Calculus 微積分

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Chapter 2

Differentiation

2.1 The Derivatives of Functions

Definition 2.1

Let f be a function defined on an open interval containing c. If the limit $\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$ exists, then the line passing through $\big(c, f(c)\big)$ with slope m is the tangent line to the graph of f at point $\big((c, f(c)\big)$.

Definition 2.2

Let f be a function defined on an open interval I containing c. f is said to be differentiable at c if the limit

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

exists. If the limit above exists, the limit is denoted by $f'(c)$ and called the derivative of f at c. When the derivative of f at each point of I exists, f is said to be differentiable on I and the derivative of f is a function denoted by f'.

• **Notation**: The prime notation $'$ is associated with a function (of one variable) and is used to denote the derivative of that function. For a given function f defined on an open interval I and x being the name of the variable, the limit operation

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

is denoted by $\dfrac{d}{dx}f(x)$ $\big($or $\dfrac{df(x)}{dx}$ or even $\dfrac{dy}{dx}$ if $y = f(x)\big)$, and the limit

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta) - f(c)}{\Delta x}$$

is denoted by $\dfrac{d}{dx}\Big|_{x=c} f(x)$ but not $\dfrac{d}{dx}f(c)$ $\big(\frac{d}{dx}f(c)$ is in fact $0\big)$. The operator $\dfrac{d}{dx}$ is a differential operator called the differentiation and is applied to functions of variable x. However, for historical (and convenient) reason, $\dfrac{d}{dx}f(x)$ is sometimes denoted by $(f(x))'$ (so that $'$ is treated as the differential operator $\dfrac{d}{dx}$) and f' is sometimes denoted by $\dfrac{df}{dx}$ (so that f is always treated as a function of variable x).

Remark 2.3. Letting $x = c + \Delta x$ in the definition of the derivatives, then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

if the limit exists.

Example 2.4. Let f be a constant function. Then f' is the zero function.

Example 2.5. Let $f(x) = x^n$, where n is a positive integer. Then

$$f(x + \Delta x) = x^n + C^n_1 x^{n-1}\Delta x + C^n_2 x^{n-2}(\Delta x)^2 + \cdots + C^n_{n-1} x(\Delta x)^{n-1} + (\Delta x)^n\,;$$

thus if $\Delta x \neq 0$,

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = nx^{n-1} + C^n_2 x^{n-2}\Delta x + \cdots + C^n_{n-1} x(\Delta x)^{n-2} + (\Delta x)^{n-1}\,.$$

The limit on the right-hand side is clearly nx^{n-1}, so we establish that

$$\frac{d}{dx}x^n = nx^{n-1}\,.$$

Example 2.6. Now suppose that $f(x) = x^{-n}$, where n is a positive integer. Then if $x + \Delta x \neq 0$,

$$f(x + \Delta x) = \frac{1}{x^n + C^n_1 x^{n-1}\Delta x + C^n_2 x^{n-2}(\Delta x)^2 + \cdots + C^n_{n-1} x(\Delta x)^{n-1} + (\Delta x)^n}\,;$$

thus if $x \neq 0$, $\Delta x \neq 0$, and $x + \Delta x \neq 0$ (which can be achieved if $|\Delta x| \ll 1$),

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{-\big[C^n_1 x^{n-1} + C^n_2 x^{n-2}\Delta x + \cdots + C^n_{n-1} x(\Delta x)^{n-2} + (\Delta x)^{n-1}\big]}{x^n\big[x^n + C^n_1 x^{n-1}\Delta x + C^n_2 x^{n-2}(\Delta x)^2 + \cdots + C^n_{n-1} x(\Delta x)^{n-1} + (\Delta x)^n\big]}\,.$$

Therefore, if $x \neq 0$,

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}$$

which shows $\dfrac{d}{dx} x^{-n} = -nx^{-n-1}$.

