# Calculus 微積分

**Ching-hsiao Arthur Cheng 鄭經斅**

# Chapter 8

# Integration Techniques and Improper Integrals

## 8.1 Basic Integration Rules

We recall the following formula:

1. Let $f, g$ be functions and $k$ be a constant. Then

$$\int kf(x)\,dx = k\int f(x)\,dx\,, \qquad \int (f+g)(x)\,dx = \int f(x)\,dx + \int g(x)\,dx\,.$$

2. Let $r$ be a real number. Then

$$\int x^r\,dx = \begin{cases} \dfrac{1}{r+1}x^{r+1} + C & \text{if } r \neq -1\,, \\ \ln x + C & \text{if } r = -1\,. \end{cases}$$

3. If $a > 0$, then $\displaystyle\int a^x\,dx = \frac{1}{\ln a}a^x + C$. In particular, $\displaystyle\int e^x\,dx = e^x + C$.

4. If $a \neq 0$, $\displaystyle\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C$, $\displaystyle\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$,
$\displaystyle\int \tan(ax)\,dx = \frac{1}{a}\ln|\sec(ax)| + C$, $\displaystyle\int \cot(ax)\,dx = \frac{1}{a}\ln|\sin(ax)| + C$,
$\displaystyle\int \sec(ax)\,dx = \frac{1}{a}\ln|\sec(ax)+\tan(ax)|+C$, $\displaystyle\int \csc x\,dx = -\frac{1}{a}\ln|\csc(ax)+\cot(ax)|+C$.

5. $\displaystyle\int \sec^2 x\,dx = \tan x + C$, $\displaystyle\int \sec x \tan x\,dx = \sec x + C$.

6. If $a > 0$, then

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C\,, \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\arctan\frac{\sqrt{x^2-a^2}}{a} + C\,.$$

**Example 8.1.** Find the indefinite integrals $\int \frac{4}{x^2+9}\,dx$, $\int \frac{4x}{x^2+9}\,dx$ and $\int \frac{4x^2}{x^2+9}\,dx$.

From the formula above, it is easy to see that

$$\int \frac{4}{x^2+9}\,dx = \frac{4}{3}\arctan\frac{x}{3} + C\,.$$

Noting that $\frac{4x}{x^2+9} = 2\frac{\frac{d}{dx}(x^2+9)}{x^2+9}$, using the formula $\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f'(x)}$, we find that

$$\int \frac{4x}{x^2+9}\,dx = 2\ln|x^2+9| + C = 2\ln(x^2+9) + C\,.$$

Finally, noting that $\frac{4x^2}{x^2+9} = \frac{4(x^2+9)-36}{x^2+9} = 4 - \frac{36}{x^2+9}$, by the formula above we find that

$$\int \frac{4x^2}{x^2+9}\,dx = 4x - 12\arctan\frac{x}{3} + C\,.$$

**Example 8.2.** Find the indefinite integrals $\int \frac{3}{\sqrt{4-x^2}}\,dx$, $\int \frac{3x}{\sqrt{4-x^2}}\,dx$ and $\int \frac{3x^2}{\sqrt{4-x^2}}\,dx$.

From the formula above,

$$\int \frac{3}{\sqrt{4-x^2}}\,dx = 3\arcsin\frac{x}{2} + C\,.$$

For the second integral, we let $4-x^2 = u$. Then $-2xdx = du$; thus

$$\int \frac{3x}{\sqrt{4-x^2}}\,dx = -\frac{3}{2}\int u^{-\frac{1}{2}}\,du = -\frac{3}{2}\frac{1}{1-\frac{1}{2}}u^{\frac{1}{2}} + C = -3(4-x^2)^{\frac{1}{2}} + C\,.$$

For the third integral, first we observe that

$$\int \frac{3x^2}{\sqrt{4-x^2}}\,dx = \int \frac{3(x^2-4)}{\sqrt{4-x^2}}\,dx + \int \frac{12}{\sqrt{4-x^2}}\,dx = -3\int \sqrt{4-x^2}\,dx + 12\arcsin\frac{x}{2}\,.$$

Let $x = 2\sin u$. Then $dx = 2\cos u\,du$; thus

$$\begin{aligned}\int \sqrt{4-x^2}\,dx &= \int \sqrt{4(1-\sin^2 u)}\cdot 2\cos u\,du = \int 4\cos^2 u\,du = \int \big[2+2\cos(2u)\big]\,du \\ &= 2u + \sin(2u) + C = 2u + 2\sin u\cos u + C \\ &= 2\arcsin\frac{x}{2} + x\sqrt{1-\frac{x^2}{4}} + C = 2\arcsin\frac{x}{2} + \frac{x\sqrt{4-x^2}}{2} + C\,.\end{aligned}$$

Therefore,

$$\int \frac{3x^2}{\sqrt{4-x^2}}\,dx = 6\arcsin\frac{x}{2} - \frac{3}{2}x\sqrt{4-x^2} + C\,.$$

**Remark 8.3.** One should add

$$\int \frac{x}{\sqrt{a^2-x^2}}\,dx = -\sqrt{a^2-x^2} + C \qquad \text{and} \qquad \int \frac{x}{\sqrt{a^2+x^2}}\,dx = \sqrt{a^2+x^2} + C$$

into the table of integrations.

**Example 8.4.** Find the indefinite integral $\displaystyle\int \frac{dx}{1+e^x}$.

Let $u = 1+e^x$. Then $du = e^x dx$ which implies that $dx = \dfrac{du}{u-1}$. Therefore,

$$\begin{aligned}\int \frac{dx}{1+e^x} &= \int \frac{du}{u(u-1)} = \int \Big(\frac{1}{u-1} - \frac{1}{u}\Big)\,du = \ln|u-1| - \ln|u| + C \\ &= x - \ln(1+e^x) + C\,.\end{aligned}$$

Another way of finding the integral is by observing that

$$\frac{1}{1+e^x} = \frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x} = 1 - \frac{\frac{d}{dx}(1+e^x)}{1+e^x}\,;$$

thus using the formula $\dfrac{d}{dx}\ln|f(x)| = \dfrac{f'(x)}{f(x)}$, we find that

$$\int \frac{dx}{1+e^x} = x - \ln(1+e^x) + C\,.$$

## 8.2 Integration by Parts - 分部積分

Suppose that $u, v$ are two differentiable functions of $x$. Then the product rule implies that

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}\,.$$

Therefore, if $\dfrac{du}{dx}v$ and $u\dfrac{dv}{dx}$ are Riemann integrable (on the interval of interests),

$$\int \frac{du}{dx}v\,dx + \int u\frac{dv}{dx}\,dx = (uv)(x) + C\,.$$

Symbolically, we write $\dfrac{du}{dx}v\,dx$ ad $v\,du$ and $u\dfrac{dv}{dx}\,dx$ as $u\,dv$, the formula above implies that

$$\int udv = uv - \int vdu\,.$$

**Theorem 8.5: Integration by Parts**

If $u$ and $v$ are functions of $x$ and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du\,.$$

**Example 8.6.** Find the indefinite integral $\displaystyle\int \ln x\,dx$. Recall that we have shown that

$$\int \ln x\,dx = x\ln x - x + C$$

using the Riemann sum. Let $u = \ln x$ and $v = x$ (so that $dv = dx$). Then integration by parts shows that

$$\int \ln x\,dx = x\ln x - \int x\,d(\ln x) = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C\,.$$

**Example 8.7.** Find the indefinite integral $\displaystyle\int x\cos x\,dx$. Recall that we have shown that

$$\int x\cos x\,dx = x\sin x + \cos x + C$$

using the Riemann sum. Let $u = x$ and $v = \sin x$ (so that $dv = \cos x\,dx$). Then integration by parts shows that

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C\,.$$

**Principles of applying integration by parts**: Choose $u$ and $v$ such that $v\,du$ has simpler form than $u\,dv$, and this is usually achieved by

1. finding $u$ such that the derivative of $u$ is a function simpler than $u$, or
2. finding $v$ such that the derivative of $v$ is more complicate than $v$.

**Example 8.8.** Find the indefinite integral $\displaystyle\int xe^x\,dx$.

Let $u = x$ and $v = e^x$ (so that $dv = e^x dx$). Then integration by parts shows that

$$\int xe^x\,dx = xe^x - \int e^x\,dx = (x-1)e^x + C\,.$$

**Example 8.9.** Find the indefinite integral $\int x^r \ln x\, dx$, where $r$ is a real number.

Suppose first that $r \neq -1$. Let $u = \ln x$ and $v = \frac{1}{r+1} x^{r+1}$. Then integration by parts shows that

$$\begin{aligned}\int x^r \ln x\, dx &= \frac{1}{r+1} x^{r+1} \ln x - \int \frac{1}{r+1} x^{r+1} \cdot \frac{1}{x}\, dx = \frac{1}{r+1} x^{r+1} \ln x - \frac{1}{r+1} \int x^r\, dx \\ &= \frac{1}{r+1} x^{r+1} \ln x - \frac{1}{(r+1)^2} x^{r+1} + C\,.\end{aligned}$$

Now if $r = -1$. Let $u = v = \ln x$. Then integration by parts implies that

$$\int x^{-1} \ln x\, dx = (\ln x)^2 - \int \ln x \cdot \frac{1}{x}\, dx = (\ln x)^2 - \int x^{-1} \ln x\, dx$$

which implies that

$$\int x^{-1} \ln x\, dx = \frac{1}{2} (\ln x)^2 + C\,.$$

Therefore,

$$\int x^r \ln x\, dx = \begin{cases} \dfrac{1}{r+1} x^{r+1} \ln x - \dfrac{1}{(r+1)^2} x^{r+1} + C & \text{if } r \neq -1\,, \\ \dfrac{1}{2} (\ln x)^2 + C & \text{if } r = -1\,. \end{cases}$$

**Example 8.10.** Find the indefinite integral $\int x^2 \cos x\, dx$.

Let $u = x^2$ and $v = \sin x$ (so that $dv = \cos x\, dx$). Then integration by parts shows that

$$\int x^2 \cos x\, dx = x^2 \sin x - \int \sin x \cdot 2x\, dx = x^2 \sin x - 2 \int x \sin x\, dx\,.$$

Integrating by parts again, we find that

$$\int x \sin x\, dx = -x \cos x + \int \cos x\, dx = -x \cos x + \sin x + C\,;$$

thus we obtain the

$$\int x^2 \cos x\, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C\,.$$

**Example 8.11.** Find the indefinite integrals $\int e^{ax} \sin(bx)\, dx$ and $\int e^{ax} \cos(bx)\, dx$, where $a, b$ are non-zero constants.

Let $u = \sin(bx)$ (or $u = \cos(ax)$) and $v = a^{-1}e^{ax}$ (so that $dv = e^{ax}\,dx$). Then

$$\int e^{ax}\sin(bx)\,dx = \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}\int e^{ax}\cos(bx)\,dx\,,$$
$$\int e^{ax}\cos(bx)\,dx = \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}\int e^{ax}\cos(bx)\,dx\,.$$

The two identities above further imply that

$$\begin{aligned}\int e^{ax}\sin(bx)\,dx &= \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}\int e^{ax}\cos(bx)\,dx \\ &= \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}\Big[\frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}\int e^{ax}\sin(bx)\,dx\Big] \\ &= \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a^2}e^{ax}\cos(bx) - \frac{b^2}{a^2}\int e^{ax}\sin(bx)\,dx\,;\end{aligned}$$

thus

$$\int e^{ax}\sin(bx)\,dx = \frac{1}{a^2+b^2}\big[ae^{ax}\sin(bx) - be^{ax}\cos(bx)\big] + C\,. \tag{8.2.1}$$

Similarly,

$$\int e^{ax}\cos(bx)\,dx = \frac{1}{a^2+b^2}\big[ae^{ax}\cos(bx) + be^{ax}\sin(bx)\big] + C\,. \tag{8.2.2}$$

**Remark 8.12.** By the Euler identity (5.9.1), $\int e^{ax}\sin(bx)\,dx$ and $\int e^{ax}\cos(bx)\,dx$ are the real and imaginary part of the integral $\int e^{ax}e^{ibx}\,dx$. By the fact that $e^{ax}e^{ibx} = e^{(a+ib)x}$ and pretending that $\int e^{cx}\,dx = \frac{1}{c}e^{cx} + C$ for complex number $c$, we find that

$$\begin{aligned}\int e^{ax}e^{ibx}\,dx &= \frac{1}{a+ib}e^{(a+ib)x} + C = \frac{1}{a+ib}e^{ax}\big[\cos(bx) + i\sin(bx)\big] + C \\ &= \frac{a-ib}{a^2+b^2}e^{ax}\big[\cos(bx) + i\sin(bx)\big] + C \\ &= \frac{e^{ax}}{a^2+b^2}\big[a\cos(bx) + b\sin(bx) + i\big(a\sin(bx) - b\cos(bx)\big)\big] + C\,;\end{aligned}$$

thus we conclude (8.2.1) and (8.2.2).

