Calculus 微積分

Ching-hsiao Arthur Cheng 鄭經斅

Chapter 12

Vector-Valued Functions

12.1 Vector-Valued Functions of One Variable

Definition 12.1

A function of the form

$$\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \qquad \text{or} \qquad \boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

is a vector-valued function of one variable, where the component function f, g and h are real-valued functions of the parameter t. Using the vector notation, vector-valued functions above are sometimes denoted by

$$\boldsymbol{r}(t) = \big(f(t), g(t)\big) \qquad \text{or} \qquad \boldsymbol{r}(t) = \big(f(t), g(t), h(t)\big)\,.$$

Remark 12.2. When $\boldsymbol{r}$ is a vector-valued function, we automatically assume that its components f, g (and h) have a common domain.

Definition 12.3: Limit of Vector-Valued Functions

1. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then

$$\lim_{t\to a} \boldsymbol{r}(t) = \Big(\lim_{t\to a} f(t)\Big)\mathbf{i} + \Big(\lim_{t\to a} g(t)\Big)\mathbf{j}$$

provided that the limits $\lim_{t\to a} f(t)$ and $\lim_{t\to a} g(t)$ exist.

2. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t\to a} \boldsymbol{r}(t) = \Big(\lim_{t\to a} f(t)\Big)\mathbf{i} + \Big(\lim_{t\to a} g(t)\Big)\mathbf{j} + \Big(\lim_{t\to a} h(t)\Big)\mathbf{k}$$

provided that the limits $\lim_{t\to a} f(t)$, $\lim_{t\to a} g(t)$ and $\lim_{t\to a} h(t)$ exist.

Remark 12.4. Using the ϵ-δ language, the limit of a vector-valued function $\boldsymbol{r}$ is defined as follows: Let I be the domain of $\boldsymbol{r}$. The notation $\lim_{t\to a}\boldsymbol{r}(t) = \mathbf{L}$ means for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\boldsymbol{r}(t) - \mathbf{L}\| < \varepsilon$ whenever $0 < |t-a| < \delta$ and $t \in I$.

Definition 12.5: Continuity of Vector-Valued Functions

A vector-valued function $\boldsymbol{r}$ is said to be continuous at a point a if the limit $\lim_{t\to a}\boldsymbol{r}(t)$ exists and $\lim_{t\to a}\boldsymbol{r}(t) = \boldsymbol{r}(a)$.

Definition 12.6: Differentiation of Vector-Valued Functions

The derivative of a vector-valued function $\boldsymbol{r}$ at a point a is

$$\boldsymbol{r}'(a) = \lim_{h\to 0}\frac{\boldsymbol{r}(a+h) - \boldsymbol{r}(a)}{h}$$

provided that the limit above exists. If $\boldsymbol{r}'(a)$ exists, then $\boldsymbol{r}$ is said to be differentiable at a and $\boldsymbol{r}'(a)$ is called the derivative of $\boldsymbol{r}$ at a. If $\boldsymbol{r}'(t)$ exists for all t in an interval I, then $\boldsymbol{r}$ is said to be differentiable on the interval I.

Theorem 12.7

1. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then
$$\boldsymbol{r}'(a) = f'(a)\mathbf{i} + g'(a)\mathbf{j}$$
provided that $f'(a)$ and $g'(a)$ exist.
2. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then
$$\boldsymbol{r}'(a) = f'(a)\mathbf{i} + g'(a)\mathbf{j} + h'(a)\mathbf{k}$$
provided that $f'(a)$, $g'(a)$ and $h'(a)$ exist.

Theorem 12.8

Let $\boldsymbol{r}$ and $\boldsymbol{u}$ be differentiable vector-valued functions and f be a differentiable real-valued function.

(a) $\dfrac{d}{dt}(f\boldsymbol{r})(t) = f'(t)\boldsymbol{r}(t) + f\boldsymbol{r}'(t)$.

(b) $\dfrac{d}{dt}\big[\boldsymbol{r}(t) \pm \boldsymbol{u}(t)\big] = \boldsymbol{r}'(t) \pm \boldsymbol{u}'(t)$.

(c) $\dfrac{d}{dt}\big[\boldsymbol{r}(t) \star \boldsymbol{u}(t)\big] = \boldsymbol{r}'(t) \star \boldsymbol{u}(t) + \boldsymbol{r}(t) \star \boldsymbol{u}'(t)$, where $\star$ is the dot product or the cross product.

(d) $\dfrac{d}{dt}\boldsymbol{r}\big(f(t)\big) = f'(t)\boldsymbol{r}'\big(f(t)\big)$.

