Fourier Analysis

富氏分析

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Contents

Chapter 1

Review on Analysis/Advanced Calculus

1.1 Pointwise and Uniform Convergence（逐點收斂與均勻收斂）

Definition 1.1. Let $I \subseteq \mathbb{R}$ be an interval, and $f_k, f : I \to \mathbb{R}$ be functions for $k = 1, 2, \cdots$. The sequence of functions $\{f_k\}_{k=1}^{\infty}$ is said to ***converge pointwise*** if $\big\{f_k(a)\big\}_{k=1}^{\infty}$ converges for all $a \in I$. In other words, $\{f_k\}_{k=1}^{\infty}$ converges pointwise if there exists a function $f : I \to \mathbb{R}$ such that

$$\lim_{k\to\infty} \big|f_k(x) - f(x)\big| = 0 \qquad \forall\, x \in I.$$

In this case, $\{f_k\}_{k=1}^{\infty}$ is said to converge pointwise to f and is denoted by $f_k \to f$ p.w..

The sequence of functions $\{f_k\}_{k=1}^{\infty}$ is said to ***converge uniformly*** on I if there exists $f : I \to \mathbb{R}$ such that

$$\lim_{k\to\infty} \sup_{x\in I} \big|f_k(x) - f(x)\big| = 0\,.$$

In this case, $\{f_k\}_{k=1}^{\infty}$ is said to converge uniformly to f on I. In other words, $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on I if for every $\varepsilon > 0$, $\exists\, N > 0$ such that

$$\big|f_k(x) - f(x)\big| < \varepsilon \qquad \forall\, k \geqslant N \text{ and } x \in I\,.$$

Proposition 1.2. *Let $I \subseteq \mathbb{R}$ be an interval, and $f_k, f : I \to \mathbb{R}$ be functions for $k = 1, 2, \cdots$. If $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on I, then $\{f_k\}_{k=1}^{\infty}$ converges pointwise to I.*

Proposition 1.3 (Cauchy criterion for uniform convergence)**.** *Let $I \subseteq \mathbb{R}$ be an interval, and $f_k : I \to \mathbb{R}$ be a sequence of functions. Then $\{f_k\}_{k=1}^{\infty}$ converges uniformly (to some* *function f) on I if and only if for every $\varepsilon > 0$, $\exists\, N > 0$ such that*

$$\big|f_k(x) - f_\ell(x)\big| < \varepsilon \qquad \forall\, k, \ell \geqslant N \text{ and } x \in I\,.$$

Theorem 1.4. *Let $I \subseteq \mathbb{R}$ be an interval, and $f_k : I \to \mathbb{R}$ be a sequence of continuous functions converging to $f : I \to \mathbb{R}$ uniformly on I. Then f is continuous on I; that is,*

$$\lim_{x\to a} f(x) = \lim_{x\to a}\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty}\lim_{x\to a} f_k(x) = f(a)\,.$$

Theorem 1.5. *Let $I \subseteq \mathbb{R}$ be a finite interval, $f_k : I \to \mathbb{R}$ be a sequence of differentiable functions, and $g : I \to \mathbb{R}$ be a function. Suppose that $\big\{f_k(a)\big\}_{k=1}^\infty$ converges for some $a \in I$, and $\{f_k'\}_{k=1}^\infty$ converges uniformly to g on I. Then*

1. *$\{f_k\}_{k=1}^\infty$ converges uniformly to some function f on I.*

2. *The limit function f is differentiable on I, and $f'(x) = g(x)$ for all $x \in I$; that is,*

$$\lim_{k\to\infty} f_k'(x) = \lim_{k\to\infty} \frac{d}{dx} f_k(x) = \frac{d}{dx}\lim_{k\to\infty} f_k(x) = f'(x)\,.$$

Theorem 1.6. *Let $f_k : [a,b] \to \mathbb{R}$ be a sequence of Riemann integrable functions which converges uniformly to f on $[a,b]$. Then f is Riemann integrable, and*

$$\lim_{k\to\infty}\int_a^b f_k(x)dx = \int_a^b \lim_{k\to\infty} f_k(x)dx = \int_a^b f(x)dx\,. \tag{1.1}$$