**Example 8.13.** Find the indefinite $\int x^n e^{ax}\,dx$, $\int x^n\sin(ax)\,dx$ and $\int x^n\cos(ax)\,dx$, where $a > 0$ is a constant.

Let $u = x^n$ and $v = a^{-1}e^{ax}$ (so that $dv = e^{ax}\,dx$), $v = -a^{-1}\cos(ax)$ (so that $dv = \sin(ax)$) and $v = a^{-1}\sin(ax)$ (so that $dv = \cos(ax)$) in these three cases. Then

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \int \frac{1}{a}e^{ax}\cdot nx^{n-1}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx\,.$$

Moreover,

$$\int x^n \sin(ax)\,dx = -\frac{1}{a}x^n\cos(ax) + \frac{n}{a}\int x^{n-1}\cos(ax)\,dx\,,$$
$$\int x^n \cos(ax)\,dx = \frac{1}{a}x^n\sin(ax) - \frac{n}{a}\int x^{n-1}\sin(ax)\,dx\,.$$

The two identities above further imply that the following recurrence relations

$$\int x^n \sin(ax)\,dx = -\frac{1}{a}x^n\cos(ax) + \frac{n}{a^2}x^{n-1}\sin(ax) - \frac{n(n-1)}{a^2}\int x^{n-2}\sin(ax)\,dx\,,$$
$$\int x^n \cos(ax)\,dx = \frac{1}{a}x^n\sin(ax) + \frac{n}{a^2}x^{n-1}\cos(ax) - \frac{n(n-1)}{a^2}\int x^{n-2}\cos(ax)\,dx\,.$$

**Example 8.14.** Using integration by parts, we have

$$\begin{aligned}\int \cos^n x\,dx &= \int \cos^{n-1}x\,d(\sin x) = \sin x\cos^{n-1}x - \int \sin x\,d(\cos^{n-1}x)\\ &= \sin x\cos^{n-1}x + (n-1)\int \sin^2 x\cos^{n-2}x\,dx\\ &= \sin x\cos^{n-1}x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x\,dx\\ &= \sin x\cos^{n-1}x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int\cos^n x\,dx\,;\end{aligned}$$

thus rearranging terms, we conclude that

$$\int \cos^n x\,dx = \frac{\sin x\cos^{n-1}x}{n} + \frac{n-1}{n}\int\cos^{n-2}x\,dx\,. \tag{8.2.3}$$

Similarly,

$$\int \sin^n x\,dx = -\frac{\cos x\sin^{n-1}x}{n} + \frac{n-1}{n}\int\sin^{n-2}x\,dx\,. \tag{8.2.4}$$

**Theorem 8.15: Wallis's Formulas**

If $n$ is a non-negative integer, then

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1}x\,dx = \int_0^{\frac{\pi}{2}}\cos^{2n+1}x\,dx = \frac{(2^n n!)^2}{(2n+1)!}$$

and

$$\int_0^{\frac{\pi}{2}} \sin^{2n}x\,dx = \int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx = \frac{(2n)!}{(2^n n!)^2}\cdot\frac{\pi}{2}\,.$$

*Proof.* Note that (8.2.3) implies that

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{\sin x \cos^{n-1} x}{n}\Big|_{x=0}^{x=\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\,dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\,dx\,.$$

Therefore,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^{2n+1} x\,dx &= \frac{2n}{2n+1}\int_0^{\frac{\pi}{2}} \cos^{2n-1} x\,dx = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\int_0^{\frac{\pi}{2}} \cos^{2n-3} x\,dx = \cdots \\ &= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdots\frac{2}{3}\int_0^{\frac{\pi}{2}} \cos x\,dx = \frac{2}{3}\cdot\frac{4}{5}\cdots\frac{2n}{2n+1} \\ &= \frac{2^2 4^2\cdots(2n)^2}{(2n+1)!} = \frac{(2^n n!)^2}{(2n+1)!}\end{aligned}$$

and

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx &= \frac{2n-1}{2n}\int_0^{\frac{\pi}{2}} \cos^{2n-2} x\,dx = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\int_0^{\frac{\pi}{2}} \cos^{2n-4} x\,dx = \cdots \\ &= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^0 x\,dx = \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\cdot\frac{\pi}{2} \\ &= \frac{(2n)!}{2^2 4^2\cdots(2n)^2}\cdot\frac{\pi}{2} = \frac{(2n)!}{(2^n n!)^2}\cdot\frac{\pi}{2}\,.\end{aligned}$$

The substitution $x = \frac{\pi}{2} - u$ shows that

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx \qquad \text{for all non-negative integers } n\,,$$

so we conclude the theorem. □

**Theorem 8.16: Stirling's Formula**

$$\lim_{n\to\infty} \frac{n!}{n^{n+0.5}e^{-n}} = \sqrt{2\pi}.$$

*Proof.* Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$. Then Wallis's formula shows that

$$I_{2n} = \frac{(2n)!}{(2^n n!)^2}\cdot\frac{\pi}{2} \qquad \text{and} \qquad I_{2n+1} = \frac{(2^n n!)^2}{(2n+1)!}\,.$$

Moreover, since $\sin^{2n+2} x \leqslant \sin^{2n+1} x \leqslant \sin^{2n} x$ on $\big[0, \frac{\pi}{2}\big]$, we also have $I_{2n+2} \leqslant I_{2n+1} \leqslant I_{2n}$ for all $n \geqslant 0$. Therefore,

$$\frac{I_{2n+2}}{I_{2n}} \leqslant \frac{I_{2n+1}}{I_{2n}} \leqslant 1 \qquad \forall\, n \geqslant 0\,.$$

Note that

$$\frac{I_{2n+2}}{I_{2n}} = \frac{I_{2(n+1)}}{I_{2n}} = \frac{\dfrac{(2(n+1))!}{2^{2(n+1)}((n+1)!)^2}}{\dfrac{(2n)!}{2^{2n}(n!)^2}} = \frac{2n+1}{2(n+1)}\,;$$

thus $\lim\limits_{n\to\infty} \frac{I_{2n+2}}{I_{2n}} = 1$. As a consequence, the Squeeze Theorem implies that $\lim\limits_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$.

Let $s_n = \frac{n!}{n^{n+0.5}e^{-n}}$. Then the fact that the function $y = \big(1 + \frac{1}{x}\big)^{x+0.5}$ is decreasing on $(0,\infty)$ (left as an exercise) and (5.4.3) show that $s_n \geqslant s_{n+1} \geqslant 0$ for all $n \in \mathbb{N}$. Therefore, the completeness of the real number (see Theorem 9.20) implies that $\lim\limits_{n\to\infty} s_n = s$ exists. Moreover,

$$\begin{aligned}\frac{I_{2n+1}}{I_{2n}} &= \frac{\dfrac{2^{2n}(n!)^2}{(2n+1)!}}{\dfrac{(2n)!}{2^{2n}(n!)^2}\dfrac{\pi}{2}} = \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!}\cdot\frac{2}{\pi}\\ &= \frac{2^{4n}(s_n n^{n+0.5}e^{-n})^4}{s_{2n}(2n)^{2n+0.5}e^{-2n}s_{2n+1}(2n+1)^{2n+1.5}e^{-2n-1}}\cdot\frac{2}{\pi}\\ &= \frac{s_n^4}{s_{2n}s_{2n+1}}\frac{e}{2\pi}\big(1+\frac{1}{2n}\big)^{-2n-1.5}\,;\end{aligned}$$

thus (5.4.3) implies that

$$1 = \lim_{n\to\infty}\frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty}\frac{s_n^4}{s_{2n}s_{2n+1}}\cdot\frac{1}{2\pi} = \frac{s^2}{2\pi}\,.$$

The theorem is then concluded by the fact that $s \geqslant 0$. □

## 8.3 Trigonometric Integrals

In this section, we are concerned with the integrals

$$\int \sin^m x\cos^n x\,dx \qquad \text{and} \qquad \int \sec^m x\tan^n x\,dx\,,$$

where $m, n$ are non-negative integers.

### 8.3.1 The integral of $\sin^m x\cos^n x$

- **The case when one of $m$ and $n$ is odd**

Suppose $m = 2k+1$ or $n = 2\ell+1$. Write

$$\int \sin^{2k+1} x\cos^n x\,dx = \int \cos^n x(1-\cos^2 x)^k\sin x\,dx = -\int \cos^n x(1-\cos^x x)^k\,d(\cos x)$$

and

$$\int \sin^m x \cos^{2\ell+1} x\, dx = \int \sin^m x (1-\sin^2 x)^\ell \cos x\, dx = \int \sin^m x (1-\sin^2 x)^\ell\, d(\sin x)$$

so that the integral can be obtained by integrating polynomials.

**Example 8.17.** Find the indefinite integral $\displaystyle\int \sin^3 x \cos^4 x\, dx$.

Let $u = \cos x$. Then $du = -\sin x\, dx$; thus

$$\begin{aligned}\int \sin^3 x \cos^4 x\, dx &= \int (1-\cos^2 x)\cos^4 x \sin x\, dx = -\int (1-u^2)u^4\, du \\ &= -\frac{1}{5}u^5 + \frac{1}{7}u^7 + C = -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C\,.\end{aligned}$$

We also write

$$\begin{aligned}\int \sin^3 x \cos^4 x\, dx &= \int (1-\cos^2 x)\cos^4 x \sin x\, dx = -\int (1-\cos^2 x)\cos^4 x\, d(\cos x) \\ &= -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C\,.\end{aligned}$$

- **The case when $m$ and $n$ are both even**

First we talk about how to integrate $\cos^n x$. We have shown the recurrence relation (8.2.3) in previous section, and there are other ways of finding the integral of $\cos^n x$ without using integration by parts. The case when $n = 2\ell+1$ can be dealt with the previous case, so we focus on the case $n = 2\ell$. Make use of the half angle formula

$$\cos^2 x = \frac{1+\cos(2x)}{2}\,,$$

we can write

$$\int \cos^{2\ell} x\, dx = \int \Big(\frac{1+\cos(2x)}{2}\Big)^\ell dx = \sum_{i=0}^{\ell} \frac{C^\ell_i}{2^\ell} \int \cos^i (2x)\, dx \overset{(u=2x)}{=} \sum_{i=0}^{\ell} \frac{C^\ell_i}{2^{\ell+1}} \int \cos^i u\, du$$

which is a linear combination of integrals of the form $\displaystyle\int \cos^i u\, du$, while the power $i$ is at most half of $n$. Keeping on applying the half angle formula for even powers of cosine, eventually integral $\displaystyle\int \cos^i u\, du$ will be reduced to sum of integrals of cosine with odd powers (which can be evaluated by the previous case).