Proof. We only prove (c) for the case $\star$ being the cross product. Write $\boldsymbol{r}(t) = r_1(t)\mathbf{i} + r_2(t)\mathbf{j} + r_3(t)\mathbf{k}$ and $\boldsymbol{u}(t) = u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k}$. By the definition of the cross product, $\big[\boldsymbol{r}(t) \times \boldsymbol{u}(t)\big]_i$, the i-th component of $\boldsymbol{r}(t) \times \boldsymbol{u}(t)$, is given by $\sum\limits_{1\leqslant j,k\leqslant 3} \varepsilon_{ijk} r_j(t) u_k(t)$. By the product rule,

$$\begin{aligned}\frac{d}{dt}\big[\boldsymbol{r}(t) \times \boldsymbol{u}(t)\big]_i &= \frac{d}{dt}\sum_{1\leqslant j,k\leqslant 3} \varepsilon_{ijk} r_j(t) u_k(t) = \sum_{1\leqslant j,k\leqslant 3} \varepsilon_{ijk} \frac{d}{dt}\big[r_j(t) u_k(t)\big] \\ &= \sum_{1\leqslant j,k\leqslant 3} \varepsilon_{ijk}\big[r_j'(t) u_k(t) + r_j(t) u_j'(t)\big] = \boldsymbol{r}'(t) \times \boldsymbol{u}(t) + \boldsymbol{r}(t) \times \boldsymbol{u}'(t)\,,\end{aligned}$$

where we have used $\boldsymbol{r}'(t) = r_1'(t)\mathbf{i} + r_2'(t)\mathbf{j} + r_3'(t)\mathbf{k}$ and $\boldsymbol{u}'(t) = u_1'(t)\mathbf{i} + u_2'(t)\mathbf{j} + u_3'(t)\mathbf{k}$ to conclude the last equality. □

Remark 12.9. The proof presented above in fact can be used to show that

$$\begin{aligned}&\frac{d}{dt}\begin{vmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t)\end{vmatrix} \\ &\quad = \begin{vmatrix} a_{11}'(t) & a_{12}'(t) & a_{13}'(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t)\end{vmatrix} + \begin{vmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}'(t) & a_{22}'(t) & a_{23}'(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t)\end{vmatrix} + \begin{vmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}'(t) & a_{32}'(t) & a_{33}'(t)\end{vmatrix}\end{aligned}$$

since the determinant of $A = \big[a_{ij}(t)\big]_{1\leqslant i,j\leqslant 3}$ is given by $\sum\limits_{1\leqslant i,j,k\leqslant 3} \varepsilon_{ijk} a_{1i}(t) a_{2j}(t) a_{3k}(t)$. The formula above shows that the differentiation of determinants is obtained by differentiating row by row (or column by column).

• Integration of vector-valued functions of one variable

Similar to the differentiation of vector-valued functions which mimics the differentiation of real-valued functions, we can also define the Riemann integral of a vector-valued function $\boldsymbol{r}$ on $[a, b]$ as the "limit" of the Riemann sum

$$\sum_{k=1}^{n} \boldsymbol{r}(\xi_k)(t_k - t_{k-1})\,, \tag{12.1.1}$$

where $\{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$. To be more precise, a vector-valued function $\boldsymbol{r} : [a, b] \to \mathbb{R}^d$, where $d = 2$ or 3, is said to be Riemann integrable on $[a, b]$ if there exists a vector $\boldsymbol{A}$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$ satisfying $\|\mathcal{P}\| < \delta$, any Riemann sum of $\boldsymbol{r}$ for $\mathcal{P}$ (given by (12.1.1)) locates in $(\boldsymbol{A}-\varepsilon, \boldsymbol{A}+\varepsilon)$, where the vector $\boldsymbol{A}\pm\varepsilon$ is the vector obtained by adding or subtracting ε from each component of $\boldsymbol{A}$. The vector $\boldsymbol{A}$, if exists, is written as $\displaystyle\int_a^b \boldsymbol{r}(t)\,dt$. Since the limit of a vector-valued function can be computed componentwise, we have the following

Theorem 12.9

1. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then
$$\int_a^b \boldsymbol{r}(t)\,dt = \Big(\int_a^b f(t)dt\Big)\mathbf{i} + \Big(\int_a^b g(t)\,dt\Big)\mathbf{j}$$
provided that $\displaystyle\int_a^b f(t)\,dt$ and $\displaystyle\int_a^b g(t)\,dt$ exist.
2. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then
$$\int_a^b \boldsymbol{r}(t)\,dt = \Big(\int_a^b f(t)dt\Big)\mathbf{i} + \Big(\int_a^b g(t)\,dt\Big)\mathbf{j} + \Big(\int_a^b h(t)\,dt\Big)\mathbf{k}$$
provided that $\displaystyle\int_a^b f(t)\,dt$, $\displaystyle\int_a^b g(t)\,dt$ and $\displaystyle\int_a^b h(t)\,dt$ exist.

The Fundamental Theorem of Calculus provides a way to compute the definite integral of vector-valued functions, and this enables us to define the indefinite integral of vector-valued functions as follows.