Combining the previous three examples, we conclude that

$$\frac{d}{dx} x^n = \begin{cases} nx^{n-1} & \forall\, x \in \mathbb{R} \quad \text{if } n \in \mathbb{N} \cup \{0\}\,, \\ nx^{n-1} & \forall\, x \neq 0 \quad \text{if } n \in \mathbb{Z} \text{ and } n < 0\,. \end{cases} \tag{2.1.1}$$

Combining Example 2.4-2.6, we conclude that

$$\frac{d}{dx} x^n = \begin{cases} nx^{n-1} & \forall\, x \in \mathbb{R} \quad \text{if } n \in \mathbb{N} \cup \{0\}\,, \\ nx^{n-1} & \forall\, x \neq 0 \quad \text{if } n \in \mathbb{Z} \text{ and } n < 0\,. \end{cases} \tag{2.1.2}$$

我們注意到當 n 是負整數時，在計算 $\dfrac{d}{dx}\Big|_{x=c} x^n$ 時，已經必須先假設 $c \neq 0$ 才能計算導數，並非最後算出來 $\dfrac{d}{dx}\Big|_{x=c} x^n = nc^{n-1}$ 時發現 c 不可為零所以不能代入。這是一個非常重要的觀念！不能搞錯順序！

Example 2.7. Let $f(x) = \sin x$. By the sum and difference formula,

$$\begin{aligned} f(x+\Delta x) - f(x) &= \sin(x+\Delta x) - \sin x = \sin x \cos \Delta x + \sin \Delta x \cos x - \sin x \\ &= \sin x(\cos \Delta x - 1) + \sin \Delta x \cos x\,; \end{aligned}$$

thus by the fact that $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\displaystyle\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$, we find that

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \left[\sin x \frac{\cos \Delta x - 1}{\Delta x} + \frac{\sin \Delta x}{\Delta x} \cos x\right] = \cos x\,. \tag{2.1.3}$$

In other words, the derivative of the sine function is cosine.

On the other hand, let $g(x) = \cos x$. Then $g(x) = -f\big(x - \frac{\pi}{2}\big)$. Then if $\Delta x \neq 0$,

$$\frac{g(x+\Delta x) - g(x)}{\Delta x} = -\frac{f\big(x - \frac{\pi}{2} + \Delta x\big) - f\big(x - \frac{\pi}{2}\big)}{\Delta x}\,;$$

thus

$$\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} = -\cos\big(x - \frac{\pi}{2}\big) = -\sin x\,.$$

In other words, the derivative of the cosine function is minus sine. To summarize,

$$\frac{d}{dx} \sin x = \cos x \qquad \text{and} \qquad \frac{d}{dx} \cos x = -\sin x\,. \tag{2.1.4}$$

Example 2.8. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x^2 & \text{if } x \text{ is rational}\,, \\ -x^2 & \text{if } x \text{ is irrational}\,. \end{cases}$$

Then $g(x) = xf(x)$, where f is given in Example 1.22. By the fact that $\lim\limits_{x\to 0} f(x) = 0$,

$$\lim_{\Delta x \to 0} \frac{g(\Delta x) - g(0)}{\Delta x} = \lim_{\Delta x \to 0} f(\Delta x) = 0\,.$$

In other words, g is differentiable at 0. Moreover, similar argument used to explain that the function f in Example 1.22 is only continuous at 0 can be used to show that the function g is only continuous at 0. Therefore, we obtain a function which is differentiable at one point but discontinuous elsewhere.

Remark 2.9. If f is a function defined on a interval I, and c is one of the end-point. Then it is possible to define the one-sided derivative. For example, if c is the left end-point of I, then we can consider the limit

$$\lim_{\Delta x \to 0^+} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

if it exists. The limit above, if exists, is called the derivatives of f at c from the right.

Theorem 2.10: 可微必連續

Let f be a function defined on an open interval I, and $c \in I$. If f is differentiable at c, then f is continuous at c.

Proof. If $x \neq c$, $f(x) - f(c) = \dfrac{f(x) - f(c)}{x - c}(x - c)$. Since the limit $\lim\limits_{x\to c} \dfrac{f(x) - f(c)}{x - c}$ exists and $\lim\limits_{x\to c}(x - c) = 0$, by Theorem 1.14 we conclude that

$$\lim_{x\to c} \big[f(x) - f(c)\big] = \Big(\lim_{x\to c} \frac{f(x) - f(c)}{x - c}\Big)\Big(\lim_{x\to c}(x - c)\Big) = 0\,.$$

Therefore, $\lim\limits_{x\to c} f(x) = f(c)$ which shows that f is continuous at c. □

Remark 2.11. When f is continuous on an open interval I, f is **not** necessary differentiable on I. For example, consider $f(x) = |x|$. Then Theorem 1.14 implies that f is continuous on I, but $\lim\limits_{\Delta x\to 0} \dfrac{f(\Delta x) - f(0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{|\Delta x|}{\Delta x}$ D.N.E.

