Calculus 微積分

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Chapter 4

Integration

• **The Σ notation**: The sum of n-terms $a_1, a_2, \cdots, a_n$ is written as $\sum_{i=1}^{n} a_i$. In other words,

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n \,.$$

Here i is called the index of summation, a_i is the i-th terms of the sum. We note that i in the sum $\sum_{i=1}^{n} a_i$ is a dummy index which can be replaced by other indices such as j, k, and etc. Therefore, $\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k$, and so on.

• **Basic properties of sums**: $\displaystyle\sum_{i=1}^{n}(ca_i + b_i) = c\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$.

Theorem 4.1: Summation Formula

1. $\sum_{i=1}^{n} c = cn$ if c is a constant;
2. $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$;
3. $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$;
4. $\sum_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$.

4.1 The Area under the Graph of a Non-negative Continuous Function

Let $f : [a, b] \to \mathbb{R}$ be a non-negative continuous function, and R be the region enclosed by the graph of the function f, the x-axis and straight lines $x = a$ and $x = b$. We consider computing $\mathcal{A}(\mathrm{R})$, the area of R. Generally speaking, since the graph of $y = f(x)$ is in general not a straight line, the computation of $\mathcal{A}(\mathrm{R})$ is not straight-forward. How do we compute the area $\mathcal{A}(\mathrm{R})$?

Partition $[a, b]$ into n sub-intervals with equal length, and let $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$. By the Extreme Value Theorem, for each $1 \leqslant i \leqslant n$ f attains its maximum and minimum on $[x_{i-1}, x_i]$; thus for $1 \leqslant i \leqslant n$, there exist $M_i, m_i \in [x_{i-1}, x_i]$ such that

$$f(M_i) = \text{the maximum of } f \text{ on } [x_{i-1}, x_i]$$

and

$$f(m_i) = \text{the minimum of } f \text{ on } [x_{i-1}, x_i].$$

The sum $S(n) \equiv \sum\limits_{i=1}^{n} f(M_i)\Delta x$ is called the upper sum of f for the partition $\{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$, and $s(n) \equiv \sum\limits_{i=1}^{n} f(m_i)\Delta x$ is called the lower sum of f for the partition $\{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$. By the definition of the upper sum and lower sum, we find that for each $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} f(m_i)\Delta x \leqslant \mathcal{A}(\mathrm{R}) \leqslant \sum_{i=1}^{n} f(M_i)\Delta x\,.$$

If the limits of the both sides exist and are identical as Δx approaches 0 (which is the same as n approaches infinity), by the Squeeze Theorem we can conclude that $\mathcal{A}(\mathrm{R})$ is the same as the limit.

Example 4.2. Let $f(x) = x^2$, and R be the region enclosed by the graph of $y = f(x)$, the X-axis, and the straight lines $x = a$ and $x = b$, where we assume that $0 \leqslant a < b$. Then the lower sum is obtained by the "left end-point rule" approximation of $\mathcal{A}(\mathrm{R})$

$$\sum_{i=1}^{n} \Big(a + \frac{(i-1)(b-a)}{n}\Big)^2 \frac{b-a}{n}$$

and the upper sum is obtained by the "right end-point rule" approximation

$$\sum_{i=1}^{n} \Big(a + \frac{i(b-a)}{n}\Big)^2 \frac{b-a}{n}\,.$$

By Theorem 4.1,

$$\begin{aligned}
\sum_{i=1}^{n} \Big(a + \frac{i(b-a)}{n}\Big)^2 \frac{b-a}{n} &= \sum_{i=1}^{n} \Big[a^2 + \frac{2a(b-a)i}{n} + \frac{a^2(b-a)^2 i^2}{n^2}\Big] \frac{b-a}{n} \\
&= a^2(b-a) + \frac{a(b-a)^2 n(n+1)}{n^2} + \frac{a^2(b-a)^3}{n^3}\frac{n(n+1)(2n+1)}{6} \\
&= a^2(b-a) + a(b-a)^2\Big(1+\frac{1}{n}\Big) + \frac{a^2(b-a)^3}{6}\Big(1+\frac{1}{n}\Big)\Big(2+\frac{1}{n}\Big)\,.
\end{aligned}$$

Letting $n \to \infty$, we find that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Big(a + \frac{i(b-a)}{n}\Big)^2 \frac{b-a}{n} = a^2(b-a) + a(b-a)^2 + \frac{a^2(b-a)^3}{3} = \frac{b^3-a^3}{3}\,.$$

Similarly,

$$\begin{aligned}\sum_{i=1}^{n} \Big(a + \frac{(i-1)(b-a)}{n}\Big)^2 \frac{b-a}{n} &= \frac{a^2(b-a)}{n} + \sum_{i=1}^{n} \Big(a + \frac{i(b-a)}{n}\Big)^2 \frac{b-a}{n} - \frac{b^2(b-a)}{n} \\ &= a^2(b-a) + \frac{a(b-a)^2 n(n+1)}{n^2} + \frac{a^2(b-a)^3}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{(a^2-b^2)(b-a)}{n}\,;\end{aligned}$$

thus

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Big(a + \frac{(i-1)(b-a)}{n}\Big)^2 \frac{b-a}{n} = \frac{b^3-a^3}{3}\,.$$

Therefore, $\mathcal{A}(\mathrm{R}) = \dfrac{b^3-a^3}{3}$.