**Example 8.18.** Find the indefinite integral $\int \cos^6 x\, dx$.

By the half angle formula,

$$\begin{aligned}
\int \cos^6 x\, dx &= \int \Big(\frac{1+\cos(2x)}{2}\Big)^3 dx = \frac{1}{8}\int \big[1 + 3\cos(2x) + 3\cos^2(2x) + \cos^3(2x)\big]\, dx \\
&= \frac{1}{8}\int \Big[1 + 3\cos(2x) + \frac{3}{2}\big(1+\cos(4x)\big) + \big(1-\sin^2(2x)\big)\cos(2x)\Big]\, dx \\
&= \frac{1}{8}\int \Big(\frac{5}{2} + 4\cos(2x) + \frac{3}{2}\cos(4x)\Big)\, dx - \frac{1}{16}\int \sin^2(2x)\, d\big(\sin(2x)\big) \\
&= \frac{1}{8}\Big[\frac{5x}{2} + 2\sin(2x) + \frac{3}{8}\sin(4x)\Big] - \frac{1}{48}\sin^3(2x) + C\,.
\end{aligned}$$

Now suppose that $m = 2k$ and $n = 2\ell$. Make use of the half angle formulas

$$\sin^2 x = \frac{1-\cos(2x)}{2} \qquad \text{and} \qquad \cos^2 x = \frac{1+\cos(2x)}{2}$$

to write

$$\int \sin^{2k} x \cos^{2\ell} x\, dx = \frac{1}{2^{k+\ell}}\int \big(1-\cos(2x)\big)^k \big(1+\cos(2x)\big)^\ell\, dx\,.$$

Expanding parenthesis, the integral above becomes the linear combination of integrals of the form $\int \cos^i(2x)\, dx$.

**Example 8.19.** Find the indefinite integral $\int \sin^2 x \cos^4 x\, dx$.

By the half angle formula,

$$\begin{aligned}
\int \sin^2 x \cos^4 x\, dx &= \int \frac{1-\cos(2x)}{2}\Big(\frac{1+\cos(2x)}{2}\Big)^2 dx \\
&= \frac{1}{8}\int \big[1-\cos(2x)\big]\big[1 + 2\cos(2x) + \cos^2(2x)\big]\, dx \\
&= \frac{1}{8}\int \big[1 + \cos(2x) - \cos^2(2x) - \cos^3(2x)\big]\, dx \\
&= \frac{1}{8}\int \Big(\frac{1-\cos(4x)}{2} + \sin^2(2x)\cos(2x)\Big]\, dx \\
&= \frac{1}{8}\Big[\frac{x}{2} - \frac{\sin(4x)}{8}\Big] + \frac{1}{48}\sin^3(2x) + C\,.
\end{aligned}$$

### 8.3.2 The integral of $\sec^m x \tan^n x$

**Rule of thumb**: make use of $\frac{d}{dx}\tan x = \sec^2 x$ and $\frac{d}{dx}\sec x = \sec x \tan x$.

- **The case when $m$ is even**

Suppose that $m = 0$ and $n \geqslant 2$. Then we obtain the recurrence relation

$$\begin{aligned}\int \tan^n x\, dx &= \int \tan^{n-2} x \tan^2 x\, dx = \int \tan^{n-2}(\sec^2 x - 1)\, dx \\ &= \int \tan^{n-2}\, d(\tan x) - \int \tan^{n-2} x\, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\, dx\,.\end{aligned}$$

Suppose that $m = 2k$ is even and positive. Using the substitution $u = \tan x$, we have

$$\int \sec^{2k} x \tan^n x\, dx = \int \sec^{2(k-1)} x \tan^n x \sec^2 x\, dx = \int (1+\tan^2 x)^{k-1} \tan^n x\, d(\tan x)$$

which can be obtained by integrating polynomials.

- **The case when $n$ is odd**

Suppose that $n = 2\ell + 1$ is odd and $m \geqslant 1$. Then

$$\int \sec^m x \tan^{2\ell+1} x\, dx = \int \sec^{m-1} x \tan^{2\ell} \sec x \tan x\, dx = \int \sec^{m-1} x (\sec^2 x - 1)^\ell\, d(\sec x)$$

which can be obtained by integrating polynomials.

- **The case when $m$ is odd and $n$ is even**

Suppose that $m = 2k+1$ and $n = 2\ell$. Then

$$\int \sec^{2k+1} x \tan^{2\ell} x\, dx = \int \sec^{2k+1} x (\sec^2 x - 1)^\ell\, dx\,;$$

thus it suffices to know how to compute $\displaystyle\int \sec^m x\, dx$.

Using integration by parts,

$$\begin{aligned}\int \sec^m x\, dx = \int \sec^{m-2} x\, d(\tan x) &= \tan x \sec^{m-2} x - \int \tan x\, d(\sec^{m-2} x) \\ &= \tan x \sec^{m-2} x - (m-2)\int \tan^2 x \sec^{m-2} x\, dx \\ &= \tan x \sec^{m-2} x - (m-2)\int (\sec^2 x - 1)\sec^{m-2} x\, dx\end{aligned}$$

thus rearranging terms we obtain the recurrence relation

$$\int \sec^m x\, dx = \frac{m-2}{m-1}\tan x \sec^{m-2} x + \frac{m-2}{m-1}\int \sec^{m-2} x\, dx\,.$$

**Example 8.20.** Find the indefinite integral $\displaystyle\int \sec^4(3x)\tan^3(3x)\,dx$.

By the discussion above,

$$\begin{aligned}\int \sec^4(3x)\tan^3(3x)\,dx &= \frac{1}{3}\int \sec^2(3x)\tan^3(3x)d(\tan(3x))\\ &= \frac{1}{3}\int \big[\tan^2(3x)+1\big]\tan^3(3x)d(\tan(3x))\\ &= \frac{1}{3}\Big[\frac{1}{6}\tan^6(3x)+\frac{1}{4}\tan^4(3x)\Big]+C\,.\end{aligned}$$

**Example 8.21.** Find the indefinite integral $\displaystyle\int \sqrt{a^2+x^2}\,dx$.

By the substitution of variable $x = a\tan\theta$ (so that $dx = a\sec^2\theta d\theta$), we find that

$$\begin{aligned}\int \sqrt{a^2+x^2}\,dx = \int a^2\sec^3\theta\,d\theta &= a^2\Big(\frac{1}{2}\tan\theta\sec\theta+\frac{1}{2}\int\sec\theta\,d\theta\Big)\\ &= \frac{a^2}{2}\big(\tan\theta\sec\theta+\ln|\sec\theta+\tan\theta|\big)+C\\ &= \frac{a^2}{2}\Big(\frac{x}{a}\cdot\frac{\sqrt{a^2+x^2}}{a}+\ln\Big|\frac{x+\sqrt{a^2+x^2}}{a}\Big|\Big)+C\\ &= \frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\ln\big(x+\sqrt{a^2+x^2}\big)+C\,. \qquad (8.3.1)\end{aligned}$$

### 8.3.3 Other techniques of integration involving trigonometric functions

- Integration by substitution (for integrand with special structures):

**Example 8.22.** Find the indefinite integral $\displaystyle\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$.

Let $u = \sin x$. Then $du = \cos x\,dx$; thus

$$\begin{aligned}\int\frac{\cos^3 x}{\sqrt{\sin x}}\,dx = \int\frac{(1-u^2)}{\sqrt{u}}\,du &= \int\big(u^{-\frac{1}{2}}-u^{\frac{3}{2}}\big)\,du\\ &= \frac{1}{1-\frac{1}{2}}u^{\frac{1}{2}}-\frac{1}{1+\frac{3}{2}}u^{\frac{5}{2}}+C = 2\sqrt{\sin x}-\frac{5}{2}\sin^{\frac{5}{2}}x+C\,.\end{aligned}$$

**Example 8.23.** Find the indefinite integral $\displaystyle\int \frac{\sec x}{\tan^2 x}\,dx$.

Rewrite the integrand into a fraction of sine and cosine, we find that

$$\int\frac{\sec x}{\tan^2 x}\,dx = \int\frac{\cos x}{\sin^2 x}\,dx = \int\frac{1}{\sin^2 x}\,d(\sin x) = -\sin^{-1}x+C = -\csc x+C\,.$$

**Example 8.24.** Find the indefinite integral $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx$.

Let $u = \sec x$. Then $du = \sec x \tan x\,dx$; thus

$$\begin{aligned}\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx &= \int \frac{(\sec^2 x - 1)\sec x \tan x}{\sec^{\frac{3}{2}} x}\,dx = \int \frac{u^2-1}{u^{\frac{3}{2}}}\,du = \int \big(u^{\frac{1}{2}} - u^{-\frac{3}{2}}\big)\,du \\ &= \frac{2}{3}u^{\frac{3}{2}} + 2u^{-\frac{1}{2}} + C = \frac{2}{3}\sec^{\frac{3}{2}} x + 2\cos^{\frac{1}{2}} x + C\,.\end{aligned}$$

• When the angular variable are different, making use of the sum and difference formula:

**Example 8.25.** Find the indefinite integral $\displaystyle\int \sin^3(5x)\cos(4x)\,dx$.

Using the sum and difference formula

$$\sin\theta\cos\phi = \frac{1}{2}\big[\sin(\theta+\phi) + \sin(\theta-\phi)\big]\,, \quad \sin\theta\sin\phi = \frac{1}{2}\big[\cos(\theta-\phi) - \sin(\theta+\phi)\big]\,,$$

we find that

$$\begin{aligned}\int \sin^3(5x)\cos(4x)\,dx &= \frac{1}{2}\int \sin^2(5x)\big[\sin(9x) + \sin x\big]\,dx \\ &= \frac{1}{4}\int \sin(5x)\big[\cos(4x) - \cos(14x) + \cos(4x) - \cos(6x)\big]\,dx \\ &= \frac{1}{8}\int \big[2\sin(9x) + 2\sin x - \sin(19x) + \sin(9x) - \sin(11x) + \sin x\big]\,dx \\ &= \frac{1}{8}\Big[-\frac{1}{3}\cos(9x) - 3\cos x + \frac{1}{19}\cos(19x) + \frac{1}{11}\cos(11x)\Big] + C\,.\end{aligned}$$

## 8.4 Partial Fractions - 部份分式

In this section, we are concerned with the integrals $\displaystyle\int \frac{N(x)}{D(x)}\,dx$, where $N, D$ are polynomial functions.

Write $N(x) = D(x)Q(x) + R(x)$, where $Q, R$ are polynomials such that the degree of $R$ is less than the degree of $D$ (such an $R$ is called a remainder). Then $\dfrac{N(x)}{D(x)} = R(x) + \dfrac{R(x)}{D(x)}$. Since it is easy to find the indefinite integral of $R$, it suffices to consider the case when the degree of the numerator is less than the degree of the denominator.

W.L.O.G., we assume that $N$ and $D$ no common factor, $\deg(N) < \deg(D)$, and the leading coefficient of $D$ is 1. Since $D$ is a polynomial with real coefficients,

$$D(x) = \Big(\prod_{j=1}^{m}(x+q_j)^{r_j}\Big)\Big(\prod_{j=1}^{n}(x^2+b_jx+c_j)^{d_j}\Big)\,,$$

where $r_j, d_j \in \mathbb{N}$, $q_j \neq q_k$ for all $j \neq k$, $b_j \neq b_k$ or $c_j \neq c_k$ for all $j \neq k$, and $b_j^2 - 4c_j < 0$ for all $1 \leqslant j \leqslant m$. In other words, $-q_j$ are zeros of $D$ with multiplicity $r_j$, and $\dfrac{-b_j \pm i\sqrt{4c_j - b_j^2}}{2}$ are zeros of $D$ with multiplicity $d_j$, here $i = \sqrt{-1}$. Therefore,

$$\frac{N(x)}{D(x)} = \sum_{j=1}^{m} \Big[ \sum_{\ell=1}^{r_j} \frac{A_{j\ell}}{(x+q_j)^\ell} \Big] + \sum_{j=1}^{n} \Big[ \sum_{\ell=1}^{r_j} \frac{B_{j\ell}x + C_{j\ell}}{(x^2+b_jx+c_j)^\ell} \Big] \tag{8.4.1}$$

for some constants $A_{j\ell}$, $B_{j\ell}$ and $C_{j\ell}$. Note that there are $\sum\limits_{j=1}^{m} r_j + 2\sum\limits_{j=1}^{n} d_j \equiv \deg(D)$ constants to be determined, and this can be done by the comparison of coefficients after the reduction of common denominator.