2.2 Rules of Differentiation

Theorem 2.12

We have the following differentiation rules:

1. If k is a constant, then $\dfrac{d}{dx}k = 0$.

2. If n is a non-zero integer, then $\dfrac{d}{dx}x^n = nx^{n-1}$ (whenever x^{n-1} makes sense).

3. $\dfrac{d}{dx}\sin x = \cos x$, $\dfrac{d}{dx}\cos x = -\sin x$.

4. If k is a constant and $f : (a,b) \to \mathbb{R}$ is differentiable at $c \in (a,b)$, then kf is differentiable at c and
$$\frac{d}{dx}\Big|_{x=c}\big[kf(x)\big] = kf'(c)\,.$$

5. If $f, g : (a,b) \to \mathbb{R}$ are differentiable at $c \in (a,b)$, then $f \pm g$ is differentiable at c and
$$\frac{d}{dx}\Big|_{x=c}\big[f(x) \pm g(x)\big] = f'(c) \pm g'(c)\,.$$

Proof of 5. Let $h(x) = f(x) + g(x)$. Then if $\Delta x \neq 0$,

$$\frac{h(c+\Delta x) - h(c)}{\Delta x} = \frac{f(c+\Delta x) - f(c)}{\Delta x} + \frac{g(c+\Delta x) - g(c)}{\Delta x}\,.$$

Since f, g are differentiable at c,

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = f'(c) \qquad \text{and} \qquad \lim_{\Delta x \to 0} \frac{g(c+\Delta x) - g(c)}{\Delta x}$$

exist. Therefore, by Theorem 1.14,

$$h'(c) = f'(c) + g'(c)\,.$$

The conclusion for the difference can be proved in the same way. □

Example 2.13. Let $f(x) = 3x^2 - 5x + 7$. Then

$$\begin{aligned}\frac{d}{dx}f(x) &= \frac{d}{dx}(3x^2 - 5x) + \frac{d}{dx}7 = \frac{d}{dx}(3x^2) - \frac{d}{dx}(5x)\\ &= 3\frac{d}{dx}x^2 - 5\frac{d}{dx}x = 3\cdot(2x) - 5 = 6x - 5\,.\end{aligned}$$

In general, for a polynomial function

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \equiv \sum_{k=0}^{n} a_k x^k \,,$$

where $a_0, a_1, \cdots, a_n \in \mathbb{R}$, by induction we can show that

$$\frac{d}{dx} p(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1 = \sum_{k=1}^{n} k a_k x^{k-1} \,.$$

Theorem 2.14: Product Rule

Let $f, g : (a, b) \to \mathbb{R}$ be real-valued functions, and $c \in (a, b)$. If f and g are differentiable at c, then fg is differentiable at c and

$$\frac{d}{dx}\Big|_{x=c} (fg)(x) = f'(c)g(c) + f(c)g'(c) \,.$$

Proof. Let $h(x) = f(x)g(x)$. Then

$$\begin{aligned} h(c+\Delta x) - h(c) &= f(c+\Delta x)g(c+\Delta x) - f(c)g(c) \\ &= f(c+\Delta x)g(c+\Delta x) - f(c)g(c+\Delta x) + f(c)g(c+\Delta x) - f(c)g(c) \\ &= \big[f(c+\Delta x) - f(c)\big]g(c+\Delta x) + f(c)\big[g(c+\Delta x) - g(c)\big] \,. \end{aligned}$$

Therefore, if $\Delta x \neq 0$,

$$\frac{h(c+\Delta x) - h(c)}{\Delta x} = \frac{f(c+\Delta x) - f(c)}{\Delta x} g(c+\Delta x) + f(c) \frac{g(c+\Delta x) - g(c)}{\Delta x} \,.$$

Since f, g are differentiable at c,

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = f'(c) \,, \lim_{\Delta x \to 0} \frac{g(c+\Delta x) - g(c)}{\Delta x} \,, \text{ and } \lim_{\Delta x \to 0} g(c+\Delta x) = g(c)$$

exist. By Theorem 1.14,

$$h'(c) = f'(c)g(c) + f(c)g'(c)$$

which concludes the product rule. □

Example 2.15. Let $f(x) = x^3 \sin x$. Then the product rule implies that

$$f'(x) = 3x^2 \sin x + x^3 \cos x \,.$$

Theorem 2.16: Quotient Rule

Let $f, g : (a, b) \to \mathbb{R}$ be real-valued functions, and $c \in (a, b)$. If f and g are differentiable at c and $g(c) \neq 0$, then $\dfrac{f}{g}$ is differentiable at c and

$$\frac{d}{dx}\Big|_{x=c} \frac{f}{g}(x) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.$$