Remark 4.3. Let R_1 be the region enclosed by $f(x) = x^2$, the x-axis and $x = a$, the R_2 be the region enclosed by $f(x) = x^2$, the x-axis and $x = b$, then intuitively $\mathcal{A}(\mathrm{R}) = \mathcal{A}(\mathrm{R}_2) - \mathcal{A}(\mathrm{R}_1)$ and this is true since $\mathcal{A}(\mathrm{R}_1) = \dfrac{a^3}{3}$ and $\mathcal{A}(\mathrm{R}_2) = \dfrac{b^3}{3}$.

If f is not continuous, then f might not attain its extrema on the interval $[x_{i-1}, x_i]$. In this case, it might be impossible to form the upper sum or the lower sum for a given partition. On the other hand, the left end-point rule $\sum\limits_{i=1}^{n} f(x_{i-1})\Delta x$ and the right end-point rule $\sum\limits_{i=1}^{n} f(x_i)\Delta x$ of approximating the area are always possible. We can even consider the "mid-point rule" approximation given by

$$\sum_{i=1}^{n} f\big(\frac{x_{i-1}+x_i}{2}\big)\Delta x$$

and consider the limit of the expression above as n approaches infinity.

4.2 Riemann Sums and Definite Integrals

In general, in order to find an approximation of $\mathcal{A}(\mathrm{R})$, the interval $[a, b]$ does not have to be divided into sub-intervals with equal length. Assume that $[a, b]$ are divided into n sub-intervals and the end-points of those sub-intervals are ordered as $a = x_0 < x_1 < x_2 < \cdots <$

$x_n = b$, here the collection of end-points $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is called a ***partition*** of $[a, b]$. Then the "left end-point rule" approximation for the partition $\mathcal{P}$ is given by

$$\ell(\mathcal{P}) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

and the "right end-point rule" approximation for the partition $\mathcal{P}$ is given by

$$r(\mathcal{P}) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}),$$

and the limit process as $n \to \infty$ in the discussion above is replaced by the limit process as the norm of partition $\mathcal{P}$, denoted by $\|\mathcal{P}\|$ and defined by $\|\mathcal{P}\| \equiv \max\left\{x_i - x_{i-1} \,\middle|\, 1 \leqslant i \leqslant n\right\}$, approaches 0. Before discussing what the limits above mean, let us look at the following examples.

Example 4.4. Consider the region bounded by the graph of $f(x) = \sqrt{x}$ and the x-axis for $0 \leqslant x \leqslant 1$. Let $x_i = \dfrac{i^2}{n^2}$ and $\mathcal{P} = \{x_0 = 0 < x_1 < \cdots < x_n = 1\}$. We note that

$$\|\mathcal{P}\| = \max\left\{\frac{i^2 - (i-1)^2}{n^2} \,\middle|\, 1 \leqslant i \leqslant n\right\} = \max\left\{\frac{2i-1}{n^2} \,\middle|\, 1 \leqslant i \leqslant n\right\} = \frac{2n-1}{n^2}$$

thus $\|\mathcal{P}\| \to 0$ is equivalent to that $n \to \infty$.

Using the right end-point rule (which is the same as the upper sum),

$$\begin{aligned} S(\mathcal{P}) &= \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{i}{n}\frac{2i-1}{n^2} = \frac{1}{n^3}\sum_{i=1}^{n}(2i^2 - i) \\ &= \frac{1}{n^3}\Big[\frac{n(n+1)(2n+1)}{3} - \frac{n(n+1)}{2}\Big] \\ &= \frac{1}{3}\Big(1 + \frac{1}{n}\Big)\Big(2 + \frac{1}{n}\Big) - \frac{1}{2n}\Big(1 + \frac{1}{n}\Big); \end{aligned}$$

thus

$$\lim_{\|\mathcal{P}\| \to 0} S(\mathcal{P}) = \lim_{n \to \infty}\Big[\frac{1}{3}\Big(1 + \frac{1}{n}\Big)\Big(2 + \frac{1}{n}\Big) - \frac{1}{2n}\Big(1 + \frac{1}{n}\Big)\Big] = \frac{2}{3}.$$

Using the left end-point rule (which is the same as the lower sum),

$$\begin{aligned} s(\mathcal{P}) &= \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{i-1}{n}\frac{2i-1}{n^2} = \frac{1}{n^3}\sum_{i=1}^{n}(2i^2 - 3i + 1) \\ &= \frac{1}{n^3}\Big[\frac{n(n+1)(2n+1)}{3} - \frac{3n(n+1)}{2} + n\Big] \\ &= \frac{1}{3}\Big(1 + \frac{1}{n}\Big)\Big(2 + \frac{1}{n}\Big) - \frac{3}{2n}\Big(1 + \frac{1}{n}\Big) + \frac{1}{n^2}; \end{aligned}$$

thus

$$\lim_{\|\mathcal{P}\|\to 0} s(\mathcal{P}) = \lim_{n\to\infty} \Big[\frac{1}{3}\Big(1+\frac{1}{n}\Big)\Big(2+\frac{1}{n}\Big) - \frac{3}{2n}\Big(1+\frac{1}{n}\Big) + \frac{1}{n^2}\Big] = \frac{2}{3}\,.$$

Therefore, the area of the region of interest is $\dfrac{2}{3}$.