Definition 12.10

1. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then the indefinite integral (anti-derivative) of $\boldsymbol{r}$ is
$$\int \boldsymbol{r}(t)\,dt = \Big(\int f(t)dt\Big)\mathbf{i} + \Big(\int g(t)\,dt\Big)\mathbf{j}$$
provided that $\displaystyle\int f(t)\,dt$ and $\displaystyle\int g(t)\,dt$ exist.
2. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then the indefinite integral (anti-derivative) of $\boldsymbol{r}$ is
$$\int \boldsymbol{r}(t)\,dt = \Big(\int f(t)dt\Big)\mathbf{i} + \Big(\int g(t)\,dt\Big)\mathbf{j} + \Big(\int h(t)\,dt\Big)\mathbf{k}$$
provided that $\displaystyle\int f(t)\,dt$, $\displaystyle\int g(t)\,dt$ and $\displaystyle\int h(t)\,dt$ exist.

Having defined the indefinite integral of vector-valued functions, by the Fundamental Theorem of Calculus and Theorem 12.7 we have

$$\frac{d}{dt}\int \boldsymbol{r}(t)\,dt = \boldsymbol{r}(t)$$

as long as $\boldsymbol{r}$ is continuous.

12.2 Curves and Parametric Equations

Definition 12.11

A subset C in the plane (or space) is called a ***curve*** if C is the image of an interval $I \subseteq \mathbb{R}$ under a continuous vector-valued function $\boldsymbol{r}$. The continuous function $\boldsymbol{r} : I \to \mathbb{R}^2$ (or $\mathbb{R}^3$) is called a ***parametrization*** of the curve, and the equation

$$(x, y) = \boldsymbol{r}(t)\,,\ t \in I \qquad \big(\text{or } (x, y, z) = \boldsymbol{r}(t),\ \ t \in I\big)$$

is called a ***parametric equation*** of the curve. A curve C is called a ***plane curve*** if it is a subset in the plane.

Since a plane can be treated as a subset of space, in the following we always assume that the curve under discussion is a curve in space (so that the parametrization of the curve is given by $\boldsymbol{r} : I \to \mathbb{R}^3$).

Definition 12.12

A curve C is called ***simple*** if it has an injective parametrization; that is, there exists $\boldsymbol{r} : I \to \mathbb{R}^3$ such that $\boldsymbol{r}(I) = C$ and $\boldsymbol{r}(x) = \boldsymbol{r}(y)$ implies that $x = y$. A curve C with parametrization $\boldsymbol{r} : I \to \mathbb{R}^3$ is called ***closed*** if $I = [a, b]$ for some closed interval $[a, b] \subseteq \mathbb{R}$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$. A ***simple closed*** curve C is a closed curve with parametrization $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$ such that $\boldsymbol{r}$ is one-to-one on (a, b). A ***smooth*** curve C is a curve with differentiable parametrization $\boldsymbol{r} : I \to \mathbb{R}^3$ such that $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all $t \in I$.

Example 12.13. The parabola $y = x^2 + 2$ on the plane is a simple smooth plane curve since $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^2$ given by $r(t) = t\mathbf{i} + (r^2 + 2)\mathbf{j}$ is an injective differentiable parametrization of this parabola. We note that $\tilde{\boldsymbol{r}} : \big(-\frac{\pi}{2}, \frac{\pi}{2}\big) \to \mathbb{R}^2$ given by $\tilde{\boldsymbol{r}}(t) = \tan t\,\mathbf{i} + (\sec^2 t + 1)\mathbf{j}$ is also an injective smooth parametrization of this parabola. In general, a curve usually has infinitely many parameterizations.

Example 12.14. Let $I \subseteq \mathbb{R}$ be an interval, and $\boldsymbol{r}: I \to \mathbb{R}^2$ be defined by $\boldsymbol{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$. Since $\boldsymbol{r}$ is continuous and the co-domain is $\mathbb{R}^2$, the image of I under $\boldsymbol{r}$, denoted by C, is a plane curve. We note that C is part of the unit circle centered at the origin. Moreover, C is a smooth curve since $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all $t \in I$.

1. If $I = [a, b]$ and $|b - a| < 2\pi$, then C is a simple curve.
2. If $I = [0, 2\pi]$, then C is not a simple curve. However, C a simple closed curve.

Example 12.15. Let $\boldsymbol{r}: [0, 2\pi] \to \mathbb{R}^2$ be defined by $\boldsymbol{r}(t) = \sin t\mathbf{i} + \sin t \cos t\mathbf{j}$. The image $\boldsymbol{r}([0, 2\pi])$ is a curve called figure eight.

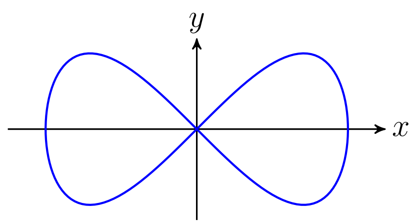


Figure 12.1: Figure eight

Example 12.16. Let $\boldsymbol{r}: \mathbb{R} \to \mathbb{R}^3$ be defined by $\boldsymbol{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$. Then the image $\boldsymbol{r}(\mathbb{R})$ is a simple smooth space curve. This curve is called a helix.