Proof. Let $h(x) = \dfrac{f(x)}{g(x)}$. Then

$$\begin{aligned} h(c + \Delta x) - h(c) &= \frac{f(c + \Delta x)}{g(c + \Delta x)} - \frac{f(c)}{g(c)} = \frac{f(c + \Delta x)g(c) - f(c)g(c + \Delta x)}{g(c)g(c + \Delta x)} \\ &= \frac{f(c + \Delta x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(c + \Delta x)}{g(c)g(c + \Delta x)} \\ &= \frac{\big[f(c + \Delta x) - f(c)\big]g(c) - f(c)\big[g(c + \Delta x) - g(c)\big]}{g(c)g(c + \Delta x)}. \end{aligned}$$

Therefore, if $\Delta x \neq 0$,

$$\frac{h(c + \Delta x) - h(c)}{\Delta x} = \frac{1}{g(c)g(c + \Delta x)}\Big[\frac{f(c + \Delta x) - f(c)}{\Delta x} g(c) - f(c)\frac{g(c + \Delta x) - g(c)}{\Delta x}\Big].$$

Since f, g are differentiable at c,

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = f'(c)\,, \lim_{\Delta x \to 0} \frac{g(c + \Delta x) - g(c)}{\Delta x}\,, \text{ and } \lim_{\Delta x \to 0} g(c + \Delta x) = g(c)$$

exist. By Theorem 1.14,

$$h'(c) = \frac{1}{g(c)^2}\Big[f'(c)g(c) - f(c)g'(c)\Big]$$

which concludes the quotient rule. □

Remark 2.17. Suppose that in addition to the assumption in Theorem 2.16 one has already known that $h = f/g$ is differentiable at c, then applying the product rule to $f = gh$ one finds that

$$f'(c) = g'(c)h(c) + g(c)h'(c) = g'(c)\frac{f(c)}{g(c)} + g(c)h'(c)$$

which, after rearranging terms, shows the quotient rule. The proof of Theorem 2.16 indeed is based on the fact that we do not know the differentiability of h at c yet.

Example 2.18. Let n be a positive integer and $f(x) = x^{-n}$. We have shown by definition that $f'(x) = -nx^{-n-1}$ if $x \neq 0$. Now we use Theorem 2.16 to compute the derivative of f: if $x \neq 0$,

$$\frac{d}{dx}x^{-n} = \frac{d}{dx}\frac{1}{x^n} = -\frac{\frac{d}{dx}x^n}{x^{2n}} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{-n-1}\,.$$

Example 2.19. Since $\tan x = \dfrac{\sin x}{\cos x}$, by Theorem 2.16 we have

$$\frac{d}{dx}\tan x = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\,.$$

Similarly, we also have

$$\begin{aligned}\frac{d}{dx}\cot x &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\csc^2 x\,,\\ \frac{d}{dx}\sec x &= -\frac{-\sin x}{\cos^2 x} = \sec x \tan x\,,\\ \frac{d}{dx}\csc x &= -\frac{\cos x}{\sin^2 x} = -\csc x \cot x\,.\end{aligned}$$

We note that without using the quotient rule, the derivative of the tangent function can be found using the sum-and-difference formula

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}\,. \tag{2.2.1}$$

Using (2.2.1), we find that

$$\tan(x+\Delta x) - \tan x = \tan \Delta x\big[1 + \tan(x+\Delta x)\tan x\big]\,;$$

thus if $\Delta x \neq 0$,

$$\frac{\tan(x+\Delta x) - \tan x}{\Delta x} = \frac{\sin \Delta x}{\Delta x} \cdot \frac{1 + \tan(x+\Delta x)\tan x}{\cos \Delta x}$$

which, using (1.2.2), shows that

$$\lim_{\Delta x \to 0}\frac{\tan(x+\Delta x) - \tan x}{\Delta x} = \Big(\lim_{\Delta x\to 0}\frac{\sin \Delta x}{\Delta x}\Big)\Big(\lim_{\Delta x\to 0}\frac{1 + \tan(x+\Delta x)\tan x}{\cos \Delta x}\Big) = \sec^2 x\,.$$

• **Higher-order derivatives**:

Let f be defined on an open interval $I = (a, b)$. If f' exists on I and possesses derivatives at every point in I, by definition we use f'' to denote the derivative of f'.In other words,

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\frac{d}{dx}f(x) \equiv \frac{d^2}{dx^2}f(x) = \frac{d^2 f(x)}{dx^2}\Big(= \frac{d^2 y}{dx^2} \text{ if } y = f(x)\Big)\,.$$

The function f'' is called the second derivative of f. Similar as the "first" derivative case, $f''(c) = \frac{d^2}{dx^2}\Big|_{x=c} f(x)$.