Example 4.5. In this example we use a different approach to compute $\mathcal{A}(\mathrm{R})$ in Example 4.2. Assume that $0 < a < b$. Let $r = \big(\frac{b}{a}\big)^{\frac{1}{n}}$, $x_i = ar^i$, and $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$.

Claim: If $c > 1$, then $c^{\frac{1}{n}} = 1$ as n approaches infinity.

Proof of the claim: If $c > 1$, then $c^{\frac{1}{n}} > 1$. Let $y_n = c^{\frac{1}{n}} - 1$. Then $c = (1+y_n)^n \geqslant 1 + ny_n$ which implies that $0 < y_n \leqslant \dfrac{c-1}{n}$ for all $n \in \mathbb{N}$. By the Squeeze Theorem, $c^{\frac{1}{n}} \to 1$ as $n \to \infty$. □

Note that the claim above implies that $r \to 1$ as $n \to \infty$. Moreover, $x_i - x_{i-1} = a(r^i - r^{i-1}) = ar^{i-1}(r-1)$; thus

$$0 < a(r-1) = x_1 - x_0 \leqslant \|\mathcal{P}\| = x_n - x_{n-1} = ar^{n-1}(r-1) < b(r-1)\,.$$

Therefore, $\|\mathcal{P}\| \to 0$ is equivalent to that $n \to \infty$.

Using the "left end-point rule" approximation of the area,

$$\begin{aligned}\mathcal{A}(\mathrm{R}) &= \lim_{n\to\infty} \sum_{i=1}^{n} x_{i-1}^2 (x_i - x_{i-1}) = \lim_{n\to\infty} \sum_{i=1}^{n} a^2 r^{2(i-1)} ar^{i-1}(r-1) = a^3 \lim_{n\to\infty} (r-1) \sum_{i=1}^{n} r^{3(i-1)} \\ &= a^3 \lim_{n\to\infty} (r-1)\frac{r^{3n}-1}{r^3-1} = a^3 \lim_{n\to\infty} \frac{\frac{b^3}{a^3}-1}{r^2+r+1} = \frac{b^3-a^3}{3}\,.\end{aligned}$$

Similarly, when applying the "right end-point rule" approximation, we obtain that

$$\lim_{n\to\infty} \sum_{i=1}^{n} x_i^2 (x_i - x_{i-1}) = a^3 \lim_{n\to\infty} (r-1) \sum_{i=1}^{n} r^{3i} = a^3 \lim_{n\to\infty} (r-1)\frac{r^{3n+3}-r^3}{r^3-1} = \frac{b^3-a^3}{3}\,.$$

This also gives the area of the region R.

To compute an approximated value of $\mathcal{A}(\mathrm{R})$, there is no reason for evaluating the function at the left end-points or the right end-points like what we have discussed above. For example, we can also consider the "mid-point rule"

$$m(\mathcal{P}) = \sum_{i=1}^{n} f\Big(\frac{x_i + x_{i-1}}{2}\Big)(x_i - x_{i-1})$$

to approximate the value of $\mathcal{A}(\mathrm{R})$, and compute the limit of the sum above as $\|\mathcal{P}\|$ approaches 0 in order to obtain $\mathcal{A}(\mathrm{R})$. In fact, we should be able to consider any point $c_i \in [x_{i-1}, x_i]$ and consider the limit of the sum

$$\lim_{\|\mathcal{P}\|\to 0} \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

if the region R does have area.

Now let us forget about the concept of the area. For a general function $f : [a, b] \to \mathbb{R}$, we can also consider the limit above as $\|\mathcal{P}\|$ approaches 0, if the limit exists. The discussion above motivates the following definitions.

Definition 4.6: Partition of Intervals and Riemann Sums

A finite set $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is said to be a partition of the closed interval $[a, b]$ if $a = x_0 < x_1 < \cdots < x_n = b$. Such a partition $\mathcal{P}$ is usually denoted by $\{a = x_0 < x_1 < \cdots < x_n = b\}$. The norm of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, is the number $\max\big\{x_i - x_{i-1} \,\big|\, 1 \leqslant i \leqslant n\big\}$; that is,

$$\|\mathcal{P}\| \equiv \max\big\{x_i - x_{i-1} \,\big|\, 1 \leqslant i \leqslant n\big\}.$$

A partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is called regular if $x_i - x_{i-1} = \|\mathcal{P}\|$ for all $1 \leqslant i \leqslant n$.

Let $f : [a, b] \to \mathbb{R}$ be a function. A Riemann sum of f for the partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ is a sum which takes the form

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}),$$

where the set $\Xi = \{c_0, c_1, \cdots, c_{n-1}\}$ satisfies that $x_{i-1} \leqslant c_i \leqslant x_i$ for each $1 \leqslant i \leqslant n$.