In the following, when a parametrization $\boldsymbol{r}: I \to \mathbb{R}^3$ of curves C is mentioned, we always assume that "there is no overlap"; that is, there are no intervals $[a, b], [c, d] \subseteq I$ satisfying that $\boldsymbol{r}([a, b]) = \boldsymbol{r}([c, d])$. If in addition

1. C is a simple curve, then $\boldsymbol{r}$ is injective, or
2. C is closed, then $I = [a, b]$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$, or
3. C is simple closed, then $I = [a, b]$ and $\boldsymbol{r}$ is injective on $[a, b)$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$.
4. C is smooth, then $\boldsymbol{r}$ is differentiable and $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all $t \in I$.

12.2.1 Polar Graphs

In Example 10.13 we talk about one particular correspondence between a curve on the $r\theta$-plane and a curve on the xy-plane. The equation $r = \cos\theta$ is called a polar equation which means an equation in polar coordinate, and the corresponding curve given by the relation $(x, y) = (r\cos\theta, r\sin\theta)$ on the xy-plane is called the polar graph of this polar equation.

Definition 12.17

Let (r, θ) be the polar coordinate. A polar equation is an equation that r and θ satisfy. The polar graph of a polar equation is the collection of points $(r\cos\theta, r\sin\theta)$ in xy-plane with (r, θ) satisfying the given polar equation.

Remark 12.18. Usually, the polar equation under consideration is of the form

$$r = f(\theta) \qquad \text{or} \qquad \theta = g(r)$$

for some functions f and g. The polar graph of the polar equation $r = f(\theta)$ is the curve parameterized by the parametrization $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^2$ given by $\boldsymbol{r}(t) = f(t)\cos t\mathbf{i} + f(t)\sin t\mathbf{j}$ (where t is the role of θ), while the polar graph of the polar equation $\theta = g(r)$ is the curve parameterized by the parametrization $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^2$ given by $\boldsymbol{r}(t) = t\cos g(t)\mathbf{i} + t\sin g(t)\mathbf{j}$ (where t is the role of r).

Example 12.19.

1. The polar graph of the polar equation $r = r_0$, where $r_0 \neq 0$ is a constant, is the circle centered at the origin with radius $|r_0|$.
2. The polar graph of the polar equation $\theta = \theta_0$, where θ_0 is a constant, is the straight line with slope $\tan\theta_0$.
3. The polar graph of the polar equation $r = \sec\theta$ is $x = 1$ (in the xy-plane).
4. The polar graph of the polar equation $r = a\cos\theta$, where a is a constant, is the circle centered at $\big(\frac{a}{2}, 0\big)$ with radius $\frac{|a|}{2}$.
5. The polar graph of the polar equation $r = a\sin\theta$, where a is a constant, is the circle centered at $\big(0, \frac{a}{2}\big)$ with radius $\frac{|a|}{2}$.

Example 12.20. A conic section (圓錐曲線) can be defined purely in terms of plane geometry: it is the locus of all points P whose distance to a fixed point F (called the focus 焦點) is a constant multiple (called the eccentricity e 離心率) of the distance from P to a fixed line L (called the directrix 準線). For $0 < \mathrm{e} < 1$ we obtain an ellipse, for $\mathrm{e} = 1$ a parabola, and for $\mathrm{e} > 1$ a hyperbola.

Now we consider the polar equation whose polar graph represents a conic section. Let the focus be the pole of a polar coordinate, and the polar axis is perpendicular to the directrix but does not intersect the directrix. Then the eccentricity e is given by

$$\mathrm{e} = \frac{d(P, F)}{d(P, L)} \qquad \text{for all points } P \text{ on the conic section,} \tag{12.2.1}$$

where $d(P, F)$ is the distance between P and the focus F, and $d(P, L)$ is the distance between P and the directrix.

Let P denote the distance between the pole and the directrix, and the polar coordinate of points P on a conic section is (r, θ). Then (12.2.1) implies that

$$\mathrm{e} = \frac{r}{r\cos\theta + \mathrm{P}}\,.$$

Therefore, the polar equation of a conic section with eccentricity e is given by

$$r = \frac{\mathrm{eP}}{1 - \mathrm{e}\cos\theta}\,.$$

In general, for a given conic section we let the principal ray denote the ray starting from the focus, perpendicular to the directrix without intersecting the directrix. Let the focus F be the pole of a polar coordinate and θ_0 be the directed angel from the polar axis to the principal ray. If (r, θ) is the polar representation of point P on the conic section, then (r, θ) satisfies

$$\mathrm{e} = \frac{r}{r\cos(\theta-\theta_0) + \mathrm{P}} \qquad \text{or equivalently,} \quad r = \frac{\mathrm{eP}}{1 - \mathrm{e}\cos(\theta-\theta_0)}\,.$$

Example 12.21 (Limaçons - 蚶線)**.** The polar graph of the polar equation $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$, where $a, b > 0$ are constants, is called a limaçon. A limaçons is also called a cardioid（心臟線）if $a = b$.