The third derivatives and even higher-order derivatives are denoted by the following: if $y = f(x)$,

$$\begin{array}{rccc}
\text{Third derivative: } y''' & f'''(x) & \dfrac{d^3}{dx^3}f(x) & \dfrac{d^3 f(x)}{dx^3} \\
\text{Fourth derivative: } y^{(4)} & f^{(4)}(x) & \dfrac{d^4}{dx^4}f(x) & \dfrac{d^4 f(x)}{dx^4} \\
 & \vdots & & \\
\text{n-th derivative: } y^{(n)} & f^{(n)}(x) & \dfrac{d^n}{dx^n}f(x) & \dfrac{d^n f(x)}{dx^n}\,.
\end{array}$$

2.3 The Chain Rule

The chain rule is used to study the derivative of composite functions.

Theorem 2.20: Chain Rule - 連鎖律

Let I, J be open intervals, $f : J \to \mathbb{R}$, $g : I \to \mathbb{R}$ be real-valued functions, and the range of g is contained in J. If g is differentiable at $c \in I$ and f is differentiable at $g(c)$, then $f \circ g$ is differentiable at c and

$$\frac{d}{dx}\Big|_{x=c}(f \circ g)(x) = f'(g(c))g'(c)\,.$$

Proof. To simplify the notation, we set $d = g(c)$.

Let $\varepsilon > 0$ be given. Since f is differentiable at d and g is differentiable at c, there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned}
\Big|\frac{f(d+k)-f(d)}{k} - f'(d)\Big| &< \frac{\varepsilon}{2(1+|g'(c)|)} \quad \text{whenever} \quad 0 < |k| < \delta_1\,, \\
\Big|\frac{g(c+h)-g(c)}{h} - g'(c)\Big| &< \min\Big\{1, \frac{\varepsilon}{2(1+|f'(d)|)}\Big\} \quad \text{whenever} \quad 0 < |h| < \delta_2\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\big|f(d+k)-f(d)-f'(d)k\big| &\leqslant \frac{\varepsilon}{2(1+|g'(c)|)}|k| \quad \text{whenever} \quad |k| < \delta_1\,, \\
\big|g(c+h)-g(c)-g'(c)h\big| &\leqslant \min\Big\{1, \frac{\varepsilon}{2(1+|f'(d)|)}\Big\}|h| \quad \text{whenever} \quad |h| < \delta_2\,.
\end{aligned}$$

By Theorem 2.10, g is continuous at c; thus $\lim\limits_{h\to 0} g(c+h) = g(c)$. This fact provides $\delta_3 > 0$ such that

$$\big|g(c+h) - g(c)\big| < \delta_1 \quad \text{whenever} \quad |h| < \delta_3\,.$$

Define $\delta = \min\{\delta_2, \delta_3\}$. Then $\delta > 0$. Moreover, if $|h| < \delta$, the number $k \equiv g(c+h) - g(c)$ satisfies $|k| < \delta_1$. As a consequence, if $|h| < \delta$,

$$\begin{aligned}
&\big|(f\circ g)(c+h) - (f\circ g)(c) - f'(d)g'(c)h\big| = \big|f(g(c+h)) - f(d) - f'(d)g'(c)h\big| \\
&\quad = \big|f(d+k) - f(d) - f'(d)g'(c)h\big| \\
&\quad = \big|f(d+k) - f(d) - f'(d)k + f'(d)k - f'(d)g'(c)h\big| \\
&\quad \leqslant \big|f(d+k) - f(d) - f'(d)k\big| + \big|f'(d)\big|\big|k - g'(c)h\big| \\
&\quad \leqslant \frac{\varepsilon}{2(1+|g'(c)|)}|k| + \big|f'(d)\big|\big|g(c+h) - g(c) - g'(c)h\big| \\
&\quad \leqslant \frac{\varepsilon}{2(1+|g'(c)|)}\big(|k - g'(c)h| + |g'(c)||h|\big) + \big|f'(d)\big|\frac{\varepsilon}{2(1+|f'(d)|)} \\
&\quad \leqslant \frac{\varepsilon}{2(1+|g'(c)|)}\big(|h| + |g'(c)||h|\big) + \big|f'(d)\big|\frac{\varepsilon|h|}{2(1+|f'(d)|)} \\
&\quad = \frac{\varepsilon}{2}|h| + \frac{\big|f'(d)\big|}{2(1+|f'(d)|)}\varepsilon|h|\,.
\end{aligned}$$