Definition 1.7. Let $I \subseteq \mathbb{R}$ be an interval. The collection of bounded continuous real-valued functions defined on I is denoted by $\mathscr{C}_b(I;\mathbb{R})$. The sup-norm of $\mathscr{C}_b(I;\mathbb{R})$, denoted by $\|\cdot\|_\infty$, is defined by

$$\|f\|_\infty = \sup_{x\in I}\big|f(x)\big| \qquad \forall\, f \in \mathscr{C}_b(I;\mathbb{R})\,.$$

If $I = [a,b] \subseteq \mathbb{R}$ is a closed interval (so that every continuous function on I is bounded), we simply use $\mathscr{C}([a,b];\mathbb{R})$ to denote $\mathscr{C}_b([a,b];\mathbb{R})$.

Having the definition above, we can rephrase Proposition 1.3 and Theorem 1.4 as follows.

Theorem 1.8. *Let $I \subseteq \mathbb{R}$ be an interval. Then $\big(\mathscr{C}_b(I;\mathbb{R}), \|\cdot\|_\infty\big)$ is a complete norm space; that is, every Cauchy sequence in $\big(\mathscr{C}_b(I;\mathbb{R}), \|\cdot\|_\infty\big)$ converges uniformly (to some limit) in $\mathscr{C}_b(I;\mathbb{R})$.*

1.2 Series of Functions and The Weierstrass M-Test

Definition 1.9. Let $I \subseteq \mathbb{R}$ be an interval, and $g_k : I \to \mathbb{R}$ (or $\mathbb{C}$) be a sequence of functions. We say that the series $\sum\limits_{k=1}^{\infty} g_k$ converges pointwise if the sequence of partial sum $\{s_n\}_{n=1}^{\infty}$ given by

$$s_n = \sum_{k=1}^{n} g_k$$

converges pointwise. We say that $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly on I if $\{s_n\}_{n=1}^{\infty}$ converges uniformly on I.

The following two corollaries are direct consequences of Proposition 1.3 and Theorem 1.4.

Corollary 1.10. *Let $I \subseteq \mathbb{R}$ be an interval, and $g_k : I \to \mathbb{R}$ be functions. Then $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly on I if and only if*

$$\forall\, \varepsilon > 0, \exists\, N > 0 \ni \Big| \sum_{k=m+1}^{n} g_k(x) \Big| < \varepsilon \qquad \forall\, n > m \geqslant N \text{ and } x \in A\,.$$

Corollary 1.11. *Let $I \subseteq \mathbb{R}$ be an interval, and $g_k, g : I \to \mathbb{R}$ be functions. If $g_k : I \to \mathbb{R}$ are continuous and $\sum\limits_{k=1}^{\infty} g_k(x)$ converges to g uniformly on I, then g is continuous.*

Theorem 1.12 (Weierstrass M-test)**.** *Let $I \subseteq \mathbb{R}$ be an interval, and $g_k : I \to \mathbb{R}$ be a sequence of functions. Suppose that $\exists\, M_k > 0$ such that $\sup\limits_{x \in I} |g_k(x)| \leqslant M_k$ for all $k \in \mathbb{N}$ and $\sum\limits_{k=1}^{\infty} M_k$ converges. Then $\sum\limits_{k=1}^{\infty} g_k$ and $\sum\limits_{k=1}^{\infty} |g_k|$ both converge uniformly on I.*

Corollary 1.13. *Let $I \subseteq \mathbb{R}$ be an interval, and $g_k : I \to \mathbb{R}$ be a sequence of continuous functions. Suppose that $\exists\, M_k > 0$ such that $\sup\limits_{x \in I} |g_k(x)| \leqslant M_k$ for all $k \in \mathbb{N}$ and $\sum\limits_{k=1}^{\infty} M_k$ converges. Then $\sum\limits_{k=1}^{\infty} g_k$ is continuous on I.*

The following two theorems are direct consequences of Theorem 1.5 and 1.6.