Definition 4.7: Riemann Integrals - 黎曼積分

Let $f : [a, b] \to \mathbb{R}$ be a function. f is said to be Riemann integrable on $[a, b]$ if there exists a real number A such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is partition of $[a, b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums for the partition $\mathcal{P}$ belongs to the interval $(A - \varepsilon, A + \varepsilon)$. Such a number A (is unique and) is called the Riemann integral of f on $[a, b]$ and is denoted by $\int_{[a,b]} f(x)\,dx$.

Remark 4.8. For conventional reason, the Riemann integral of f over the interval with left end-point a and right-end point b is written as $\int_a^b f(x)\,dx$, and is called the definite integral of f from a to b. The function f sometimes is called the integrand of the integral.

We also note that here in the representation of the integral, x is a dummy variable; that is, we can use any symbol to denote the independent variable; thus

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

and etc.

The following example shows that no all functions are Riemann integrable.

Example 4.9. Consider the Dirichlet function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational}\,, \\ 1 & \text{if } x \text{ is irrational}\,, \end{cases}$$

on the interval $[1,2]$. By partitioning $[1,2]$ into n sub-intervals with equal length, the Riemann sum given by the right end-point rule is always zero since the right end-point of each sub-interval is rational. On the other hand, by partitioning $[1,2]$ into n sub-intervals using geometric sequence $1, r, r^2, \cdots, r^{n-1}, 2$, where $r = 2^{\frac{1}{n}}$, by the fact that $r^i \notin \mathbb{Q}$ for each $1 \leqslant i \leqslant n-1$ the Riemann sum of f for this partition given by the right end-point rule is

$$\begin{aligned}\sum_{i=1}^{n} f(r^i)(r^i - r^{i-1}) &= \sum_{i=1}^{n-1} (r^i - r^{i-1}) = r^1 - r^0 + r^2 - r^1 + \cdots + r^{n-1} - r^{n-2} \\ &= r^{n-1} - r^0 = \frac{2}{r} - 1\end{aligned}$$

which approaches 1 as r approaches 1. Therefore, f is not integrable on $[1,2]$ since there are two possible limits of Riemann sums which means that the Riemann sums cannot concentrate around any firxed real number.

Theorem 4.10

If $f : [a,b] \to \mathbb{R}$ is continuous, then f is Riemann integrable on $[a,b]$.

Example 4.11. In this example we compute $\displaystyle\int_a^b x^q\,dx$ when $q \neq -1$ is a rational number and $0 < a < b$. Since $f(x) = x^q$ is continuous on $[a,b]$, by Theorem 4.10 to find the integral it suffices to find the limit of the Riemann sum given by the left end-point rule as $\|\mathcal{P}\|$ approaches 0.

We follow the idea in Example 4.5. Let $r = \left(\frac{b}{a}\right)^{\frac{1}{n}}$ and $x_i = ar^i$, as well as the partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$. Then the Riemann sum of f for the partition $\mathcal{P}$ given by left end-point rule is

$$\begin{aligned} L(\mathcal{P}) &= \sum_{i=1}^{n} (ar^{i-1})^q (ar^i - ar^{i-1}) = a^{q+1}(r-1)\sum_{i=1}^{n} r^{(i-1)(q+1)} = a^{q+1}(r-1)\frac{r^{n(q+1)}-1}{r^{q+1}-1} \\ &= \frac{r-1}{r^{q+1}-1}\left(b^{q+1} - a^{q+1}\right). \end{aligned}$$

Since $\left.\frac{d}{dr}\right|_{r=1} r^{q+1} = (q+1)$, we have

$$\lim_{r\to 1} \frac{r^{q+1}-1}{r-1} = \left.\frac{d}{dr}\right|_{r=1} r^{q+1} = q+1\,;$$

thus by the fact that $r \to 1$ as $n \to \infty$ (or $\|\mathcal{P}\| \to 0$), we find that

$$\lim_{\|\mathcal{P}\|\to 0} L(\mathcal{P}) = \lim_{\|\mathcal{P}\|\to 0} L(\mathcal{P}) = \frac{b^{q+1} - a^{q+1}}{q+1}\,.$$

Therefore, $\int_a^b x^q\,dx = \frac{b^{q+1} - a^{q+1}}{q+1}$ if $q \neq 1$ is a rational number and $0 < a < b$.

Example 4.12. Since the sine function is continuous on any closed interval $[a, b]$, to find $\int_a^b \sin x\,dx$ we can partition $[a, b]$ into sub-intervals with equal length, use the right end-point rule to find an approximated value of the integral, and finally find the integral by passing the number of sub-intervals to the limit.

Let $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. The right end-point rule gives the approximation

$$\sum_{i=1}^{n} \sin x_i \Delta x = \sum_{i=1}^{n} \sin(a + i\Delta x)\Delta x = \Delta x \sum_{i=1}^{n} \sin(a + i\Delta x)$$

of the integral.