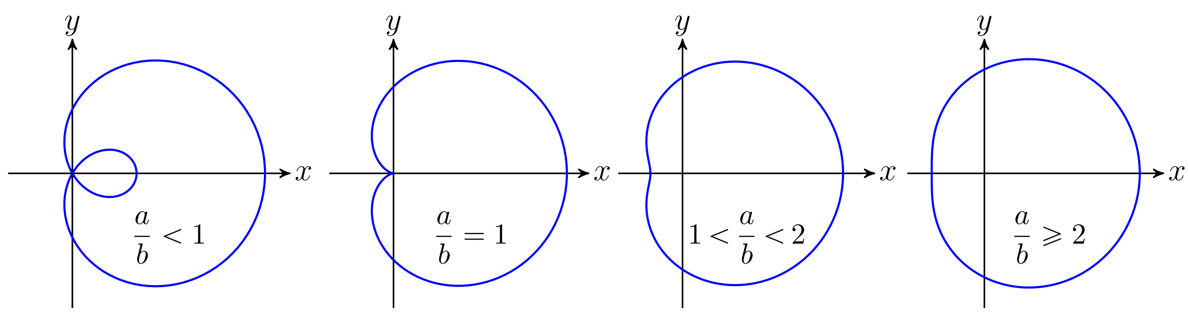


Figure 12.2: Limaçons $r = a \pm b\cos\theta$ with the ratio $\frac{a}{b}$ in different regions

(1) There is an inner loop when $\frac{a}{b} < 1$. (2) When $a = b$ it is also called the cardioid. (3) When $1 < \frac{a}{b} < 2$, the region enclosed by the limaçon is not convex. This kind of limaçons is called dimpled limaçons. (4) When $\frac{a}{b} \geqslant 2$, it is called convex limaçons.

Example 12.22 (Rose curves)**.** The polar graph of the polar equation $r = a\cos n\theta$ or $r = a\sin n\theta$, where $a > 0$ is a given number and $n \geqslant 2$ is an integer, is called a rose curve.

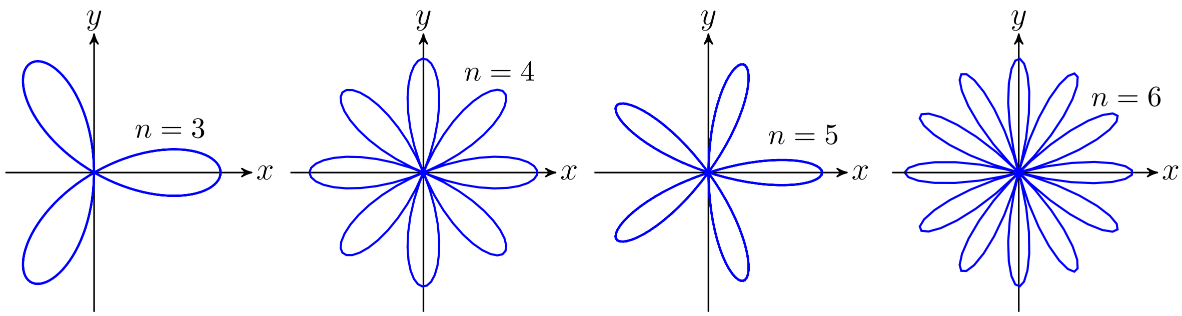


Figure 12.3: Rose curves $r = a\cos n\theta$: n petals when n is odd and $2n$ petals when n is even

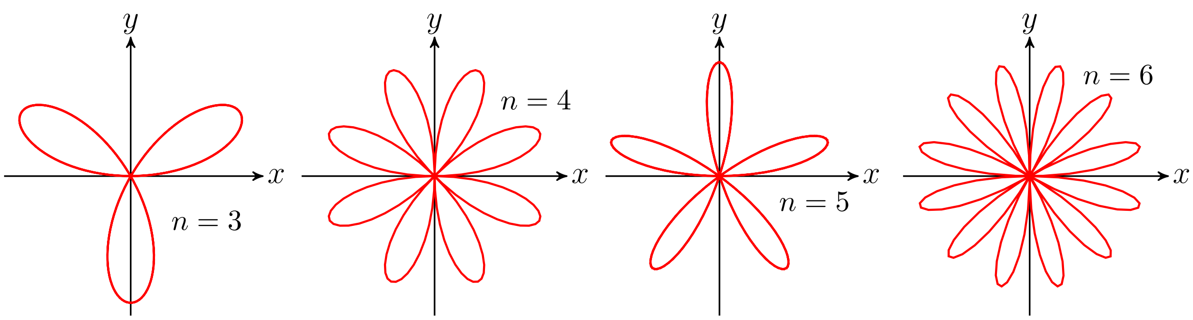


Figure 12.4: Rose curves $r = a\sin n\theta$: n petals when n is odd and $2n$ petals when n is even

Example 12.23 (Lemniscates - 雙紐線)**.** The polar graph of the polar equation $r^2 = a^2\sin 2\theta$ or $r^2 = a^2\cos 2\theta$ is called a lemniscate.

