of ω^j and the j-th term of the DFT of $\{x_1, x_3, x_5, x_7\}$ for $0 \leqslant j \leqslant 3$). Nevertheless, instead of computing the DFT of $\{x_0, x_2, x_4, x_6\}$ and $\{x_1, x_3, x_5, x_7\}$ directly using matrix multiplication $\boldsymbol{X} = M\boldsymbol{x}$, we again divide the sequence of length 4 into further shorter sequence $\{x_0, x_4\}$, $\{x_2, x_6\}$, $\{x_1, x_5\}$ and $\{x_3, x_7\}$. Once the DFT of those sequence of length 2 are computed, it requires another $2 \times 2 = 4$ multiplications to compute the DFT of $\{x_0, x_2, x_4, x_6\}$ and $\{x_1, x_3, x_5, x_7\}$. By Example 2.35, it does not require any multiplications to compute the DFT of sequences of length 2; thus the total multiplications required to compute the DFT of $\{x_0, x_1, \cdots, x_7\}$ is $4 + 4 = 8$.

2.7 Fourier Series for Functions of Two Variables

In this section we briefly introduce the Fourier series of complex-valued functions defined on $\Omega \equiv [-L_1, L_1] \times [-L_2, L_2]$. Let

$$L^2(\Omega) = \Big\{ f : \Omega \to \mathbb{C} \,\Big|\, \int_\Omega \big|f(x_1, x_2)\big|^2 \, d(x_2, x_2) < \infty \Big\} \Big/ \sim$$

equipped with the inner product

$$\langle f, g\rangle \equiv \frac{1}{\nu(\Omega)} \int_\Omega f(x_1, x_2)\overline{g(x_1, x_2)}\, d(x_1, x_2)\,,$$

where $\nu(\Omega)$ denotes the area of Ω and $\sim$ again denotes the equivalence relation defined by $f \sim g$ if and only if $f - g = 0$ except on a set of measure zero. Let $\mathbf{e}_{k\ell}(\boldsymbol{x}) = e^{i\pi(\frac{k}{L}, \frac{\ell}{M})\cdot \boldsymbol{x}}$, here $\boldsymbol{x} = (x_1, x_2)$. Then $\{\mathbf{e}_{k\ell}\}_{k,\ell\in\mathbb{Z}}$ is a complete orthonormal set in $L^2(\Omega)$; that is, for each $f \in L^2(\Omega)$, by defining the partial sum

$$s_{n,m}(f, \boldsymbol{x}) = \sum_{k=-n}^{n} \sum_{\ell=-m}^{m} \langle f, \mathbf{e}_{k\ell}\rangle \mathbf{e}_{k\ell}(\boldsymbol{x})$$

we have

$$\lim_{n,m\to\infty} \big\| f - s_{n,m}(f, \cdot)\big\|_{L^2(\Omega)} = 0\,,$$

where $\|\cdot\|_{L^2(\Omega)}$ is the norm induced by the inner product $\langle\cdot,\cdot\rangle$. The limit of $s_{n,m}(f,\cdot)$, as $n,m\to\infty$, in the inner product space $\big(L^2(\Omega),\langle\cdot,\cdot\rangle\big)$ is denoted by

$$s(f,\cdot)=\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}\langle f,\mathbf{e}_{k\ell}\rangle\,\mathbf{e}_{k\ell}$$

and is called the Fourier series of f. One should expect that

Given a collection of data $\{x_{mn}\}_{0\leqslant n\leqslant M-1,0\leqslant n\leqslant N-1}$, the discrete Fourier transform (or simply DFT) of $\{x_{mn}\}_{0\leqslant n\leqslant M-1,0\leqslant n\leqslant N-1}$ is a double sequence $\{X_{k\ell}\}_{k,\ell\in\mathbb{Z}}$ defined by

$$X_{k\ell}=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}x_{mn}\,\omega_{M}^{mk}\omega_{N}^{n\ell}\,,$$

where $\omega_M=e^{-\frac{2\pi i}{M}}$ and $\omega_N=e^{-\frac{2\pi i}{N}}$. The double sequence $\{X_{k\ell}\}_{k,\ell\in\mathbb{Z}}$ is doubly periodic satisfying $X_{k+M,\ell+N}$ for all $k,\ell\in\mathbb{Z}$; thus we usually only focus on the terms $\{X_{k\ell}\}_{0\leqslant k\leqslant M-1,0\leqslant\ell\leqslant N-1}$. The discrete inverse Fourier transform of a double sequence $\{X_{k\ell}\}_{0\leqslant k\leqslant M-1,0\leqslant\ell\leqslant N-1}$ is a double sequence $\{x_{mn}\}_{m,n\in\mathbb{Z}}$ defined by

$$x_{mn}=\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{\ell=0}^{N-1}X_{k\ell}\,\overline{\omega_M}^{\,mk}\,\overline{\omega_N}^{\,n\ell}\,,$$

where $\overline{\omega_M}$ and $\overline{\omega_N}$ are complex conjugate of ω_M and ω_N defined above.

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