Theorem 1.14. *Let $g_k : [a, b] \to \mathbb{R}$ be a sequence of Riemann integrable functions. If $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly on $[a, b]$, then*

$$\int_a^b \sum_{k=1}^{\infty} g_k(x)dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)dx\,.$$

Theorem 1.15. *Let $g_k : (a,b) \to \mathbb{R}$ be a sequence of differentiable functions. Suppose that $\sum\limits_{k=1}^{\infty} g_k(c)$ converges for some $c \in (a,b)$, and $\sum\limits_{k=1}^{\infty} g_k'$ converges uniformly on (a,b). Then*

$$\sum_{k=1}^{\infty} g_k'(x) = \frac{d}{dx}\sum_{k=1}^{\infty} g_k(x)\,.$$

1.3 Analytic Functions and the Stone-Weierstrass Theorem

Theorem 1.16. *Let $f : (a,b) \to \mathbb{R}$ be an infinitely differentiable functions; that is, $f^{(k)}(x)$ exists for all $k \in \mathbb{N}$ and $x \in (a,b)$. Let $c \in (a,b)$ and suppose that for some $0 < h < \infty$, $\big|f^{(k)}(x)\big| \leqslant M$ for all $x \in (c-h, c+h) \subseteq (a,b)$. Then*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \qquad \forall\, x \in (c-h, c+h)\,.$$

Moreover, the convergence is uniform.

Proof. First, we claim that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^n \int_c^x \frac{(y-x)^n}{n!} f^{(n+1)}(y)dy \qquad \forall\, x \in (a,b)\,. \tag{1.2}$$

By the fundamental theorem or Calculus it is clear that (1.2) holds for $n = 0$. Suppose that (1.2) holds for $n = m$. Then

$$\begin{aligned} f(x) &= \sum_{k=0}^{m} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^m \Big[\frac{(y-x)^{m+1}}{(m+1)!} f^{(m+1)}(y)\Big|_{y=c}^{y=x} - \int_c^x \frac{(y-x)^{m+1}}{(m+1)!} f^{(m+2)}(y)dy\Big] \\ &= \sum_{k=0}^{m+1} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^{m+1} \int_c^x \frac{(y-x)^{m+1}}{(m+1)!} f^{(m+2)}(y)dy \end{aligned}$$

which implies that (1.2) also holds for $n = m+1$. By induction (1.2) holds for all $n \in \mathbb{N}$.

Letting $s_n(x) = \sum\limits_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$, then if $x \in (c-h, c+h)$,

$$\big|s_n(x) - f(x)\big| \leqslant \Big|\int_c^x \frac{h^n}{n!} M dy\Big| \leqslant \frac{h^{n+1}}{n!} M\,.$$

Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} \frac{h^{n+1}}{n!} M = 0$, $\exists\, N > 0$ such that $\Big|\frac{h^{n+1}}{n!}\Big| M < \varepsilon$ if $n \geqslant N$. As a consequence, if $n \geqslant N$, $\big|s_n(x) - f(x)\big| < \varepsilon$ whenever $n \geqslant N$. ▫

Definition 1.17. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be ***real analytic*** at $a \in \text{int}(I)$ if $f(x) = \sum\limits_{k=0}^{\infty} \dfrac{f^{(k)}(a)}{k!}(x-a)^k$ in a neighborhood of a.

Theorem 1.18 (Weierstrass)**.** *For every given $f \in \mathscr{C}([0,1];\mathbb{R})$ there exists a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ such that $\{p_n\}_{n=1}^{\infty}$ converges uniformly to f on $[0,1]$. In other words, the collection of all polynomials is dense in the space $\big(\mathscr{C}([0,1];\mathbb{R}), \|\cdot\|_\infty\big)$.*

Proof. Let $r_k(x) = C^n_k x^k (1-x)^{n-k}$. By looking at the partial derivatives with respect to x of the identity $(x+y)^n = \sum\limits_{k=0}^{n} C^n_k x^k y^{n-k}$, we find that

1. $\sum\limits_{k=0}^{n} r_k(x) = 1$; 2. $\sum\limits_{k=0}^{n} k r_k(x) = nx$; 3. $\sum\limits_{k=0}^{n} k(k-1) r_k(x) = n(n-1)x^2$.