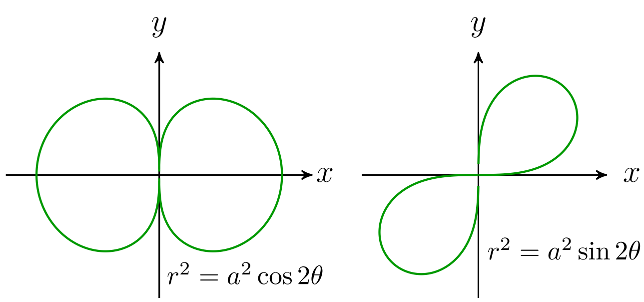


Figure 12.5: Lemniscate $r^2 = a^2\cos 2\theta$ or $r^2 = a^2\sin 2\theta$

12.3 Physical and Geometric Meanings of the Derivative of Vector-Valued Functions

Let $I \subseteq \mathbb{R}$ be an interval and $\boldsymbol{r} : I \to \mathbb{R}^3$ be a differentiable vector-valued function.

12.3.1 Physical meaning

Treat I as the time interval, and $\boldsymbol{r}(t)$ as the position of an object at time t. For $a, b \in I$ and

$a < b$, $\dfrac{\boldsymbol{r}(b) - \boldsymbol{r}(a)}{b - a}$ is the average velocity of the object in the time interval $[a, b]$. Therefore,

$$\boldsymbol{r}'(c) = \lim_{h \to 0} \frac{\boldsymbol{r}(c + h) - \boldsymbol{r}(c)}{h},$$

is the instantaneous velocity at $t = c$, and $\|\boldsymbol{r}'(c)\|$ is the instantaneous speed at $t = c$. If $\boldsymbol{r}$ is twice differentiable, then the derivative of the velocity vector $\boldsymbol{r}'$ is the acceleration.

Definition 12.24

Let $I \subseteq \mathbb{R}$ be the time interval and $\boldsymbol{r} : I \to \mathbb{R}^3$ be the position vector. The velocity vector, acceleration vector and the speed at time t are

$$\begin{aligned} \textbf{Velocity} &= \boldsymbol{v}(t) = \boldsymbol{r}'(t), \\ \textbf{Acceleration} &= \boldsymbol{a}(t) = \boldsymbol{r}''(t), \\ \textbf{Speed} &= \|\boldsymbol{v}(t)\| = \|\boldsymbol{r}'(t)\|. \end{aligned}$$

Example 12.25. Suppose a satellite is under uniform circular motion（等速率圓周運動）and the position of the satellite is given by

$$\boldsymbol{r}(t) = \big(R\cos(\omega t), R\sin(\omega t)\big),$$

where R is the distance between the satellite and the center of Earth, and ω is the angular velocity. Then

$$\boldsymbol{r}'(t) = R\omega\big(-\sin(\omega t), \cos(\omega t)\big) \quad \text{and} \quad \boldsymbol{r}''(t) = -R\omega^2\big(\cos(\omega t), \sin(\omega t)\big);$$

thus

$$\|\boldsymbol{a}(t)\| = \|\boldsymbol{r}''(t)\| = R\omega^2 = \frac{\|\boldsymbol{r}'(t)\|^2}{R} = \frac{\|\boldsymbol{v}(t)\|^2}{R}$$

which gives the famous formula for the centripetal acceleration（向心加速度）.

Example 12.26. In this example we consider the motion of a planet around a single sun. In the plane on which the planet moves, we introduce a polar coordinate system and a Cartesian coordinate system as follows:

1. Let the sun be the pole of the polar coordinate system, and fixed a polar axis on this plane.

2. Let $\mathbf{i}$ be the unit vector in the direction of the polar axis, and $\mathbf{j}$ be the corresponding unit vector obtained by rotating $\mathbf{i}$ counterclockwise by $\dfrac{\pi}{2}$.

Suppose the position of the planet on the plane at time $t \in I$ is given by $\boldsymbol{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. For each $t \in I$, let $(r(t), \theta(t))$ be the polar representation of $(x(t), y(t))$ in the trajectory. We would like to determine the relation that $r(t)$ and $\theta(t)$ satisfy.