**Remark 8.26.** In this remark we "show" that a rational function $N/D$ with $\deg(N) < \deg(D)$ can always be written as the sum of partial fractions (8.4.1). Suppose that $\alpha$ is a zero of $D$ with multiplicity $k$ so that $D(x) = (x-\alpha)^k f(x)$, where $f(x)$ is a polynomial and $f(\alpha) \neq 0$. Since

$$\frac{N(x)}{D(x)} - \frac{N(\alpha)}{(x-\alpha)^k f(\alpha)} = \frac{N(x)f(\alpha) - f(x)N(\alpha)}{(x-\alpha)^k f(x) f(\alpha)} = \frac{g(x)}{(x-\alpha)^k f(x)},$$

where $g(x) = N(x) - f(x)\dfrac{N(\alpha)}{f(\alpha)}$. Since $g$ vanishes at $x = \alpha$, $g(x) = (x-\alpha)^m h(x)$ for some polynomial $h$ satisfying $h(\alpha) \neq 0$ (and we remark that here $m$ is not necessarily less than $k$). Therefore, with $\beta$ denoting the constant $\dfrac{N(\alpha)}{f(\alpha)}$, we obtain that

$$\frac{N(x)}{D(x)} - \frac{\beta}{(x-\alpha)^k} = \frac{(x-\alpha)^m h(x)}{(x-\alpha)^k f(x)} = \frac{h_1(x)}{(x-\alpha)^{k_1} f(x)},$$

where $k_1 \geqslant 0$ and $h_1(\alpha) \neq 0$ if $k_1 > 0$. We note that $f$ and $h_1$ are both polynomials satisfying $\deg h_1 < k_1 + \deg(f)$ and $f(\alpha) \neq 0$. Applying the process continuously, we obtain that

$$\frac{N(x)}{D(x)} = \sum_{i=1}^{k} \frac{C_k}{(x-\alpha)^k} + \frac{N_1(x)}{D_1(x)}$$

for some polynomials $N_1$, $D_1 (= f)$ with $\deg(N_1) < \deg(D_1) = \deg(D) - k$ and some sequence of constants $C_1, C_2, \cdots, C_k$, where $D_1(\alpha) \neq 0$. This explains the presence of the first sum on the right-hand side of (8.4.1) (and also shows how to find the coefficient $A_{jr_j}$ in the highest order term $\dfrac{1}{(x+q_j)^{r_j}}$ for each $j$).

**Example 8.27.** Write $\dfrac{5x^2+20x+6}{x^3+2x^2+x}$ in the form of (8.4.1).

Note that $x^3+2x^2+x = x(x^2+2x+1) = x(x+1)^2$; thus to write the rational function above in the form of (8.4.1), we must have

$$\frac{5x^2+20x+6}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

for some constant $A, B, C$.

Multiplying both sides of the equality above by $x(x+1)^2$, we find that

$$5x^2+20x+6 = A(x+1)^2 + Bx(x+1) + Cx = (A+B)x^2 + (2A+B+C)x + A\,;$$

thus $A, B, C$ satisfy

$$\begin{aligned} A+B &= 5 \\ 2A+B+C &= 20 \\ A &= 6\,. \end{aligned}$$

Therefore, $A = 6$, $B = -1$ and $C = 9$; thus

$$\frac{5x^2+20x+6}{x^3+2x^2+x} = \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\,.$$

**Example 8.28.** Write $\dfrac{1}{x^4+1}$ in the form of (8.4.1).

Note that $x^4+1 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$, so

$$\frac{1}{x^4+1} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}\,.$$

Multiplying both sides of the equality above by $x^4+1$, we have

$$\begin{aligned} 1 &= (Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1) \\ &= (A+C)x^3 + (-\sqrt{2}A+B+\sqrt{2}C+D)x^2 + (A-\sqrt{2}B+C+\sqrt{2}D)x + (B+D)\,; \end{aligned}$$

thus comparing the coefficients, we find that $A, B, C, D$ satisfy

$$\begin{aligned} A+C &= 0 \\ -\sqrt{2}A+B+\sqrt{2}C+D &= 0 \\ A-\sqrt{2}B+C+\sqrt{2}D &= 0 \\ B+D &= 1\,. \end{aligned}$$

Therefore, the first and the third equations imply that $A = -C$ and $B = D$; thus the second and the fourth equation shows that $A = -C = \dfrac{1}{2\sqrt{2}}$ and $B = D = \dfrac{1}{2}$. As a consequence,

$$\frac{1}{x^4+1} = \frac{1}{2\sqrt{2}}\Big[\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} + \frac{-x+\sqrt{2}}{x^2-\sqrt{2}x+1}\Big].$$

In order to find the integral of $\dfrac{N(x)}{D(x)}$, by writing $\dfrac{N(x)}{D(x)}$ in the form of (8.4.1), it suffices to find the integral of $\dfrac{B_{j\ell}x + C_{j\ell}}{(x^2+b_jx+c_j)^\ell}$ for

$$\int \frac{A_{j\ell}}{(x+q_j)^\ell}\,dx = \begin{cases} \dfrac{A_{j\ell}}{1-\ell}(x+q_j)^{1-\ell} + C & \text{if } \ell \neq 1\,, \\ A_{j\ell}\ln|x+q_j| + C & \text{if } \ell = 1\,. \end{cases}$$

Note that

$$\frac{B_{j\ell}x + C_{j\ell}}{(x^2+b_jx+c)^\ell} = \frac{B_{j\ell}}{2}\frac{2x+b_j}{(x^2+b_jx+c_j)^\ell} + \big(C_{j\ell} - \frac{b_jB_{j\ell}}{2}\big)\frac{1}{(x^2+b_jx+c_j)^\ell}$$

and

$$\int \frac{2x+b_j}{(x^2+b_jx+c_j)^\ell}\,dx = \begin{cases} \dfrac{1}{1-\ell}(x^2+b_jx+c_j)^{1-\ell} + C & \text{if } \ell \neq 1\,, \\ \ln(x^2+b_jx+c_j) + C & \text{if } \ell = 1\,; \end{cases}$$

thus to find the integral of $\dfrac{B_{j\ell}x + C_{j\ell}}{(x^2+b_jx+c_j)^\ell}$, it suffices to compute $\displaystyle\int \frac{1}{(x^2+b_jx+c_j)^\ell}\,dx$. Nevertheless, with $a$ denoting the number $\dfrac{4c_j - b_j^2}{4}$,

$$\int \frac{1}{(x^2+b_jx+c_j)^\ell}\,dx = \int \frac{1}{\big[\big(x-\frac{b_j}{2}\big)^2 + \frac{4c_j-b_j^2}{4}\big]^\ell}\,dx = \int \frac{1}{\big[\big(x-\frac{b_j}{2}\big)^2 + a^2\big]^\ell}\,d\big(x-\frac{b_j}{2}\big)$$

which can be computed through the substitution $x - \dfrac{b_j}{2} = a\tan u$:

$$\int \frac{1}{\big[\big(x-\frac{b_j}{2}\big)^2 + a^2\big]^\ell}\,d\big(x-\frac{b_j}{2}\big) = a^{1-2\ell}\int \cos^{2\ell-2}u\,du\,.$$

**Example 8.29.** Find the indefinite integral $\displaystyle\int \frac{dx}{x^4+1}$.

Using the conclusion from Example 8.28, we find that

$$\begin{aligned}
\int \frac{dx}{x^4+1} &= \frac{1}{2\sqrt{2}} \int \Big[\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} + \frac{-x+\sqrt{2}}{x^2-\sqrt{2}x+1}\Big]\,dx \\
&= \frac{1}{2\sqrt{2}} \int \Big[\frac{1}{2}\cdot\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{1}{2}\cdot\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\Big]\,dx \\
&\quad + \frac{1}{2\sqrt{2}} \int \Big[\frac{1}{2}\cdot\frac{\sqrt{2}}{x^2+\sqrt{2}x+1} + \frac{1}{2}\cdot\frac{\sqrt{2}}{x^2-\sqrt{2}x+1}\Big]\,dx \\
&= \frac{1}{4\sqrt{2}} \int \Big[\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} + \frac{\sqrt{2}}{\big(x+\frac{1}{\sqrt{2}}\big)^2+(\frac{1}{\sqrt{2}})^2} - \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1} + \frac{\sqrt{2}}{\big(x-\frac{1}{\sqrt{2}}\big)^2+(\frac{1}{\sqrt{2}})^2}\Big]\,dx \\
&= \frac{1}{4\sqrt{2}}\Big[\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + 2\arctan(\sqrt{2}x+1) + 2\arctan(\sqrt{2}x-1)\Big] + C\,.
\end{aligned}$$

**Example 8.30.** Find the indefinite integral $\displaystyle\int \frac{\sec x}{\tan^3 x}\,dx$.

Let $u = \sec x$. Then $du = \sec x \tan x$; thus

$$\int \frac{\sec x}{\tan^3 x}\,dx = \int \frac{\sec x \tan x}{\tan^4 x}\,dx = \int \frac{du}{(u^2-1)^2} = \int \frac{du}{(u+1)^2(u-1)^2}\,.$$

Write $\dfrac{1}{(u+1)^2(u-1)^2}$ is the form of (8.4.1):

$$\frac{1}{(u+1)^2(u-1)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2}\,,$$

where $A, B, C, D$ satisfy

$$A(u+1)(u-1)^2 + B(u-1)^2 + C(u-1)(u+1)^2 + D(u+1)^2 = 1\,.$$

Therefore, $A, B, C, D$ satisfy

$$\begin{aligned}
A + C &= 0 \\
-A + B + C + D &= 0 \\
-A - 2B - C + 2D &= 0 \\
A + B - C + D &= 1
\end{aligned}$$

which implies that $A = B = -C = D = \dfrac{1}{4}$. As a consequence,

$$\begin{aligned}
\int \frac{du}{(u+1)^2(u-1)^2} &= \frac{1}{4}\int \Big[\frac{1}{u+1} + \frac{1}{(u+1)^2} - \frac{1}{u-1} + \frac{1}{u-1)^2}\Big]\,du \\
&= \frac{1}{4}\Big[\ln|u+1| - \frac{1}{u+1} - \ln|u-1| - \frac{1}{u-1}\Big] + C \\
&= \frac{1}{4}\Big[\ln\Big|\frac{u+1}{u-1}\Big| - \frac{2u}{u^2-1}\Big] + C\,;
\end{aligned}$$

thus

$$\int \frac{\sec x}{\tan^3 x}\,dx = \frac{1}{4}\Big[\ln\Big|\frac{\sec x+1}{\sec x-1}\Big| - \frac{2\sec x}{\tan^2 x}\Big] + C\,.$$

**Example 8.31.** Find the indefinite integral $\displaystyle\int \sqrt{\tan x}\,dx$.

Let $u = \sqrt{\tan x}$. Then $u^2 = \tan x$ which implies that $2udu = \sec^2 x\,dx$ or $\dfrac{2udu}{1+u^4} = dx$. Therefore,

$$\begin{aligned}
\int \sqrt{\tan x}\,dx &= \int \frac{2u^2}{1+u^4}\,du = \frac{1}{\sqrt{2}}\int \Big[\frac{u}{u^2-\sqrt{2}u+1} - \frac{u}{u^2+\sqrt{2}u+1}\Big]\,du \\
&= \frac{1}{2\sqrt{2}}\ln\Big|\frac{u^2-\sqrt{2}u+1}{u^2+\sqrt{2}u+1}\Big| + \frac{1}{2}\int \Big[\frac{1}{u^2-\sqrt{2}u+1} + \frac{1}{u^2+\sqrt{2}u+1}\Big]\,du \\
&= \frac{1}{2\sqrt{2}}\ln\Big|\frac{u^2-\sqrt{2}u+1}{u^2+\sqrt{2}u+1}\Big| + \frac{\sqrt{2}}{2}\arctan(\sqrt{2}u-1) + \arctan(\sqrt{2}u+1) + C \\
&= \frac{1}{2\sqrt{2}}\ln\Big|\frac{\tan x-\sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1}\Big| + \frac{\sqrt{2}}{2}\arctan\frac{\sqrt{2\tan x}}{1-\tan x} + C\,,
\end{aligned}$$

where we have use the fact that

$$\arctan x + \arctan y = \arctan\frac{x+y}{1-xy} + C$$

to conclude the last equality.