The inequality above implies that if $0 < |h| < \delta$,

$$\Big|\frac{(f\circ g)(c+h) - (f\circ g)(c)}{h} - f'(d)g'(c)\Big| \leqslant \frac{\varepsilon}{2} + \frac{\big|f'(d)\big|}{2(1+|f'(d)|)}\varepsilon < \varepsilon$$

which concludes the chain rule. □

How to memorize the chain rule? Let $y = g(x)$ and $u = f(y)$. Then the derivative $u = (f\circ g)(x)$ is $\dfrac{du}{dx} = \dfrac{du}{dy}\dfrac{dy}{dx}$.

Example 2.21. Let $f(x) = (3x - 2x^2)^3$. Then $f'(x) = 3(3x - 2x^2)^2(3 - 4x)$.

Example 2.22. Let $f(x) = \Big(\dfrac{3x-1}{x^2+3}\Big)^2$. Then

$$\begin{aligned}
f'(x) &= 2\Big(\frac{3x-1}{x^2+3}\Big)^{2-1}\frac{d}{dx}\frac{3x-1}{x^2+3} = \frac{2(3x-1)}{x^2+3}\cdot\frac{3(x^2+3) - 2x(3x-1)}{(x^2+3)^2} \\
&= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3}\,.
\end{aligned}$$

Example 2.23. Let $f(x) = \tan^3\left[(x^2-1)^2\right]$. Then

$$\begin{aligned} f'(x) &= \left\{3\tan^2\left[(x^2-1)^2\right]\sec^2\left[(x^2-1)^2\right]\right\} \times \left[2(x^2-1)\cdot(2x)\right] \\ &= 12x(x^2-1)\tan^2\left[(x^2-1)^2\right]\sec^2\left[(x^2-1)^2\right]. \end{aligned}$$

Example 2.24. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then if $x \neq 0$, by the chain rule we have

$$\begin{aligned} f'(x) &= \Big(\frac{d}{dx}x^2\Big)\sin\frac{1}{x} + x^2\Big(\frac{d}{dx}\sin\frac{1}{x}\Big) = 2x\sin\frac{1}{x} + x^2\cos\frac{1}{x}\Big(\frac{d}{dx}\frac{1}{x}\Big) \\ &= 2x\sin\frac{1}{x} + x^2\cos\frac{1}{x}\Big(-\frac{1}{x^2}\Big) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}\,. \end{aligned}$$

Next we compute $f'(0)$. If $\Delta x \neq 0$, we have

$$\Big|\frac{f(\Delta x) - f(0)}{\Delta x}\Big| = \Big|\Delta x \sin\frac{1}{\Delta x}\Big| \leqslant |\Delta x|\,;$$

thus $-|\Delta x| \leqslant \dfrac{f(\Delta x) - f(0)}{\Delta x} \leqslant |\Delta x|$ for all $\Delta x \neq 0$ and the Squeeze Theorem implies that

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x} = 0\,.$$

Therefore, we conclude that

$$f'(x) = \begin{cases} 2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Definition 2.25

Let f be a function defined on an open interval I. f is said to be continuously differentiable on I if f is differentiable on I and f' is continuous on I.

The function f given in Example 2.24 is differentiable on $\mathbb{R}$ but not continuously differentiable since $\lim\limits_{x \to 0} f'(x)$ D.N.E.

2.4 Implicit Differentiation

An implicit function is a function that is defined implicitly by an equation that x and y satisfy, by associating one of the variables (the value y) with the others (the arguments x). For example, $x^2 + y^2 = 1$ and $x = \cos y$ are implicit functions. Sometimes we know how to express y in terms of x from the equation (such as the first case above $y = \sqrt{1-x^2}$ or $y = -\sqrt{1-x^2}$), while in most cases there is no way to know what the function y of x exactly is.