Using the sum and difference formula, we find that

$$\cos\left[a + \left(i - \frac{1}{2}\right)\Delta x\right] - \cos\left[a + \left(i + \frac{1}{2}\right)\Delta x\right] = 2\sin(a + i\Delta x)\sin\frac{\Delta x}{2}\,;$$

thus if $\sin\dfrac{\Delta x}{2} \neq 0$,

$$\sum_{i=1}^{n}\sin(a+i\Delta x) = \frac{1}{2\sin\frac{\Delta x}{2}}\Big[\Big(\cos\big(a+\frac{1}{2}\Delta x\big) - \cos\big(a+\frac{3}{2}\Delta x\big)\Big) + \Big(\cos\big(a+\frac{3}{2}\Delta x\big) - \cos\big(a+\frac{5}{2}\Delta x\big)\Big) + \cdots + \cos\big[a+(n-\frac{1}{2})\Delta x\big] - \cos\big[a+(n+\frac{1}{2})\Delta x\big]\Big]$$

which, by the fact that $a + \big(n+\frac{1}{2}\Delta x\big) = b + \frac{1}{2}\Delta x$, implies that

$$\sum_{i=1}^{n}\sin x_i\Delta x = \frac{\frac{\Delta x}{2}}{\sin\frac{\Delta x}{2}}\Big[\cos\big(a+\frac{1}{2}\Delta x\big) - \cos\big(b+\frac{1}{2}\Delta x\big)\Big].$$

By the fact that $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$ and the continuity of the cosine function, we conclude that

$$\int_a^b \sin x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\sin x_i\Delta x = \cos a - \cos b\,.$$

Theorem 4.13

Let $f : [a,b] \to \mathbb{R}$ be a non-negative and continuous function. The area of the region enclosed by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is $\displaystyle\int_a^b f(x)\,dx$.

Example 4.14. In this example we use the integral notation to denote the areas of some common geometric figures (without really doing computations):

1. $\displaystyle\int_{-2}^{2}\sqrt{4-x^2}\,dx = 2\pi\,;$ 2. $\displaystyle\int_{-1}^{1}\sqrt{4-x^2}\,dx = \frac{2\pi}{3}+\sqrt{3}\,;$ 3. $\displaystyle\int_{-1}^{\sqrt{3}}\sqrt{4-x^2}\,dx = \pi+\sqrt{3}.$

4.2.1 Properties of Definite Integrals

Definition 4.15

1. If f is defined at $x = a$, then $\displaystyle\int_a^a f(x)\,dx = 0$.
2. If f is integrable on $[a,b]$, then $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx = -\int_{[a,b]} f(x)\,dx$.

Remark 4.16. By the definition above, if f is Riemann integrable on $[a,b]$, $\displaystyle\int_b^a f(x)\,dx$ is the limit of the sum

$$\sum_{i=1}^{n} f(x_i)(x_i - x_{i-}) \quad \textit{and} \quad \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

as $\max\big\{|x_i - x_{i-1}|\,\big|\,1 \leqslant i \leqslant n\big\} \to 0$, where $x_0 = b > x_1 > x_2 > \cdots > x_n = a$.

Theorem 4.17

If f is Riemann integrable on the three closed intervals determined by a, b and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,.$$

Theorem 4.18

Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$ and k be a constant. Then the function $kf \pm g$ are Riemann integrable on $[a,b]$, and

$$\int_a^b (kf \pm g)(x)\,dx = k\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx\,.$$

Theorem 4.19

If f is non-negative and Riemann integrable on $[a,b]$, then $\displaystyle\int_a^b f(x)\,dx \geqslant 0$.

Corollary 4.20

If f, g are Riemann integrable on $[a,b]$ and $f(x) \leqslant g(x)$ for all $a \leqslant x \leqslant b$, then

$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx\,.$$

Theorem 4.21

If f is Riemann integrable on $[a,b]$, then $|f|$ is Riemann integrable on $[a,b]$ and

$$\Big|\int_a^b f(x)\,dx\Big| \leqslant \int_a^b \big|f(x)\big|\,dx\,.$$

Theorem 4.22: 可積必有界

Let $f : [a,b] \to \mathbb{R}$ be a function. If f is Riemann integrable on $[a,b]$, then f is bounded on $[a,b]$; that is, there exists $M > 0$ such that

$$\big|f(x)\big| \leqslant M \qquad \text{whenever} \quad x \in [a,b]\,.$$

Proof. Let f be Riemann integrable on $[a,b]$. Then there exists $A \in \mathbb{R}$ and $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a,b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sum of f for $\mathcal{P}$ belongs to $(A-1, A+1)$. Choose $n \in \mathbb{N}$ so that $\dfrac{b-a}{n} < \delta$. Then the regular partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$, where $x_i = a + \dfrac{b-a}{n} i$, satisfies $\|\mathcal{P}\| < \delta$.