Define two vectors $\widehat{r}(t) = \cos\theta(t)\mathbf{i} + \sin\theta(t)\mathbf{j}$ and $\widehat{\theta}(t) = -\sin\theta(t)\mathbf{i} + \cos\theta(t)\mathbf{j}$. Then $\boldsymbol{r} = r\widehat{r}$. Moreover, let M and m be the mass of the sun and the planet, respectively. Then Newton's second law of motion implies that

$$-\frac{GMm}{r^2}\widehat{r} = m\boldsymbol{r}''\,. \tag{12.3.1}$$

By the fact that $\widehat{r}' = \theta'\widehat{\theta}$ and $\widehat{\theta}' = -\theta'\widehat{r}$, we find that

$$\begin{aligned}\boldsymbol{r}'' &= \frac{d}{dt}\big(r'\widehat{r} + r\theta'\widehat{\theta}\big) = r''\widehat{r} + r'\theta'\widehat{\theta} + r'\theta'\widehat{\theta} + r\theta''\widehat{\theta} - r(\theta')^2\widehat{r} \\ &= \big[r'' - r(\theta')^2\big]\widehat{r} + \big[2r'\theta' + r\theta''\big]\widehat{\theta}\,.\end{aligned}$$

Therefore, (12.3.1) implies that

$$-\frac{GM}{r^2}\widehat{r} = \big[r'' - r(\theta')^2\big]\widehat{r} + \big[2r'\theta' + r\theta''\big]\widehat{\theta}\,.$$

Since $\widehat{r}$ and $\widehat{\theta}$ are linearly independent, we must have

$$-\frac{GM}{r^2} = r'' - r(\theta')^2\,, \tag{12.3.2a}$$

$$2r'\theta' + r\theta'' = 0\,. \tag{12.3.2b}$$

Note that (12.3.2b) implies that $(r^2\theta')' = 0$; thus $r^2\theta'$ is a constant. Since $mr^2\theta'$ is the angular momentum, (12.3.2b) implies that the angular momentum is a constant, so-called the conservation of angular momentum（角動量守恆）.

12.3.2 Geometric meaning

Suppose that the image $r(I)$ is a curve C. Since $\boldsymbol{r}(c+h) - \boldsymbol{r}(c)$ is the vector pointing from $\boldsymbol{r}(c)$ to $\boldsymbol{r}(c+h)$, we expect that $\boldsymbol{r}'(c)$, if it is not zero, is tangent to the curve at the point $\boldsymbol{r}(c)$. This motivates the following

Definition 12.27

Let C be a smooth curve represented by $\boldsymbol{r}$ on an interval I. The unit tangent vector $\mathbf{T}$ (associated with the parametrization $\boldsymbol{r}$) is defined as

$$\mathbf{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}\,.$$

Remark 12.28. Since there are infinitely many parameterizations of a given smooth curve, different parameterizations of a smooth curve might provide different unit tangent vector. However, this is not the case and there are only two unit tangent vectors.

Theorem 12.29

Let $I \subseteq \mathbb{R}$ be an interval, and $\boldsymbol{r} : I \to \mathbb{R}^3$ be a differentiable vector-valued function. If $\|\boldsymbol{r}(t)\|$ is a constant function on I, then

$$\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t) = 0 \qquad \forall\, t \in I\,.$$

Proof. Suppose that $\|\boldsymbol{r}(t)\| = C$ for some constant C. Since $\|\boldsymbol{r}(t)\|^2 = \boldsymbol{r}(t) \cdot \boldsymbol{r}(t)$,

$$\boldsymbol{r}(t) \cdot \boldsymbol{r}(t) = C^2 \qquad \forall\, t \in I\,;$$

thus by the fact that $\boldsymbol{r}$ is differentiable, Theorem 12.8 implies that

$$\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t) = \frac{1}{2}\Big[\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t) + \boldsymbol{r}'(t) \cdot \boldsymbol{r}(t)\Big] = \frac{1}{2}\frac{d}{dt}\big[\boldsymbol{r}(t) \cdot \boldsymbol{r}(t)\big] = 0 \qquad \forall\, t \in I\,. \qquad \square$$

Corollary 12.30

Let C be a smooth curve represented by $\boldsymbol{r}$ on an interval I, and $\mathbf{T}(t) = \dfrac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}$ be the unit tangent vector (associated with the parametrization r). If $\boldsymbol{T}$ is differentiable at t, then

$$\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0 \qquad \forall\, t \in I\,.$$

Definition 12.31

Let C be a smooth curve represented by $\boldsymbol{r}$ on an interval I, and $\mathbf{T}(t) = \dfrac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}$ be the unit tangent vector (associated with r). If $\mathbf{T}'(t)$ exists and $\mathbf{T}'(t) \neq \mathbf{0}$, then the ***principal unit normal vector*** (associated with the parametrization $\boldsymbol{r}$) at t is defined as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}\,.$$

Theorem 12.32

Let C be a smooth curve represented by $\boldsymbol{r}$ on an interval I, and the principal unit normal vector $\mathbf{N}$ exists, then the acceleration vector $\boldsymbol{a}$ lies in the plane determined by the unit tangent vector $\mathbf{T}$ and $\mathbf{N}$.