**Example 8.32.** Find the indefinite integral $\displaystyle\int \frac{dx}{(1+x^n)^{\frac{1}{n}}}$, where $n$ is a positive integer.

Let $1 + x^{-n} = u^n$. Then $x^n = \dfrac{1}{u^n-1}$ and $-x^{-n-1}\,dx = u^{n-1}\,du$; thus

$$\int \frac{dx}{(1+x^n)^{\frac{1}{n}}} = \int \frac{dx}{x(1+x^{-n})^{\frac{1}{n}}} = \int \frac{-x^n}{(1+x^{-n})^{\frac{1}{n}}}(-x^{-n-1})\,dx = -\int \frac{u^{n-2}}{u^n-1}\,du$$

which is the indefinite integral of a rational function of $u$ and we know how to compute it. In particular, when $n = 4$,

$$\frac{u^2}{u^4-1} = \frac{u^2}{(u-1)(u+1)(u^2+1)} = \frac{1}{4}\cdot\frac{1}{u-1} - \frac{1}{4}\cdot\frac{1}{u+1} + \frac{1}{2}\cdot\frac{1}{u^2+1}\,;$$

thus

$$\int \frac{u^2}{u^4-1}\,du = \frac{1}{4}\ln|u-1| - \frac{1}{4}\ln|u+1| + \frac{1}{2}\arctan u + C$$

which further implies that

$$\int \frac{dx}{(1+x^4)^{\frac{1}{4}}} = \frac{1}{4}\ln\left|\frac{(1+x^{-4})^{\frac{1}{4}}-1}{(1+x^{-4})^{\frac{1}{4}}+1}\right| + \frac{1}{2}\arctan\big[(1+x^{-4})^{\frac{1}{4}}\big] + C\,.$$

- **The substitution of** $t = \tan\frac{x}{2}$

In Section 5.3 we have introduced the substitution $t = \tan\frac{x}{2}$ to find the anti-derivative of trigonometric functions. We recall that if $t = \tan\frac{x}{2}$, then

$$\sin x = \frac{2t}{1+t^2}\,, \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \text{and} \qquad dx = \frac{2dt}{1+t^2}\,.$$

Using this substitution, the anti-derivative of rational functions of sine and cosine can be computed via the integration of rational functions.

**Example 8.33.** Find the indefinite integral $\displaystyle\int \frac{\sec x}{\tan^3 x}\,dx$.

Rewriting the integrand, we have

$$\int \frac{\sec x}{\tan^3 x}\,dx = \int \frac{\cos^2 x}{\sin^3 x}\,dx\,.$$

Let $t = \tan\frac{x}{2}$. Then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$; thus

$$\begin{aligned}
\int \frac{\sec x}{\tan^3 x}\,dx &= \int \frac{\frac{(1-t^2)^2}{(1+t^2)^2}}{\frac{(2t)^3}{(1+t^2)^3}}\,\frac{2dt}{1+t^2} = \frac{1}{4}\int \frac{(1-t^2)^2}{t^3}\,dt = \frac{1}{4}\int \big(t^{-3} - 2t^{-1} + t\big)\,dt \\
&= \frac{1}{4}\Big[-\frac{1}{2}t^{-2} - 2\ln|t| + \frac{1}{2}t^2\Big] + C \\
&= \frac{1}{8}\Big[\tan^2\frac{x}{2} - \cot^2\frac{x}{2}\Big] - \frac{1}{2}\ln\Big|\tan\frac{x}{2}\Big| + C\,.
\end{aligned}$$

**Example 8.34.** Find the indefinite integral $\displaystyle\int \frac{1}{2+\sin x}\,dx$.

Let $t = \tan\frac{x}{2}$. Then $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$ and $dx = \dfrac{2dt}{1+t^2}$; thus

$$\begin{aligned}\int \frac{1}{2+\sin x}\,dx &= \int \frac{1}{2+\frac{2t}{1+t^2}}\frac{2dt}{1+t^2} = \int \frac{dt}{t^2+t+1} = \int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \frac{2}{\sqrt{3}}\arctan\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C = \frac{2}{\sqrt{3}}\arctan\frac{2t+1}{\sqrt{3}} + C \\ &= \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\tan\frac{x}{2}+\frac{1}{\sqrt{3}}\right) + C\,.\end{aligned}$$

## 8.5 Improper Integrals - 瑕積分

Recall that given a non-negative continuous function $f : [a, b] \to \mathbb{R}$, the area of the region enclosed by the graph of $f$, the $x$-axis and lines $x = a$, $x = b$ is given by $\displaystyle\int_a^b f(x)\,dx$.What happened when

1. the function under consideration is non-negative and continuous on the whole real line and we would like to know, for example, the area of the region enclosed by the graph of $f$ and the $x$-axis and is on the right-hand (or left-hand) side of the line $x = c$?

2. the function under consideration blows up at a point $c \in [a, b]$; that is, $\displaystyle\lim_{x\to c^{\pm}} f(x)$ diverges to $\infty$ or $-\infty$ (so that $f$ is not continuous at $c$ but everywhere else) and we would like to know the area of the region enclosed by the graph of $f$, the $x$-axis and lines $x = a$ and $x = b$?

Note that the definition of a definite integral $\displaystyle\int_a^b f(x)\,dx$ requires that the interval $[a, b]$ be finite and $f$ be bounded. Therefore, $\displaystyle\int_a^\infty f(x)\,dx$, $\displaystyle\int_{-\infty}^b f(x)\,dx$ and $\displaystyle\int_a^b f(x)\,dx$ when $f$ is unbounded are meaningless in the sense of Riemann integrals. How do we compute the area of those unbounded regions?

**Definition 8.34: Improper Integrals with Infinite Integration Limits**

1. If $f$ is Riemann integrable on the interval $[a, b]$ for all $a < b$, then
$$\int_a^\infty f(x)\,dx \equiv \lim_{b\to\infty} \int_a^b f(x)\,dx\,.$$
2. If $f$ is Riemann integrable on the interval $[a, b]$ for all $a < b$, then
$$\int_{-\infty}^b f(x)\,dx \equiv \lim_{a\to-\infty} \int_a^b f(x)\,dx\,.$$
3. If $f$ is Riemann integrable on the interval $[a, b]$ for all $a < b$, then
$$\int_{-\infty}^\infty f(x)\,dx \equiv \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx\,,$$
where $c$ is any real number.

In the first two cases, the improper integral converges when the limit exists. Otherwise, the improper integral diverges. If the limits, as $b$ approaches $\infty$ (or $a$ approaches $-\infty$), approaches $\infty$ or $-\infty$, then the improper integral diverges to $\infty$ or $-\infty$. In the third case, the improper integral on the left converges when both of the improper integrals on the right converges, and diverges when either of the improper integrals on the right diverges. The improper integral on the left diverges to $\infty$ (or $-\infty$) if it diverges and the improper integrals on the right is $\infty+\infty$, $\infty + C$ or $C + \infty$ (or $(-\infty) + (-\infty)$, $(-\infty) + C$ or $C + (-\infty)$).

**Example 8.35.** Evaluate $\displaystyle\int_0^\infty e^{-x}\,dx$ and $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,dx$.

Since an anti-derivative of the function $y = e^{-x}$ and $y = \dfrac{1}{x^2+1}$ is $y = -e^{-x}$ and $y = \arctan x$, the Fundamental Theorem of Calculus implies that

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}(-e^{-x})\Big|_{x=0}^{x=b} = \lim_{b\to\infty}(1-e^{-b}) = 1 - \lim_{b\to\infty} e^{-b} = 1$$

and
$$\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\arctan x\Big|_{x=0}^{x=b} = \lim_{b\to\infty}\arctan b = \frac{\pi}{2}\,.$$

**Example 8.36.** Evaluate $\displaystyle\int_1^\infty (1-x)e^{-x}\,dx$.

Let $u = 1 - x$ and $v = -e^{-x}$ (so that $dv = e^{-x}\,dx$). For any real number $b$, integration by parts implies that

$$\int_1^b (1-x)e^{-x}\,dx = \Big[(1-x)(-e^{-x})\Big]\Big|_{x=1}^{x=b} - \int_1^b (-e^{-x})(-dx) = -(1-b)e^{-b} - \int_1^b e^{-x}\,dx$$
$$= -(1-b)e^{-b} + e^{-x}\Big|_{x=1}^{x=b} = -(1-b)e^{-b} + e^{-b} - e^{-1} = be^{-b} - e^{-1}\,.$$

Therefore,

$$\int_1^\infty (1-x)e^{-x}\,dx = \lim_{b\to\infty}\int_1^b (1-x)e^{-x}\,dx = \lim_{b\to\infty}(be^{-b}-e^{-1}) = -e^{-1}\,.$$

**Example 8.37.** Evaluate $\displaystyle\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$.

To evaluate the integral above, we evaluate the two integrals

$$\int_0^\infty \frac{e^x}{1+e^{2x}}\,dx \quad\text{and}\quad \int_{-\infty}^0 \frac{e^x}{1+e^{2x}}\,dx\,.$$

By the substitution of variable $u = e^x$, we find that $du = e^x\,dx$; thus

$$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{1}{1+u^2}\,du = \arctan u + C = \arctan(e^x) + C\,.$$

Therefore,

$$\int_0^\infty \frac{e^x}{1+e^{2x}}\,dx = \lim_{b\to\infty}\int_0^b \frac{e^x}{1+e^{2x}}\,dx = \lim_{b\to\infty}\arctan(e^x)\Big|_{x=0}^{x=b}$$
$$= \lim_{b\to\infty}\Big[\arctan(e^b) - \arctan 1\Big] = \frac{\pi}{4}$$

and similarly,

$$\int_{-\infty}^0 \frac{e^x}{1+e^{2x}}\,dx = \lim_{a\to-\infty}\int_a^0 \frac{e^x}{1+e^{2x}}\,dx = \lim_{a\to-\infty}\arctan(e^x)\Big|_{x=a}^{x=0}$$
$$= \lim_{a\to-\infty}\Big[\arctan 1 - \arctan(e^a)\Big] = \frac{\pi}{4}\,.$$

The two integrals above implies that $\displaystyle\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx = \frac{\pi}{4}+\frac{\pi}{4} = \frac{\pi}{2}$.

**Example 8.38.** The improper integral $\displaystyle\int_0^\infty x\,dx$ diverges to $\infty$, and the improper integral $\displaystyle\int_{-\infty}^\infty (\sin x - 1)\,dx$ diverges to $-\infty$. The improper integral $\displaystyle\int_0^\infty \sin x\,dx$ diverges, but not diverges to $\infty$ or $-\infty$, and the improper integrals $\displaystyle\int_{-\infty}^\infty x\,dx$ diverges but not diverges to $\infty$ or $-\infty$.

**Example 8.39.** The improper integral $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ converges although it is not obvious what its value is. In fact,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}\,.$$

**Theorem 8.40**

1. If $f$ is Riemann integrable on the interval $[a,b]$ for all $a<b$, then
$$\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx \qquad \forall\, a<c\,,$$
provided that the improper integrals on both sides converge or diverge to $\infty$ (or $-\infty$).

2. If $f$ is Riemann integrable on the interval $[a,b]$ for all $a<b$, then
$$\int_{-\infty}^b f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^b f(x)\,dx \qquad \forall\, c<b\,,$$
provided that the improper integrals on both sides converge or diverge to $\infty$ (or $-\infty$).

3. If $f$ is Riemann integrable on the interval $[a,b]$ for all $a<b$ and $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ converges or diverges to $\infty$ (or $-\infty$), then
$$\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx \qquad \forall\, a,b\in\mathbb{R}\,.$$

*Proof.* We only prove 1 and 3, for the proof of 2 is similar to the proof of 1.