Suppose the contrary that f is not bounded. Then there exists $x^* \in [a,b]$ such that

$$\big|f(x^*)\big| > \frac{n(|A|+1)}{b-a} + \sum_{i=1}^{n} \big|f(x_i)\big|\,.$$

Suppose that $x^* \in [x_{k-1}, x_k]$. By the fact that $\sum\limits_{\substack{i=1 \\ i\neq k}}^{n} f(x_i)(x_i - x_{i-1}) + f(x^*)(x_k - x_{k-1})$ is a Riemann sum of f for $\mathcal{P}$, we have

$$A - 1 < \sum_{\substack{i=1 \\ i\neq k}}^{n} f(x_i)(x_i - x_{i-1}) + f(x^*)(x_k - x_{k-1}) < A + 1\,.$$

Since $x_i - x_{i-1} = \dfrac{b-1}{n}$ for all $1 \leqslant i \leqslant n$, the inequality above shows that

$$\frac{n(A-1)}{b-a} - \sum_{\substack{i=1 \\ i\neq k}}^{n} f(x_i) < f(x^*) < \frac{n(A+1)}{b-a} - \sum_{\substack{i=1 \\ i\neq k}}^{n} f(x_i)$$

and the triangle inequality further implies that

$$-\Big[\frac{n(|A|+1)}{b-a} + \sum_{\substack{i=1 \\ i\neq k}}^{n} \big|f(x_i)\big|\Big] < f(x^*) < \frac{n(|A|+1)}{b-a} + \sum_{\substack{i=1 \\ i\neq k}}^{n} \big|f(x_i)\big|\,.$$

Therefore, we conclude that

$$\big|f(x^*)\big| < \frac{n(|A|+1)}{b-a} + \sum_{\substack{i=1 \\ i\neq k}}^{n} \big|f(x_i)\big| \leqslant \frac{n(|A|+1)}{b-a} + \sum_{i=1}^{n} \big|f(x_i)\big|\,,$$

a contradiction. □

Example 4.23. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \in (0, 1]\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then f has only one discontinuity in $[0, 1]$ but f is not Riemann integrable on $[0, 1]$ since f is not bounded.

4.3 The Fundamental Theorem of Calculus

In this section, we develop a theory which shows a systematic way of finding integrals if the integrand is a continuous function.

Definition 4.24

A function F is an anti-derivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem 4.25

If F is an anti-derivative of f on an interval I, then G is an anti-derivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$ for all x in I, where C is a constant. (導函數相同的函數相差一常數)

Proof. It suffices to show the "⇒" (only if) direction. Suppose that $F' = G' = f$ on I. Then the function $h = F - G$ satisfies $h'(x) = 0$ for all $x \in I$. By the mean value theorem, for any $a, b \in I$ with $a \neq b$, there exists c in between a and b such that

$$h(b) - h(a) = h'(c)(b - a)\,.$$

Since $h'(x) = 0$ for all $x \in I$, $h(a) = h(b)$ for all $a, b \in I$; thus h is a constant function. □

Theorem 4.26: Mean Value Theorem for Integrals - 積分均值定理

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a)\,.$$

Proof. By the Extreme Value Theorem, f has a maximum and a minimum on $[a, b]$. Let $M = f(x_1)$ and $m = f(x_2)$, where $x_1, x_2 \in [a, b]$, denote the maximum and minimum of f

on $[a,b]$, respectively. Then $m \leqslant f(x) \leqslant M$ for all $x \in [a,b]$; thus Corollary 4.20 implies that

$$m(b-a) = \int_a^b m\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b M\,dx = M(b-a)\,.$$

Therefore, the number $\frac{1}{b-a}\int_a^b f(x)\,dx \in [m,M]$. By the Intermidiate Value Theorem, there exists c in between x_1 and x_2 such that $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$. □

Theorem 4.27: Fundamental Theorem of Calculus - 微積分基本定理

Let $f:[a,b]\to\mathbb{R}$ be a continuous function, and F be an anti-derivative of f on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Moreover, if $G(x) = \int_a^x f(t)\,dt$ for $x \in [a,b]$, then G is an anti-derivative of f.

We note that for $x \in [a,b]$, f is continuous on $[a,x]$; thus f is Riemann integrable on $[a,x]$ which shows that $G(x) = \int_a^x f(t)\,dt$ is well-defined.

Proof of the Fundamental Theorem of Calculus. Note that for $h \neq 0$ such that $x+h \in [a,b]$, we have

$$\frac{G(x+h)-G(x)}{h} = \frac{1}{h}\Big[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\Big] = \frac{1}{h}\int_x^{x+h} f(t)\,dt\,.$$

By the Mean Value Theorem for Integrals, there exists $c = c(h)$ in between x and $x+h$ such that $\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c)$. Since f is continuous on $[a,b]$, $\lim\limits_{h\to0} f(c) = \lim\limits_{c\to x} f(c) = f(x)$; thus

$$\lim_{h\to0}\frac{G(x+h)-G(x)}{h} = \lim_{h\to0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to0} f(c) = f(x)$$

which shows that G is an anti-derivative of f on $[a,b]$.