As a consequence,

$$\sum_{k=0}^{n} (k-nx)^2 r_k(x) = \sum_{k=0}^{n} \big[k(k-1) + (1-2nx)k + n^2x^2\big] r_k(x) = nx(1-x)\,.$$

Let $\varepsilon > 0$ be given. Since $f : [0,1] \to \mathbb{R}$ is continuous on a compact set $[0,1]$, f is uniformly continuous on $[0,1]$; thus

$$\exists\, \delta > 0 \ni \big|f(x) - f(y)\big| < \frac{\varepsilon}{2} \qquad \text{if } |x-y| < \delta,\ x, y \in [0,1]\,.$$

Consider the ***Bernstein polynomial*** $p_n(x) = \sum\limits_{k=0}^{\infty} f\big(\frac{k}{n}\big) r_k(x)$. Note that

$$\begin{aligned}
\big|f(x) - p_n(x)\big| &= \Big|\sum_{k=0}^{n} \Big(f(x) - f\big(\frac{k}{n}\big)\Big) r_k(x)\Big| \leqslant \sum_{k=0}^{n} \Big|f(x) - f\big(\frac{k}{n}\big)\Big| r_k(x) \\
&\leqslant \Big(\sum_{|k-nx|<\delta n} + \sum_{|k-nx|\geqslant \delta n}\Big) \Big|f(x) - f\big(\frac{k}{n}\big)\Big| r_k(x) \\
&< \frac{\varepsilon}{2} + 2\|f\|_\infty \sum_{|k-nx|\geqslant \delta n} \frac{(k-nx)^2}{(k-nx)^2} r_k(x) \\
&\leqslant \frac{\varepsilon}{2} + \frac{2\|f\|_\infty}{n^2\delta^2} \sum_{k=0}^{n} (k-nx)^2 r_k(x) \leqslant \frac{\varepsilon}{2} + \frac{2\|f\|_\infty}{n\delta^2} x(1-x) \leqslant \frac{\varepsilon}{2} + \frac{\|f\|_\infty}{2n\delta^2}\,.
\end{aligned}$$

Choose N large enough such that $\dfrac{\|f\|_\infty}{2N\delta^2} < \dfrac{\varepsilon}{2}$. Then for all $n \geqslant N$,

$$\|f - p_n\|_\infty = \sup_{x\in[0,1]} \big|f(x) - p_n(x)\big| < \varepsilon\,. \qquad \square$$

Remark 1.19. A polynomial of the form $p_n(x) = \sum\limits_{k=0}^{n} \beta_k r_k(x)$ is called a ***Bernstein polynomial of degree*** n, and the coefficients β_k are called Bernstein coefficients.

Corollary 1.20. *The collection of polynomials on* $[a,b]$ *is dense in* $\big(\mathscr{C}([a,b];\mathbb{R}), \|\cdot\|_\infty\big)$*; that is, for every* $f \in \mathscr{C}([a,b];\mathbb{R})$ *there exists a sequence of polynomials* $\{p_k\}_{k=1}^\infty$ *such that* $\{p_k\}_{k=1}^\infty$ *converges uniformly to* f *on* $[a,b]$*.*

Proof. We note that $g \in \mathscr{C}([a,b];\mathbb{R})$ if and only if $f(x) = g\big(x(b-a)+a\big) \in \mathscr{C}([0,1];\mathbb{R})$; thus

$$\big|f(x) - p(x)\big| < \varepsilon \ \forall\, x \in [0,1] \Leftrightarrow \Big|g(x) - p\big(\frac{x-a}{b-a}\big)\Big| < \varepsilon \ \forall\, x \in [a,b]. \qquad \square$$

1.4 Trigonometric Polynomials and the Space of 2π-Periodic Continuous Functions

In this section, we focus on the approximations of a special class of functions, the collection of 2π-periodic continuous function. Let $\mathscr{C}(\mathbb{T})$ denote the collection of 2π-periodic continuous function (defined on $\mathbb{R}$):

$$\mathscr{C}(\mathbb{T}) = \big\{ f \in \mathscr{C}(\mathbb{R};\mathbb{R}) \,\big|\, f(x+2\pi) = f(x) \quad \forall\, x \in \mathbb{R} \big\}\,.$$

The sup-norm on $\mathscr{C}(\mathbb{T})$ is denoted by $\|\cdot\|_{L^\infty(\mathbb{T})}$; that is, $\|f\|_{L^\infty(\mathbb{T})} \equiv \sup\limits_{x\in\mathbb{R}} |f(x)|$ if $f \in \mathscr{C}(\mathbb{T})$.