Given an implicit function (without solving for y in terms of x from the equation), can we find the derivative of y? This is the main topic of this section. We first focus on implicit functions of the form $f(x) = g(y)$. If $f(a) = g(b)$, we are interested in how the set $\{(x,y) \,|\, f(x) = g(y)\}$ looks like "mathematically" near (a,b).

Theorem 2.26: Implicit Function Theorem - 隱函數定理簡單版

Let f, g be continuously differentiable functions defined on some open intervals, and $f(a) = g(b)$. If $g'(b) \neq 0$, then there exists a unique continuously differentiable function $y = h(x)$, defined in an open interval containing a, satisfying that $b = h(a)$ and $f(x) = g(h(x))$.

Example 2.27. Let us compute the derivative of $h(x) = x^r$, where $r = \dfrac{p}{q}$ for some $p, q \in \mathbb{N}$ and $(p,q) = 1$. Write $y = h(x)$. Then $y^q = x^p$. Since $\dfrac{d}{dy} y^q = q y^{q-1} \neq 0$ if $y \neq 0$, by the Implicit Function Theorem we find that h is differentiable at every x satisfying $x \neq 0$. Since $h(x)^q = x^p$, by the chain rule we find that

$$q h(x)^{q-1} h'(x) = p x^{p-1} \qquad \forall\, x \neq 0\,;$$

thus

$$h'(x) = \frac{p}{q} h(x)^{1-q} x^{p-1} = \frac{p}{q} x^{\frac{p}{q}(1-q)+p-1} = r x^{r-1} \qquad \forall\, x \neq 0\,.$$

If r is a negative rational number, we can apply the quotient and find that

$$\frac{d}{dx} x^r = \frac{d}{dx} \frac{1}{x^{-r}} = \frac{r x^{-r-1}}{x^{-2r}} = r x^{r-1} \qquad \forall\, x \neq 0\,.$$

Therefore, we conclude that

$$\frac{d}{dx} x^r = r x^{r-1} \qquad \forall\, x \neq 0\,. \tag{2.4.1}$$

Remark 2.28. The derivative of x^r can also be computed by first finding the derivative of $x^{\frac{1}{p}}$ $\big($that is, find the limit $\displaystyle\lim_{\Delta x\to 0}\frac{(x+\Delta x)^{\frac{1}{p}}-x^{\frac{1}{p}}}{\Delta x}\big)$ and then apply the chain rule.

Example 2.29. Suppose that y is an implicit function of x given that $y^3+y^2-5y-x^2=-4$.

1. Find $\dfrac{dy}{dx}$.
2. Find the tangent line passing through the point $(3,-1)$.

Let $f(x)=x^2-4$ and $g(y)=y^3+y^2-5y$. Then $g'(y)=3y^2+2y-5$; thus if $y\neq 1$ or $y\neq -\dfrac{5}{3}$ $\big($or equivalently, $x\neq\pm 1$ or $x\neq\pm\sqrt{\dfrac{283}{27}}\big)$,

$$\frac{dy}{dx}=\frac{2x}{3y^2+2y-5}\,.$$

Since $(1,-3)$ satisfies the relation $y^3+y^2-5y-x^2=-4$, the slope of the tangent line passing through $(3,-1)$ is $\dfrac{2\cdot 3}{3(-1)^2+2(-1)-5}=-\dfrac{3}{2}$; thus the desired tangent line is

$$y=-\frac{3}{2}(x-3)-1\,.$$

Example 2.30. Find $\dfrac{dy}{dx}$ implicitly for the equation $\sin y=x$.

Let $f(x)=x$ and $g(y)=\sin y$. Then $g'(y)=\cos y$; thus if $y\neq n\pi+\dfrac{\pi}{2}$ (or equivalently, $x\neq\pm 1$),

$$\frac{dy}{dx}=\frac{1}{\cos y}\,. \tag{2.4.2}$$

Similarly, for function y defined implicitly by $\cos y=x$, we find that if $y\neq n\pi$ (or equivalently, $x\neq\pm 1$),