By Theorem 4.25, $G(x) = F(x) + C$ for all $x \in [a,b]$. By the fact that $G(a) = 0$, $C = -F(a)$; thus

$$\int_a^b f(x)\,dx = G(b) = F(b) - F(a)$$

which concludes the theorem. □

Example 4.28. Since an anti-derivative of the function $y = x^q$, where $q \neq -1$ is a rational number, is $y = \dfrac{x^{q+1}}{q+1}$, we find that

$$\int_a^b x^q\,dx = \frac{x^{q+1}}{q+1}\Big|_{x=b} - \frac{x^{q+1}}{q+1}\Big|_{x=a} = \frac{b^{q+1}-a^{q+1}}{q+1}\,.$$

Example 4.29. Since an anti-derivative of the sine function is negative of cosine, we find that

$$\int_a^b \sin x\,dx = (-\cos)(b) - (-\cos)(b) = \cos b - \cos a\,.$$

Example 4.30. Find $\dfrac{d}{dx}\displaystyle\int_0^{\sqrt{x}} \sin^{100} t\,dt$ for $x > 0$.

Let $F(x) = \displaystyle\int_0^x \sin^{100} t\,dt$. Then by the chain rule,

$$\frac{d}{dx}F(\sqrt{x}) = F'(\sqrt{x})\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}F'(\sqrt{x})\,.$$

By the Fundamental Theorem of Calculus, $F'(x) = \sin^{100} x$; thus

$$\frac{d}{dx}\int_0^{\sqrt{x}} \sin^{100} t\,dt = \frac{d}{dx}F(\sqrt{x}) = \frac{\sin^{100}\sqrt{x}}{2\sqrt{x}}\,.$$

Theorem 4.31

Let $f : [a, b] \to \mathbb{R}$ be continuous and f is differentiable on (a, b). If f' is Riemann integrable on $[a, b]$, then

$$\int_a^b f'(x)\,dx = f(b) - f(a)\,.$$

Proof. Let $\varepsilon > 0$ be given, and define $A = \displaystyle\int_a^b f'(x)\,dx$. By the definition of the integrability there exists $\delta > 0$ such that if $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a, b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums of f for $\mathcal{P}$ belongs to the interval $(A-\varepsilon, A+\varepsilon)$.

Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$ satisfying that $\|\mathcal{P}\| < \delta$. Then by the mean value theorem, for each $1 \leqslant i \leqslant n$ there exists $x_{i-1} < c < x_i$ such that $f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1})$. Since

$$\sum_{i=1}^n f'(c_i)(x_i - x_{i-1})$$

is a Riemann sum of f for $\mathcal{P}$, we must have

$$\Big|\sum_{i=1}^{n} f'(c_i)(x_i - x_{i-1}) - A\Big| < \varepsilon\,.$$

On the other hand, by the fact that

$$\begin{aligned}\sum_{i=1}^{n} f'(c_i)(x_i - x_{i-1}) &= \sum_{i=1}^{n}\big[f(x_i) - f(x_{i-1})\big]\\ &= f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots + f(x_n) - f(x_{n-1})\\ &= f(x_n) - f(x_0) = f(b) - f(a)\,,\end{aligned}$$

we conclude that

$$\Big|f(b) - f(a) - \int_a^b f'(x)\,dx\Big| < \varepsilon\,.$$

Since $\varepsilon > 0$ is chosen arbitrarily, we find that $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$. □

Remark 4.32. If f' is continuous on $[a, b]$, then the theorem above is simply a direct consequence of the Fundamental Theorem of Calculus. The theorem above can be viewed as a generalization of the Fundamental Theorem of Calculus.

Theorem 4.27 and 4.31 can be combined as follows:

Theorem 4.33

Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function and F be an anti-derivative of f on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Moreover, if in addition f is continuous on $[a, b]$, then $G(x) = \displaystyle\int_a^x f(t)\,dt$ is differentiable on $[a, b]$ and

$$G'(x) = f(x) \qquad \text{for all } x \in [a, b]\,.$$

Definition 4.34

An anti-derivative of f, if exists, is denoted by $\displaystyle\int f(x)\,dx$, and sometimes is also called an indefinite integral of f.

- Basic Rules of Integration:

Differentiation Formula	Anti-derivative Formula
$\frac{d}{dx}C = 0$	$\int 0\,dx = C$
$\frac{d}{dx}x^r = rx^{r-1}$	$\int x^q\,dx = \frac{x^{q+1}}{q+1} + C$ if $q \neq -1$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}\big[kf(x) + g(x)\big] = kf'(x) + g'(x)$	$\int \big[kf'(x) + g'(x)\big]\,dx = kf(x) + g(x) + C$

4.4 Integration by Substitution - 變數變換

Suppose that $g : [a, b] \to \mathbb{R}$ is differentiable, and $f : \text{range}(g) \to \mathbb{R}$ is differentiable. Then the chain rule implies that $f \circ g$ is an anti-derivative of $(f' \circ g)g'$; thus provided that

1. $(f \circ g)'$ is Riemann integrable on $[a, b]$,
2. f' is Riemann integrable on the range of g,

then Theorem 4.31 implies that

$$\begin{aligned}\int_a^b f'\big(g(x)\big)g'(x)\,dx &= \int_a^b (f \circ g)'(x)\,dx = (f \circ g)(b) - (f \circ g)(a) \\ &= f\big(g(b)\big) - f\big(g(a)\big) = \int_{g(a)}^{g(b)} f'(u)\,du\,. \end{aligned} \tag{4.4.1}$$

Replacing f' by f in the identity above shows the following

Theorem 4.35

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$, and f is continuous on the range of g, then

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du\,.$$

The anti-derivative version of Theorem 4.35 is stated as follows.