We note that $\mathscr{C}(\mathbb{T})$ can be treated as the collection of continuous functions defined on the unit circle $\mathbb{S}^1$ in the sense that every $f \in \mathscr{C}(\mathbb{T})$ corresponds to a unique $F \in \mathscr{C}(\mathbb{S}^1;\mathbb{R})$ such that

$$f(x) = F(\cos x, \sin x) \qquad \forall\, x \in \mathbb{R} \tag{1.3}$$

and vice versa.

Definition 1.21. A family of functions $\big\{\varphi_n \in \mathscr{C}(\mathbb{T}) \,\big|\, n \in \mathbb{N}\big\}$ is said to be ***an approximation of the identity*** if

(1) $\varphi_n(x) \geqslant 0$;

(2) $\displaystyle\int_{\mathbb{T}} \varphi_n(x)\,dx = 1$ for every $n \in \mathbb{N}$, here we identify $\mathbb{T}$ with the interval $[-\pi,\pi]$;

(3) $\displaystyle\lim_{n\to\infty} \int_{\delta \leqslant |x| \leqslant \pi} \varphi_n(x)\,dx = 0$ for every $\delta > 0$.

Definition 1.22 (Convolutions on $\mathbb{T}$)**.** The convolution of two (continuous) function $f, g : \mathbb{T} \to \mathbb{R}$ is the function $f \star g : \mathbb{T} \to \mathbb{C}$ defined by the integral

$$(f \star g)(x) = \int_{\mathbb{T}} f(x-y)g(y)\,dy\,.$$

Theorem 1.23. *If $\{\varphi_n\}_{n=1}^{\infty}$ is an approximation of the identity and $f \in \mathscr{C}(\mathbb{T})$, then $\varphi_n \star f$ converges uniformly to f as $n \to \infty$.*

Proof. Without loss of generality, we may assume that $f \not\equiv 0$. By the definition of the convolution,

$$\big|(\varphi_n \star f)(x) - f(x)\big| = \int_{\mathbb{T}} \varphi_n(x-y)f(y)\,dy - f(x) = \int_{\mathbb{T}} \varphi_n(x-y)\big(f(x)-f(y)\big)dy\,,$$

where we use (2) of Definition 1.21 to obtain the last equality. Now given $\varepsilon > 0$. Since $f \in \mathscr{C}(\mathbb{T})$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \dfrac{\varepsilon}{2}$ whenever $|x-y| < \delta$. Therefore,

$$\begin{aligned}
&|(\varphi_n \star f)(x) - f(x)| \\
&\quad\leqslant \int_{|x-y|<\delta} \varphi_n(x-y)\big|f(x)-f(y)\big|dy + \int_{\delta\leqslant|x-y|} \varphi_n(x-y)\big|f(x)-f(y)\big|dy \\
&\quad\leqslant \frac{\varepsilon}{2}\int_{\mathbb{T}} \varphi_n(x-y)\,dy + 2\max_{\mathbb{T}}|f| \int_{\delta\leqslant|z|\leqslant\pi} \varphi_n(z)\,dz\,.
\end{aligned}$$

By (3) of Definition 1.21, there exists $N > 0$ such that if $n \geqslant N$,

$$\int_{\delta\leqslant|z|\leqslant\pi} \varphi_n(z)\,dx < \frac{\varepsilon}{4\max_{\mathbb{T}}|f|}\,.$$

Therefore, for $n \geqslant N$, $\big|(\varphi_n \star f)(x) - f(x)\big| < \varepsilon$ for all $x \in \mathbb{T}$. ◻

Definition 1.24. A trigonometric polynomial $p(x)$ of degree n is a finite sum of the form

$$p(x) = \frac{c_0}{2} + \sum_{k=1}^{n}(c_k \cos kx + s_k \sin kx) \qquad x \in \mathbb{R}\,.$$

The collection of all trigonometric polynomial of degree n is denoted by $\mathscr{P}_n(\mathbb{T})$, and the collection of all trigonometric polynomials is denoted by $\mathscr{P}(\mathbb{T})$; that is, $\mathscr{P}(\mathbb{T}) = \bigcup\limits_{n=0}^{\infty} \mathscr{P}_n(\mathbb{T})$.