1. By the properties of the definite integrals, for $a<c$ we have
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,;$$
thus
$$\begin{aligned}\int_a^\infty f(x)\,dx &= \lim_{b\to\infty}\int_a^b f(x)\,dx = \lim_{b\to\infty}\Big[\int_a^c f(x)\,dx + \int_c^b f(x)\,dx\Big]\\ &= \int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx\,.\end{aligned}$$

3. If $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ converges or diverges to $\infty$ (or $-\infty$), then both improper integrals $\displaystyle\int_{c}^{\infty} f(x)\,dx$ and $\displaystyle\int_{-\infty}^{c} f(x)\,dx$ converge or diverge to $\infty$ (or $-\infty$). Therefore,

$$\begin{aligned}\int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx &= \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx \\ &= \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx\,. \end{aligned}$$

□

**Definition 8.41: Improper integrals with Infinite Discontinuities**

1. If $f$ is Riemann integrable on $[a, c]$ for all $a < c < b$, and $f$ has an infinite discontinuity at $b$; that is, $\displaystyle\lim_{x\to b^-} f(x) = \infty$ or $-\infty$, then
$$\int_a^b f(x)\,dx \equiv \lim_{c\to b^-} \int_a^c f(x)\,dx\,.$$
2. If $f$ is Riemann integrable on $[c, b]$ for all $a < c < b$, and $f$ has an infinite discontinuity at $a$; that is, $\displaystyle\lim_{x\to a^+} f(x) = \infty$ or $-\infty$, then
$$\int_a^b f(x)\,dx \equiv \lim_{c\to a^+} \int_c^b f(x)\,dx\,.$$
3. Suppose that $a < c < b$. If $f$ is Riemann integrable on $[a, c-\epsilon]$ and $[c+\epsilon, b]$ for all $0 < \epsilon \ll 1$, and $f$ has an infinite discontinuity at $c$; that is $\displaystyle\lim_{x\to c^+} f(x) = \infty$ or $-\infty$ and $\displaystyle\lim_{x\to c^-} f(x) = \infty$ or $-\infty$, then
$$\int_a^b f(x)\,dx \equiv \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,.$$

The convergence and divergence of the improper integrals with infinite discontinuities are similar to the statements in Definition 8.34.

**Example 8.42.** Evaluate $\displaystyle\int_0^1 x^{-\frac{1}{3}}\,dx$.

We observe that the integrand has an infinite discontinuity at 0. Therefore,

$$\int_0^1 x^{-\frac{1}{3}}\,dx = \lim_{a\to 0^+} \int_a^1 x^{-\frac{1}{3}}\,dx = \lim_{a\to 0^+} \frac{3}{2}x^{\frac{2}{3}}\Big|_{x=a}^{x=1} = \lim_{a\to 0^+} \frac{3}{2}(1 - a^{\frac{2}{3}}) = \frac{3}{2}\,.$$

**Example 8.43.** Evaluate $\displaystyle\int_0^2 x^{-3}\,dx$.

We observe that the integrand has an infinite discontinuity at 0. Therefore,

$$\int_0^2 x^{-3}\,dx = \lim_{a\to 0^+}\int_a^2 x^{-3}\,dx = \lim_{a\to 0^+}\Big(\frac{-x^{-2}}{2}\Big)\Big|_{x=a}^{x=2} = \lim_{a\to 0^+}\Big(-\frac{1}{8}+\frac{1}{2a^2}\Big) = \infty\,;$$

thus the improper integral $\displaystyle\int_0^2 x^{-3}\,dx$ diverges to $\infty$.

**Example 8.44.** Evaluate $\displaystyle\int_{-1}^2 x^{-3}\,dx$.

Since the integrand has an infinite discontinuity at 0,

$$\int_{-1}^2 x^{-3}\,dx = \int_{-1}^0 x^{-3}\,dx + \int_0^2 x^{-3}\,dx\,.$$

We have shown in previous example that the second integral on the right-hand side diverges to $\infty$. Similarly, the first integral on the right-hand side diverges to $-\infty$ since

$$\int_{-1}^0 x^{-3}\,dx = \lim_{b\to 0^-}\int_{-1}^b x^{-3}\,dx = \lim_{b\to 0^-}\frac{-x^{-2}}{2}\Big|_{x=-1}^{x=b} = \lim_{b\to 0^-}\Big(-\frac{1}{2b^2}+\frac{1}{2}\Big) = -\infty\,;$$

thus the improper integral $\displaystyle\int_{-1}^2 x^{-3}\,dx$ diverges (but not diverges to $\infty$ or $-\infty$).

**Remark 8.45.** Even though $y = -\dfrac{x^{-2}}{2}$ is an anti-derivative of the function $y = x^{-3}$, you cannot apply the "Fundamental Theorem of Calculus" to conclude that

$$\int_{-1}^2 x^{-3}\,dx = \frac{x^{-2}}{-2}\Big|_{x=-1}^{x=2} = -\frac{1}{8}+\frac{1}{2} = \frac{3}{8}$$

since $y = x^{-3}$ is not Riemann integrable on $[-1,2]$.

Similar to Theorem 8.40, we also have the following

**Theorem 8.46**

If $f$ is Riemann integrable on $[a,c]$ for all $a<c<b$, and $f$ has an infinite discontinuity at $a$ or $b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad \forall\, a<c<b\,,$$

provided that the improper integrals on both sides converge or diverge to $\infty$ (or $-\infty$).

We can also consider improper integral $\displaystyle\int_a^b f(x)\,dx$ in which $a=-\infty$ or $b=\infty$, and $f$ has an infinite discontinuity at $c$ for $a<c<b$. In this case, we define

$$\int_a^\infty f(x)\,dx = \int_a^d f(x)\,dx + \int_d^\infty f(x)\,dx \qquad \forall\, d>c\,,$$
$$\int_{-\infty}^b f(x)\,dx = \int_{-\infty}^d f(x)\,dx + \int_d^b f(x)\,dx \qquad \forall\, d<c\,,$$

and etc. In other words, when the integrand and the domain of integration are unbounded, we divide the integral into improper integrals of one type and compute those integrals separately, pretending that the summing rule

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \cdots + \int_{c_{n-1}}^{c_n} f(x)\,dx + \int_{c_n}^b f(x)\,dx$$

also holds for improper integrals.

**Example 8.47.** Evaluate $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$.

We observe that the integrand has an infinite discontinuity at 0, and the domain of integration is unbounded. Therefore,

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \int_0^1 \frac{dx}{\sqrt{x}(x+1)} + \int_1^\infty \frac{dx}{\sqrt{x}(x+1)}\,.$$

By the substitution $u=\sqrt{x}$, $du=\dfrac{dx}{2\sqrt{x}}$; thus

$$\int \frac{dx}{\sqrt{x}(x+1)} = \int \frac{2du}{u^2+1} = 2\arctan u + C = 2\arctan\sqrt{x} + C\,.$$

Therefore,

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{x}(x+1)} &= \lim_{a\to0^+}\int_a^1 \frac{dx}{\sqrt{x}(x+1)} = \lim_{a\to0^+} 2\arctan\sqrt{x}\Big|_{x=a}^{x=1}\\ &= \lim_{a\to0^+}\Big(2\cdot\frac{\pi}{4} - 2\arctan\sqrt{a}\Big) = \frac{\pi}{2}\end{aligned}$$

and

$$\begin{aligned}\int_1^\infty \frac{dx}{\sqrt{x}(x+1)} &= \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}(x+1)} = \lim_{b\to\infty} 2\arctan\sqrt{x}\Big|_{x=1}^{x=b}\\ &= \lim_{b\to\infty}\Big(2\arctan\sqrt{b} - 2\cdot\frac{\pi}{4}\Big) = \pi - \frac{\pi}{2} = \frac{\pi}{2}\,.\end{aligned}$$

As a consequence,

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \frac{\pi}{2} + \frac{\pi}{2} = \pi\,.$$

**Definition 8.48**

Let $\displaystyle\int_a^b f(x)\,dx$, where $a, b$ could be infinite, be an improper integral.

1. The improper integral $\displaystyle\int_a^b f(x)\,dx$ is said to be absolutely convergent or converge absolutely if $\displaystyle\int_a^b \big|f(x)\big|\,dx$ converges.

2. The improper integral $\displaystyle\int_a^b f(x)\,dx$ is said to be conditionally convergent or converge conditionally if $\displaystyle\int_a^b f(x)\,dx$ converges but $\displaystyle\int_a^b \big|f(x)\big|\,dx$ diverges (to $\infty$).

**Remark 8.49.** Even though it is not required in the definition that an absolutely convergent improper integral has to converge, it is in fact true an absolutely convergent improper integral converges.

**Example 8.50.** The improper integral $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ is conditionally convergent but not absolutely convergent. To see that the improper integral is not absolutely convergent, we note that if $n \in \mathbb{N}$,

$$\begin{aligned}\int_0^{2n\pi} \Big|\frac{\sin x}{x}\Big|\,dx &= \sum_{k=1}^{n} \int_{2(k-1)\pi}^{2k\pi} \Big|\frac{\sin x}{x}\Big|\,dx = \sum_{k=1}^{n} \int_0^{2\pi} \Big|\frac{\sin\big[x+2(k-1)\pi\big]}{x+2(k-1)\pi}\Big|\,dx \\ &= \sum_{k=1}^{n} \int_0^{2\pi} \frac{|\sin x|}{|x+2(k-1)\pi|}\,dx = \sum_{k=1}^{n} \int_0^{2\pi} \frac{|\sin x|}{2k\pi}\,dx \geqslant \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k}\,;\end{aligned}$$

thus by the fact that

$$\begin{aligned}\sum_{k=1}^{2^n}\frac{1}{k} &= 1+\frac{1}{2}+\Big(\frac{1}{3}+\frac{1}{4}\Big)+\Big(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\Big)+\cdots+\Big(\frac{1}{2^{n-1}+1}+\frac{1}{2^{n-1}+2}+\cdots+\frac{1}{2^n}\Big) \\ &\geqslant 1+\frac{1}{2}+\Big(\frac{1}{4}+\frac{1}{4}\Big)+\Big(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\Big)+\cdots+\Big(\underbrace{\frac{1}{2^n}+\frac{1}{2^n}+\cdots+\frac{1}{2^n}}_{2^{n-1}\text{ terms}}\Big) \\ &= 1+\underbrace{\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}}_{n\text{ terms}} = \frac{n}{2}+1 \geqslant \frac{n}{2}\,,\end{aligned}$$

we find that

$$\int_0^{2^{n+1}\pi} \Big|\frac{\sin x}{x}\Big|\,dx \geqslant \frac{2}{\pi}\sum_{k=1}^{2^n}\frac{1}{k} \geqslant \frac{n}{\pi}$$

which approaches $\infty$ as $n \to \infty$.

**Theorem 8.51: A special type of improper integral**

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1} & \text{if } p > 1\,, \\ \text{diverges to } \infty & \text{if } p \leqslant 1\,. \end{cases}$$

• **Comparison Test for Improper Integrals**

In the last part of this section, we consider some criteria which can be used to judge if an improper integral converges or diverges, without evaluating the exact value of the improper integral.

**Theorem 8.52: Direct Comparison Test**

Let $f$ and $g$ be continuous functions and $0 \leqslant g(x) \leqslant f(x)$ on the interval $[a, \infty)$.

1. If the improper integral $\displaystyle\int_a^\infty f(x)\,dx$ converges, then the improper integral $\displaystyle\int_a^\infty g(x)\,dx$ converges.
2. If the improper integral $\displaystyle\int_a^\infty g(x)\,dx$ diverges to $\infty$, then the improper integral $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

Similar result also holds for improper integrals given by other two cases in Definition 8.34 and the case with infinite discontinuities.

*Proof.* For $b > a$, define $G(b) = \displaystyle\int_a^b g(x)\,dx$ and $F(b) = \displaystyle\int_a^b f(x)\,dx$. By the Fundamental Theorem of Calculus, $F, G : [a, \infty) \to \mathbb{R}$ is differentiable (hence continuous). Since $0 \leqslant g(x) \leqslant f(x)$ on $[a, \infty)$, for all $b > a$ we have $0 \leqslant G(b) \leqslant F(b)$, and $F, G$ are monotone increasing.