$$\frac{dy}{dx}=-\frac{1}{\sin y}\,. \tag{2.4.3}$$

Remark 2.31. The curve consisting of points (x,y) satisfying the relation $\sin y=x$ cannot be the graph of a function since one x may corresponds to several y; however, the curve consisting of points (x,y) satisfying the relation $\sin y=x$ as well as $-\dfrac{\pi}{2}<y<\dfrac{\pi}{2}$ is the graph of a function called arcsin. In other words, for each $x\in(-1,1)$, there exists a unique $y\in\big(-\dfrac{\pi}{2},\dfrac{\pi}{2}\big)$ satisfying $\sin y=x$, and such y is denoted by $\arcsin x$. Since for $y\in\big(-\dfrac{\pi}{2},\dfrac{\pi}{2}\big)$ we must have $\cos y>0$, by the fact that $\sin^2 y+\cos^2 y=1$, using (2.4.2) we find that

$$\frac{d}{dx}\arcsin x=\frac{1}{\sqrt{1-x^2}}\qquad\forall\,x\in(-1,1)\,. \tag{2.4.4}$$

Similarly, the curve consisting of points (x, y) satisfying the relation $\cos y = x$ as well as $0 < y < \pi$ is the graph of a function called arccos, and (2.4.3) implies that

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}} \qquad \forall\, x \in (-1, 1)\,. \tag{2.4.5}$$

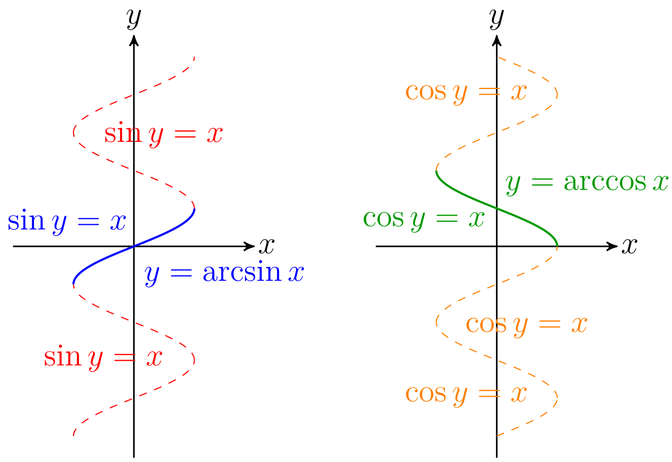

Figure 2.1: The graph of functions $y = \arcsin x$ and $y = \arccos x$

There are, unfortunately, many implicit functions that are not given by the equation of the form $f(x) = g(y)$. Nevertheless, there is a more powerful version of the Implicit Function Theorem that guarantees the continuous differentiability of the implicit functions defined through complicated relations between x and y (written in the form $f(x, y) = 0$). In the following, we always assume that the implicit function given by the equation that x and y satisfy is differentiable.

Example 2.32. Find the second derivative of the implicit function given by the equation $y = \cos(5x - 3y)$.

Differentiate in x once, we find that $\dfrac{dy}{dx} = -\sin(5x-3y)\cdot\big(5 - 3\dfrac{dy}{dx}\big)$; thus

$$\frac{dy}{dx} = \frac{-5\sin(5x-3y)}{1-3\sin(5x-3y)} = \frac{5}{3}\Big[1 - \frac{1}{1-3\sin(5x-3y)}\Big]\,. \tag{2.4.6}$$

Differentiate the equation above in x, we obtain that

$$\frac{d^2y}{dx^2} = -\frac{5}{3}\cdot\frac{3\cos(5x-3y)(5-3y')}{\big[1-3\sin(5x-3y)\big]^2} = -\frac{5\cos(5x-3y)(5-3y')}{\big[1-3\sin(5x-3y)\big]^2}$$

and (2.4.6) further implies that $\dfrac{d^2y}{dx^2} = -\dfrac{25\cos(5x-3y)}{\big[1-3\sin(5x-3y)\big]^3}\,.$

Example 2.33. Show that if it is possible to draw three normals from the point $(a, 0)$ to the parabola $x = y^2$, then $a > \frac{1}{2}$.

Suppose that the line L connecting $(a, 0)$ and (b^2, b), where $b \neq 0$, is normal to the parabola $x = y^2$. The derivative of the function defined implicitly by $x = y^2$ satisfies that

$$1 = 2y\frac{dy}{dx}\,;$$

thus the slope of the tangent line passing through (b^2, b) is $\frac{1}{2b}$. Since line L is perpendicular to the tangent line passing through (b^2, b), we must have

$$\frac{1}{2b} \cdot \frac{b - 0}{b^2 - a} = -1\,.$$

Therefore, $a = \frac{1}{2} + b^2$. Since $b \neq 0$, $a > \frac{1}{2}$.