On account of the Euler identity $e^{i\theta} = \cos\theta + i\sin\theta$, a trigonometric polynomial of degree n can also be written as

$$p(x) = \sum_{k=-n}^{n} a_k e^{ikx} \quad \text{with} \quad a_k = \frac{c_{|k|} - is_{|k|}}{2}\,,$$

where s_0 is taken to be 0. Therefore, every trigonometric polynomial of degree n is associated to a unique function of the form $\sum\limits_{k=-n}^{n} a_k e^{ikx}$ and vice versa.

Having defined trigonometric polynomials, we can show that every 2π-periodic function can be approximated by a sequence of trigonometric polynomials in the sense of uniform convergence.

Theorem 1.25. *The collection of all trigonometric polynomials $\mathscr{P}(\mathbb{T})$ is dense in $\mathscr{C}(\mathbb{T})$ with respect to the sup-norm; that is, for every $f \in \mathscr{C}(\mathbb{T})$ there exists a sequence $\{p_n\}_{n=1}^{\infty} \subseteq \mathscr{P}(\mathbb{T})$ such that $\{p_n\}_{n=1}^{\infty}$ converges uniformly to f on $\mathbb{T}$.*

Proof. Let $\varphi_n(x) = c_n(1+\cos x)^n$, where c_n is chosen so that $\int_{\mathbb{T}} \varphi_n(x)\,dx = 1$. By the residue theorem,

$$\int_{\mathbb{T}} (1+\cos x)^n dx = \oint_{\mathbb{S}^1} \Big(1 + \frac{z^2+1}{2z}\Big)^n \frac{dz}{iz} = \frac{1}{i2^n}\oint_{\mathbb{S}^1} \frac{(z+1)^{2n}}{z^{n+1}} dz = \frac{\pi}{2^{n-1}}\binom{2n}{n};$$

thus $c_n = \dfrac{2^{n-1}}{\pi}\dfrac{(n!)^2}{(2n)!}$.

Now $\{\varphi_n\}_{n=1}^{\infty}$ is clearly non-negative and satisfies (2) of Definition 1.21 for all $n \in \mathbb{N}$. Let $\delta > 0$ be given.

$$\int_{\delta \leqslant |x| \leqslant \pi} \varphi_n(x)\,dx \leqslant \int_{\delta \leqslant |x| \leqslant \pi} c_n(1+\cos\delta)^n dx \leqslant 2^{2n}\Big(\frac{1+\cos\delta}{2}\Big)^n \frac{(n!)^2}{(2n)!}\,.$$

By Stirling's formula $\lim\limits_{n\to\infty} \dfrac{n!}{\sqrt{2\pi n}n^n e^{-n}} = 1$,

$$\begin{aligned}\lim_{n\to\infty} \int_{\delta \leqslant |x| \leqslant \pi} \varphi_n(x)\,dx &\leqslant \lim_{n\to\infty} 2^{2n}\Big(\frac{1+\cos\delta}{2}\Big)^n \frac{\big(\sqrt{2\pi n}n^n e^{-n}\big)^2}{\sqrt{2\pi(2n)}(2n)^{2n}e^{-2n}} \\ &= \lim_{n\to\infty} \sqrt{\pi n}\Big(\frac{1+\cos\delta}{2}\Big)^n = 0\,.\end{aligned}$$

So $\{\varphi_n\}_{n=1}^{\infty}$ is an approximation of the identity. By Theorem 1.23, $\varphi_k \star f$ converges uniformly to f if $f \in \mathscr{C}(\mathbb{T})$, while $\varphi_n \star f$ is a trigonometric function. □

Remark 1.26. Theorem 1.25 can also be proved using the abstract version of the Stone-Weierstrass Theorem and the identification (1.3). See Theorem 7.32 in "Principles of Mathematics Analysis" by W. Rudin or Theorem 5.8.2 in Elementary Classical Analysis by J. Marsden and M. Hoffman for the Stone-Weierstrass Theorem.

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