1. If the improper integral $\displaystyle\int_a^\infty f(x)\,dx$ converges, the $\displaystyle\lim_{b\to\infty} F(b) = M$ exists. Since $F$ is monotone increasing, $F(b) \leqslant M$ for all $b > a$; thus $G(b) \leqslant M$ for all $b > a$. By the monotonicity of $G$, $\displaystyle\lim_{b\to\infty} G(b)$ exists.
2. If the improper integral $\displaystyle\int_a^\infty g(x)\,dx$ diverges to $\infty$, $\displaystyle\lim_{b\to\infty} G(b) = \infty$; thus the fact that $G(b) \leqslant F(b)$ implies that $\displaystyle\lim_{b\to\infty} F(b) = \infty$. □

**Example 8.53.** Consider the improper integral $\displaystyle\int_1^\infty e^{-x^2}\,dx$. Note that $e^{-x^2} \leqslant e^{-x}$ for all

$x \in [1, \infty)$. Since

$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}(-e^{-x})\Big|_{x=1}^{x=b} = \lim_{b\to\infty}(e^{-b} - e^{-1}) = -e^{-1}\,,$$

by Theorem 8.52 we find that the improper integral $\displaystyle\int_1^\infty e^{-x^2}\,dx$ converges.

**Example 8.54.** Consider the improper integral $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$. Note that $\dfrac{\sin^2 x}{x^2} \leqslant \dfrac{1}{x^2}$ for all $x \in [1, \infty)$. Since

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\Big(-\frac{1}{x}\Big)\Big|_{x=1}^{x=b} = \lim_{b\to\infty}\Big(\frac{1}{b} - 1\Big) = -1\,,$$

by Theorem 8.52 we find that the improper integral $\displaystyle\int_1^\infty e^{-x^2}\,dx$ converges.

**Example 8.55** (The Gamma Function)**.** The Gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt\,.$$

We note that for each $x \in \mathbb{R}$, the integrand $f(t) = t^{x-1}e^{-t}$ is positive on $[0, \infty)$.

1. If $x \geqslant 1$, the function $y = t^{x-1}e^{-\frac{t}{2}}$ is differentiable on $[0, \infty)$ and has a maximum at the point $t = 2(x-1)$. Therefore,

   $$0 \leqslant f(t) \leqslant 2^{x-1}(x-1)^{x-1}e^{-\frac{t}{2}} \qquad \forall\, t \geqslant 0\,.$$

   By the fact that

   $$\int_0^\infty e^{-\frac{t}{2}}\,dt = \lim_{b\to\infty}\int_0^b e^{-\frac{t}{2}}\,dt = \lim_{b\to\infty}\Big(-2e^{-\frac{t}{2}}\Big)\Big|_{t=0}^{t=b} = \lim_{b\to\infty}\Big(2 - 2e^{-\frac{b}{2}}\Big) = 2,$$

   we find that the improper integral $\displaystyle\int_0^\infty t^{x-1}e^{-t}\,dt$ converges.

2. If $0 < x < 1$, the function $f$ has an infinite discontinuity at 0. Therefore,

   $$\int_0^\infty t^{x-1}e^{-t}\,dt = \int_0^1 t^{x-1}e^{-t}\,dt + \int_1^\infty t^{x-1}e^{-t}\,dt\,.$$

   Again, the function $y = t^{x-1}e^{-\frac{t}{2}}$ is bounded from above by $2^{x-1}(x-1)^{x-1}$; thus the same reason as above show that the improper integral $\int_1^\infty t^{x-1}e^{-t}\,dt$ converges.

On the other hand, note that $f(t) \leqslant t^{x-1}$ for all $t \in [0,1]$. By the fact that

$$\int_0^1 t^{x-1}\,dt = \lim_{a\to 0^+}\int_a^1 t^{x-1}\,dx = \lim_{a\to 0^+}\frac{t^x}{x}\Big|_{t=a}^{t=1} = \lim_{a\to 0^+}\frac{1-a^x}{x} = \frac{1}{x}\,,$$

we find that the improper integral $\int_0^1 t^{x-1}e^{-t}\,dt$ converges. Therefore, the improper integral $\int_0^\infty t^{x-1}e^{-t}\,dt$ converges.

3. If $x \leqslant 0$, then $t^{x-1}e^{-t} \geqslant t^{x-1}e^{-1}$ for all $t \in [0,1]$. By the fact that

$$\int_0^1 t^{x-1}e^{-1}\,dt = \lim_{a\to 0^+}\int_a^1 t^{x-1}e^{-1}\,dt = \infty\,,$$

Theorem 8.52 implies that the improper integral $\int_0^1 t^{x-1}e^{-t}\,dt$ diverges to $\infty$. This implies that the improper integral $\int_0^\infty t^{x-1}e^{-t}\,dt$ diverges to $\infty$ as well.

**Theorem 8.56: Limit Comparison Test**

Let $f$ and $g$ be positive continuous functions on the interval $[a,\infty)$. If the limit $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = L$ for some $0 < L < \infty$, then

$$\int_a^\infty f(x)\,dx \ \text{ converges if and only if } \int_a^\infty g(x)\,dx \ \text{ converges.}$$

Similar result also holds for improper integrals given by other two cases in Definition 8.34 and the case with infinite discontinuities.

*Proof.* By the fact $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = L$, there exists $M > a$ such that

$$\Big|\frac{f(x)}{g(x)} - L\Big| < \frac{L}{2} \qquad \text{whenever} \quad x > M\,.$$

Therefore,

$$0 < \frac{L}{2}g(x) < f(x) < \frac{3L}{2}g(x) \qquad \text{whenever} \quad x > M\,.$$

By the direct comparison test,

$$\int_M^\infty f(x)\,dx \ \text{ converges if and only if } \int_M^\infty g(x)\,dx \ \text{ converges.}$$

The theorem is then concluded since $\int_a^M f(x)\,dx$ and $\int_a^M g(x)\,dx$ are both finite. $\square$

**Example 8.57.** Consider the improper integral $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$. Since $\lim_{x\to\infty} \frac{(1+e^{-x})/x}{1/x} = 1$, the limit comparison test implies that

$$\int_1^\infty \frac{1+e^{-x}}{x}\,dx \text{ converges if and only if } \int_1^\infty \frac{dx}{x} \text{ converges.}$$

By Theorem 8.51, we find that the integral $\int_1^\infty \frac{dx}{x}$ diverges; thus the improper integral $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ diverges.

**Example 8.58.** Consider the improper integral $\int_0^{\frac{\pi}{4}} \frac{dx}{x+\tan x}$. Note that this is an improper integral with infinite discontinuity at $x = 0$. Since

$$\lim_{x\to 0^+} \frac{x+\tan x}{x} = 1 + \lim_{x\to 0^+} \frac{\tan x}{x} = 1 + \lim_{x\to 0^+} \frac{\sin x}{x\cos x} = 2\,,$$

the limit comparison test implies that

$$\int_0^{\frac{\pi}{4}} \frac{dx}{x+\tan x} \text{ converges if and only if } \int_0^{\frac{\pi}{4}} \frac{dx}{x} \text{ converges.}$$

Since the improper integral $\int_0^{\frac{\pi}{4}} \frac{dx}{x}$ diverges (to $\infty$), we must have $\int_0^{\frac{\pi}{4}} \frac{dx}{x+\tan x}$ diverges.

**Example 8.59.** Determine the convergence of the improper integral $\int_0^\infty \frac{dx}{\sqrt[3]{x^4-x^2}}$.

Note that $\frac{1}{\sqrt[3]{x^4-x^2}} = x^{-\frac{2}{3}}(x+1)^{-\frac{1}{3}}(x-1)^{-\frac{1}{3}}$. In the interval $[0,\infty)$, the integrand has singular points at 0 and 1. Write

$$\int_0^\infty \frac{dx}{\sqrt[3]{x^4-x^2}} = \int_0^{\frac{1}{2}} \frac{dx}{\sqrt[3]{x^4-x^2}} + \int_{\frac{1}{2}}^1 \frac{dx}{\sqrt[3]{x^4-x^2}} + \int_1^2 \frac{dx}{\sqrt[3]{x^4-x^2}} + \int_2^\infty \frac{dx}{\sqrt[3]{x^4-x^2}}\,. \tag{8.5.1}$$

1. Let $f(x) = -x^{-\frac{2}{3}}(x+1)^{-\frac{1}{3}}(x-1)^{-\frac{1}{3}}$ and $g(x) = x^{-\frac{2}{3}}$. Then $f, g$ are positive continuous on $\big[a, \frac{1}{2}\big]$ for all $a > 0$. Moreover,

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \big[-(x+1)^{-\frac{1}{3}}(x-1)^{-\frac{1}{3}}\big] = 1 > 0\,,$$

and

$$\int_0^{\frac{1}{2}} g(x)\,dx = \lim_{a\to 0^+}\int_a^{\frac{1}{2}} x^{-\frac{2}{3}}\,dx = \lim_{a\to 0^+} 3x^{\frac{1}{3}}\Big|_{x=a}^{x=\frac{1}{2}} = \frac{3}{\sqrt[3]{2}}$$

which shows that the improper integral $\int_0^{\frac{1}{2}} g(x)\,dx$ converges. Therefore, the limit comparison test implies that $\int_0^{\frac{1}{2}} f(x)\,dx = -\int_0^{\frac{1}{2}} \frac{dx}{\sqrt[3]{x^4-x^2}}$ converges.

2. Let $f(x) = -x^{-\frac{2}{3}}(x+1)^{-\frac{1}{3}}(x-1)^{-\frac{1}{3}}$ and $g(x) = -(x-1)^{\frac{1}{3}}$. Then $f, g$ are positive continuous on $\left[\frac{1}{2}, b\right]$ for all $\frac{1}{2} < b < 1$. Moreover,

$$\lim_{x\to 1^-}\frac{f(x)}{g(x)} = \lim_{x\to 1^-} x^{-\frac{2}{3}}(x+1)^{-\frac{1}{3}} = 2^{-\frac{1}{3}} > 0\,,$$

and

$$\int_{\frac{1}{2}}^{1} g(x)\,dx = -\lim_{b\to 1^-}\int_{\frac{1}{2}}^{b} (x-1)^{-\frac{1}{3}}\,dx = -\lim_{b\to 1^-}\frac{3}{2}(x-1)^{\frac{2}{3}}\Big|_{x=\frac{1}{2}}^{x=b} = \frac{3}{2\sqrt[3]{4}}$$

which shows that the improper integral $\int_{\frac{1}{2}}^{1} g(x)\,dx$ converges. Therefore, the limit comparison test implies that $\int_{\frac{1}{2}}^{1} f(x)\,dx = -\int_{\frac{1}{2}}^{1} \frac{dx}{\sqrt[3]{x^4-x^2}}$ converges.

3. Similar to the previous case, we let $f(x) = x^{-\frac{2}{3}}(x+1)^{-\frac{1}{3}}(x-1)^{-\frac{1}{3}}$ and $g(x) = (x-1)^{\frac{1}{3}}$. Then $f, g$ are positive continuous on $[a, 2]$ for all $1 < a < 2$. Moreover,

$$\lim_{x\to 1^+}\frac{f(x)}{g(x)} = \lim_{x\to 1^+} x^{-\frac{2}{3}}(x+1)^{-\frac{1}{3}} = 2^{-\frac{1}{3}} > 0\,,$$

and

$$\int_1^2 g(x)\,dx = \lim_{a\to 1^+}\int_a^2 (x-1)^{-\frac{1}{3}}\,dx = -\lim_{a\to 1^+}\frac{3}{2}(x-1)^{\frac{2}{3}}\Big|_{x=a}^{x=2} = \frac{3}{2}$$

which shows that the improper integral $\int_1^2 g(x)\,dx$ converges. Therefore, the limit comparison test implies that $\int_1^2 f(x)\,dx = \int_1^2 \frac{dx}{\sqrt[3]{x^4-x^2}}$ converges.