Theorem 4.36

Let g be a function with range I and f be a continuous function on I. If g is differentiable on its domain and F is an anti-derivative of f on I, then

$$\int f\big(g(x)\big)g'(x)\,dx = F\big(g(x)\big) + C$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C\,.$$

Example 4.37. Find $\displaystyle\int (x^2+1)^2(2x)\,dx$.

Let $u = x^2+1$. Then $du = 2xdx$; thus

$$\int (x^2+1)^2(2x)\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(x^2+1)^3 + C\,.$$

Example 4.38. Find $\displaystyle\int \cos(5x)\,dx$.

Let $u = 5x$. Then $du = 5dx$; thus

$$\int \cos(5x)\,dx = \frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin(5x) + C\,.$$

Example 4.39. Find $\displaystyle\int \sec^2 x(\tan x + 3)\,dx$.

Let $u = \tan x$. Then $du = \sec^2 x dx$; thus

$$\int \sec^2 x(\tan x+3)\,dx = \int (u+3)\,du = \frac{1}{2}u^2 + 3u + C = \frac{1}{2}\tan^2 x + 3\tan x + C\,.$$

On the other hand, let $v = \tan x + 3$. Then $dv = \sec^2 x\,dx$; thus

$$\begin{aligned}\int \sec^2 x(\tan x+3)\,dx &= \int v\,dv = \frac{1}{2}v^2 + C = \frac{1}{2}(\tan x+3)^2 + C\\ &= \frac{1}{2}\tan^2 x + 3\tan x + \frac{9}{2} + C\,.\end{aligned}$$

We note that even though the right-hand side of the two indefinite integrals look different, they are in fact the same since C could be any constant, and $\dfrac{9}{2} + C$ is also any constant.

Example 4.40. Find $\displaystyle\int \frac{2zdz}{\sqrt[3]{z^2+1}}$.

Method 1: Let $x = z^2+1$. Then $dx = 2zdz$; thus

$$\int \frac{2zdz}{\sqrt[3]{z^2+1}} = \int \frac{dx}{\sqrt[3]{x}} = \int x^{-\frac{1}{3}}\,dx = \frac{3}{2}x^{\frac{2}{3}} + C = \frac{3}{2}(z^2+1)^{\frac{2}{3}} + C\,.$$

Method 2: Let $y = \sqrt[3]{z^2+1}$. Then $y^3 = z^2+1$; thus $3y^2dy = 2zdz$. Therefore,

$$\int \frac{2zdz}{\sqrt[3]{z^2+1}} = \int \frac{3y^2dy}{y} = \int 3y\,dy = \frac{3}{2}y^2 + C = \frac{3}{2}(z^2+1)^{\frac{2}{3}} + C\,.$$

Example 4.41. Find $\displaystyle\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx$.

Let $u = 2+\tan^3 x$. Then $du = 3\tan^2 x\sec^x\,dx$; thus

$$\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx = \int \frac{6du}{u^2} = 6\int u^{-2}\,du = -6u^{-1} + C = -\frac{6}{2+\tan^3 x} + C\,.$$

Sometimes an definite integral can be evaluated even though the anti-derivative of the integrand cannot be found. In such kind of cases, we have to look for special structures so that we can simplify the integrals. There is no general rule for this, and we have to do this case by case.

Example 4.42. Find $\displaystyle\int_0^\pi \frac{2x\sin x}{3+\cos(2x)}\,dx$.

Let the integral be I. By the substitution $u = \pi - x$, we find that

$$\begin{aligned} \mathrm{I} &= \int_\pi^0 \frac{2(\pi-u)\sin(\pi-u)}{3+\cos(2(\pi-u))}(-1)\,du = \int_0^\pi \frac{2(\pi-u)\sin u}{3+\cos 2u}\,du \\ &= \int_0^\pi \frac{2\pi\sin u}{3+\cos 2u}\,du - \int_0^\pi \frac{2u\sin u}{3+\cos 2u}\,du = 2\pi\int_0^\pi \frac{\sin u}{3+\cos 2u}\,du - \mathrm{I}\,; \end{aligned}$$

thus

$$\begin{aligned} \mathrm{I} &= \pi\int_0^\pi \frac{\sin u}{3+\cos 2u}\,du = -\pi\int_0^\pi \frac{d(\cos u)}{3+2\cos^2 u - 1} = -\frac{\pi}{2}\int_1^{-1} \frac{dv}{v^2+1} \\ &= \frac{\pi}{2}\int_{-1}^1 \frac{dv}{v^2+1} = \frac{\pi}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 y}{\tan^2 y+1}\,dy = \frac{\pi}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dy = \frac{\pi^2}{4}\,. \end{aligned}$$