4. Let $f(x) = x^{-\frac{2}{3}}(x+1)^{-\frac{1}{3}}(x-1)^{-\frac{1}{3}}$ and $g(x) = x^{-\frac{4}{3}}$. Then $f, g$ are positive continuous on $[2, b]$ for all $b > 2$. Moreover,

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x^{-\frac{2}{3}}(x+1)^{-\frac{1}{3}}(x-1)^{-\frac{1}{3}}}{x^{-\frac{4}{3}}} = \lim_{x\to\infty}\sqrt[3]{\frac{x^2}{(x-1)(x+1)}} = 1 > 0\,,$$

and

$$\int_2^\infty g(x)\,dx = \lim_{b\to\infty}\int_2^b x^{-\frac{4}{3}}\,dx = -\lim_{b\to\infty} 3x^{-\frac{1}{3}}\Big|_{x=2}^{x=b} = 3$$

which shows that the improper integral $\int_2^\infty g(x)\,dx$ converges. Therefore, the limit comparison test implies that $\int_2^\infty f(x)\,dx = \int_2^\infty \frac{dx}{\sqrt[3]{x^4-x^2}}$ converges.

Since the four improper integrals on the right-hand side of (8.5.1) converges, we find that the improper integral $\int_0^\infty \frac{dx}{\sqrt[3]{x^4-x^2}}$ converges.

### 8.5.1 The Laplace transform（補充，不考）

**Definition 8.60: Laplace Transform**

Let $f:[0,\infty)\to\mathbb{R}$ be continuous. The Laplace transform of $f$, denoted by $\mathscr{L}(f)$, is the function defined by

$$\mathscr{L}(f)(s) = \int_0^\infty e^{-st}f(t)\,dt \left( = \lim_{R\to\infty}\int_0^R e^{-st}f(t)dt\right),$$

and the domain of $\mathscr{L}(f)$ is the set consisting of all numbers $s$ for which the integral converges.

**Remark 8.61.** In general, the Laplace transform of $f$ can be defined, without assuming that $f$ is continuous on $[0,\infty)$, as long as the integral $\int_0^\infty e^{-st}f(t)\,dt$ makes sense. Moreover, if $f$ is continuous and satisfies

$$\big|f(t)\big| \leqslant Me^{\alpha t} \qquad \forall\, t\in[0,\infty)\,, \tag{8.5.2}$$

then $\mathscr{L}(f)(s)$ exists for all $s>\alpha$. A function $f$ is said to be of exponential order $\alpha$ if there exist $M>0$ such that the growth condition (8.5.2) holds.

**Example 8.62.** Let $f:[0,\infty)\to\mathbb{R}$ be given by $f(t)=t^p$ for some $p>-1$. Recall that the Gamma function $\Gamma:(0,\infty)\to\mathbb{R}$ is defined by

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt\,.$$

We note that if $-1 < p < 0$, $f$ is not of exponential order $a$ for all $a \in \mathbb{R}$; however, the Laplace transform of $f$ still exists. In fact, for $s > 0$,

$$\mathscr{L}(f)(s) = \lim_{R\to\infty} \int_0^R e^{-st} t^p \, dt = \lim_{R\to\infty} \int_0^{sR} e^{-t} \Big(\frac{t}{s}\Big)^p \frac{dt}{s} = \frac{\Gamma(p+1)}{s^{p+1}}\,.$$

In particular, if $p = n \in \mathbb{N} \cup \{0\}$, then

$$\mathscr{L}(f)(s) = \frac{n!}{s^{n+1}} \qquad \forall\, s > 0\,.$$

**Example 8.63.** Let $g : [0,\infty) \to \mathbb{R}$ be given by $g(t) = e^{at}\sin(bt)$ for some $b \neq 0$. Using (8.2.1), we find that

$$\int e^{(a-s)t}\sin(bt)\,dt = \frac{1}{(s-a)^2+b^2}\Big[(a-s)e^{(a-s)t}\sin(bt) - be^{(a-s)t}\cos(bt)\Big] + C\,.$$

Therefore, for $s > a$,

$$\begin{aligned}\mathscr{L}(g)(s) &= \int_0^\infty e^{(a-s)t}\sin(bt)\,dt \\ &= \lim_{b\to\infty} \frac{1}{(s-a)^2+b^2}\Big[(a-s)e^{(a-s)t}\sin(bt) - be^{(a-s)t}\cos(bt)\Big]\Big|_{t=0}^{t=b} \\ &= \frac{b}{(s-a)^2+b^2}\,.\end{aligned}$$

Similarly, if $h(t) = e^{at}\cos(bt)$, using (8.2.2) we find that for $s > a$,

$$\begin{aligned}\mathscr{L}(h)(s) &= \int_0^\infty e^{(a-s)t}\cos(bt)\,dt \\ &= \lim_{b\to\infty} \frac{1}{(s-a)^2+b^2}\Big[(a-s)e^{(a-s)t}\cos(bt) + be^{(a-s)t}\sin(bt)\Big]\Big|_{t=0}^{t=b} \\ &= \frac{s-a}{(s-a)^2+b^2}\,.\end{aligned}$$

**Theorem 8.65: Linearity of the Laplace transform**

Let $f, g : [0,\infty) \to \mathbb{R}$ be functions whose Laplace transform exist for $s > \alpha$ and $c$ be a constant. Then for $s > \alpha$,

1. $\mathscr{L}(f+g)(s) = \mathscr{L}(f)(s) + \mathscr{L}(g)(s)$.
2. $\mathscr{L}(cf)(s) = c\mathscr{L}(f)(s)$.

**Theorem 8.66**

Suppose that $f : [0,\infty) \to \mathbb{R}$ is a function such that $f, f', f'', \cdots, f^{(n-1)}$ are continuous of exponential order $\alpha$, and $f^{(n)}$ is piecewise continuous. Then $\mathscr{L}(f^{(n)})(s)$ exists for all $s > \alpha$, and

$$\mathscr{L}(f^{(n)})(s) = s^n\mathscr{L}(f)(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)\,. \quad (8.5.3)$$

*Proof.* We prove by induction. Suppose that $f$ is continuously differentiable on $[0,\infty)$ and is of exponential order $\alpha$. Then for $s > \alpha$,

$$\begin{aligned}\int_0^\infty e^{-st}f'(t)\,dt = \lim_{b\to\infty}\int_0^b e^{-st}f'(t)\,dt = \lim_{b\to\infty}\Big[e^{-st}f(t)\Big|_{t=0}^{t=b} + s\int_0^b e^{-st}f(t)\,dt\Big] \\ = s\int_0^\infty e^{-st}f(t)\,dt\Big] - f(0) + \lim_{b\to\infty}e^{-sb}f(b) = s\mathscr{L}(f)(s) - f(0)\end{aligned}$$

which shows that (8.5.3) holds for $n = 1$ and all continuously differentiable $f$.

Now suppose that (8.5.3) holds for all $k$-times continuously differentiable function $f$. Then if $s > \alpha$ and $f$ is $(k+1)$-times continuously differentiable function on $[0,\infty)$,

$$\begin{aligned}\mathscr{L}(f^{(k+1)})(s) &= \mathscr{L}\big((f')^{(k)}\big)(s) \\ &= s^k\mathscr{L}(f')(s) - s^{k-1}f'(0) - s^{k-2}(f')'(0) - \cdots - s(f')^{(n-2)}(0) - (f')^{(n-1)}(0) \\ &= s^k\big[s\mathscr{L}(f)(s) - f(0)\big] - s^{k-1}f'(0) - s^{k-2}f''(0) - \cdots - sf^{(n-1)}(0) - f^{(n)}(0) \\ &= s^{k+1}\mathscr{L}(f)(s) - s^kf(0) - s^{k-1}f'(0) - s^{k-2}f''(0) - \cdots - sf^{(n-1)}(0) - f^{(n)}(0)\end{aligned}$$

which implies that (8.5.3) holds for the case $n = k+1$. The theorem is then concluded by induction. □

- **Applications in solving the ordinary differential equations**

Let $a_0, a_1, \cdots, a_{n-1}$, $y_0, y_1, \cdots, y_{n-1}$ be given numbers, and $g : [0,\infty) \to \mathbb{R}$ be a continuous function of exponential order. The idea of solving an ordinary differential equation (here $y$ is the unknown function to be solved) of the form

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = g(s)\,, \quad (8.5.4a)$$

$$y(0) = y_0, y'(0) = y_1, \cdots, y^{(n-1)}(0) = y_{n-1}\,, \quad (8.5.4b)$$

using the method of the Laplace transform is based on the following facts:

1. The Laplace transform is a one-to-one mapping in the sense that if $f$ and $g$ are continuous function such that $\mathscr{L}(f) = \mathscr{L}(g)$ for $s > \alpha$, then $f = g$ on $[0, \infty)$.

2. The solution of (8.5.4) is of exponential order $\alpha$ (so that the Laplace transform of derivatives of $y$ can be computed using Theorem 8.66).

Under these two facts, we then take the Laplace transform of (8.5.4a) and apply Theorem 8.65 and 8.66 to obtain, by letting $Y(s) = \mathscr{L}(y)(s)$, that

$$\begin{aligned}
&a_n\big[s^n Y(s) - s^{n-1}y_0 - s^{n-2}y_1 - \cdots - sy_{n-2} - y_{n-1}\big] \\
&\quad + a_{n-1}\big[s^{n-1}Y(s) - s^{n-2}y_0 - s^{n-3}y_1 - \cdots - sy_{n-3} - y_{n-2}\big] \\
&\quad + a_{n-2}\big[s^{n-2}Y(s) - s^{n-3}y_0 - s^{n-4}y_1 - \cdots - sy_{n-4} - y_{n-3}\big] \\
&\quad + \cdots + a_1\big[sY(s) - y_0\big] + a_0 Y(s) = \mathscr{L}(g)(s)\,;
\end{aligned}$$

thus

$$\begin{aligned}
Y(s) = {} & \frac{1}{a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0} \times \\
& \times \Big[\mathscr{L}(g)(s) + y_0(a_n s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_2 s + a_1) \\
& \quad + y_1(a_n s^{n-2} + a_{n-1}s^{n-2} + \cdots + a_3 s + a_2) + \cdots\cdots + y_{n-2}(a_n s + a_{n-1}) + y_{n-1}\Big] \\
= {} & \frac{1}{a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-3} + \cdots + a_1 s + a_0}\Big[\mathscr{L}(g)(s) + \sum_{j=0}^{n-1} y_j \sum_{\ell=0}^{n-j-1} a_{n-\ell}s^{n-j-\ell-1}\Big].
\end{aligned}$$

The final step is to identify which function gives the Laplace transform above.

**Example 8.64.** Find the function $y$ satisfying

$$y'' + 2y' + 5y = \sin t\,, \qquad y(0) = 1\,,\ y'(0) = 0\,.$$

Using the result in Example 8.63 and Theorem 8.66, with $Y$ denoting $\mathscr{L}(y)$ we find that

$$s^2 Y(s) - s + 2\big[sY(s) - 1\big] + 5Y(s) = \frac{1}{s^2+1} \qquad \forall\, s > a$$

for some $a$. Therefore,

$$Y(s) = \frac{1}{s^2+2s+5}\Big(\frac{1}{s^2+1} + s + 2\Big) = \frac{s+2}{(s+1)^2+2^2} + \frac{1}{(s^2+2s+5)(s^2+1)}\,.$$

Writing the last term as the sum of partial fractions, we have

$$\frac{1}{(s^2+2s+5)(s^2+1)} = \frac{1}{10}\Big(\frac{s}{s^2+2s+5} - \frac{s-2}{s^2+1}\Big)\,;$$

thus

$$\begin{aligned}
Y(s) &= \frac{s+2}{(s+1)^2+2^2} + \frac{1}{10}\frac{s}{(s+1)^2+2^2} - \frac{1}{10}\frac{s-2}{s^2+1} \\
&= \frac{11}{10}\frac{s+1}{(s+1)^2+2^2} + \frac{9}{20}\frac{2}{(s+1)^2+2^2} - \frac{1}{10}\frac{s}{s^2+1} + \frac{1}{5}\frac{1}{s^2+1}\,.
\end{aligned}$$

Therefore, Fact 1 and Example 8.63 imply that

$$y(t) = \frac{11}{10}e^{-t}\cos(2t) + \frac{9}{20}e^{-t}\sin(2t) - \frac{1}{10}\cos t + \frac{1}{5}\sin t\,.$$