Proof. Let $\boldsymbol{v} = \boldsymbol{r}'$ be the velocity vector. Then

$$\boldsymbol{v} = \|\boldsymbol{v}\| \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \|\boldsymbol{v}\| \frac{\boldsymbol{r}'}{\|\boldsymbol{r}'\|} = \|\boldsymbol{v}\| \mathbf{T}\,.$$

Therefore,

$$\boldsymbol{a} = \boldsymbol{v}' = \|\boldsymbol{v}\|'\mathbf{T} + \|\boldsymbol{v}\|\mathbf{T}' = \|\boldsymbol{v}\|'\mathbf{T} + \|\boldsymbol{v}\|\|\mathbf{T}'\|\mathbf{N}\,.$$

The equation above implies that $\boldsymbol{a}$ is written as a linear combination of $\mathbf{T}$ and $\mathbf{N}$, it follows that $\boldsymbol{a}$ lies in the plane determined by $\mathbf{T}$ and $\mathbf{N}$. □

Remark 12.33. The coefficients of $\mathbf{T}$ and $\mathbf{N}$ in the proof above are called the ***tangential and normal components of acceleration*** and are denoted by

$$a_{\mathbf{T}} = \|\boldsymbol{v}\|' \qquad \text{and} \qquad a_{\mathbf{N}} = \|\boldsymbol{v}\|\|\mathbf{T}'\|$$

so that $\boldsymbol{a}(t) = a_{\mathbf{T}}(t)\mathbf{T}(t) + a_{\mathbf{N}}(t)\mathbf{N}(t)$. Moreover, we note that the formula for $a_{\mathbf{N}}$ above shows that $a_{\mathbf{N}} \geqslant 0$. The normal component of acceleration is also called the ***centripetal component of acceleration***.

The following theorem provides some convenient formulas for computing $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$.

Theorem 12.34

Let C be a smooth curve represented by $\boldsymbol{r}$ on an interval I, and the principal unit normal vector $\mathbf{N}$ exists. Then the tangential and normal components of acceleration are given by

$$a_{\mathbf{T}} = \|\boldsymbol{v}\|' = \boldsymbol{a} \cdot \mathbf{T} = \frac{\boldsymbol{v} \cdot \boldsymbol{a}}{\|\boldsymbol{v}\|}\,,$$

$$a_{\mathbf{N}} = \|\boldsymbol{v}\|\|\mathbf{T}'\| = \boldsymbol{a} \cdot \mathbf{N} = \frac{\|\boldsymbol{v} \times \boldsymbol{a}\|}{\|\boldsymbol{v}\|} = \sqrt{\|\boldsymbol{a}\|^2 - a_{\mathbf{T}}^2}\,.$$

Proof. It suffices to show the formula $a_{\mathbf{N}} = \dfrac{\|\boldsymbol{v} \times \boldsymbol{a}\|}{\|\boldsymbol{v}\|}$. Since $\boldsymbol{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$, we find that

$$\boldsymbol{a} \times \mathbf{T} = a_{\mathbf{N}}(\mathbf{N} \times \mathbf{T})\,;$$

thus using the fact that $a_{\mathbf{N}} \geqslant 0$, by Theorem 10.6 we find that

$$a_{\mathbf{N}} = |a_{\mathbf{N}}| = \frac{\|a \times \mathbf{T}\|}{\|\mathbf{N} \times \mathbf{T}\|} = \frac{\|\boldsymbol{a} \times \mathbf{T}\|}{\|\mathbf{N}\|\|\mathbf{T}\| \sin \frac{\pi}{2}} = \|\boldsymbol{a} \times \mathbf{T}\| = \frac{\|\boldsymbol{v} \times \boldsymbol{a}\|}{\|\boldsymbol{v}\|}\,.$$ □

12.4 Arc Length and Area

12.4.1 Arc length

12.4.2 Area enclosed by simple closed curves

Let C be a simple closed curve in the plane parameterized by $\boldsymbol{r} : [a, b] \to \mathbb{R}^2$. Suppose that

1. $\boldsymbol{r}(t) = \big(x(t), y(t)\big)$ moves **counter-clockwise** (that is, the region enclosed by C is on the left-hand side when moving along C) as t increases.

2. There exists $c \in (a, b)$ such that x is strictly **increasing** on $[a, c]$ and is strictly **decreasing** on $[c, b]$ (this implies that every vertical line intersects with the curve C at at most two points)

3. $x'y$ is Riemann integrable on $[a, b]$ (for example, x is continuously differentiable on $[a, b]$).

Based on the assumption above, in the following we "prove" that

$$\text{the area of the region enclosed by } C \text{ is } -\int_a^b x'(t)y(t)\,dt\,. \tag{12.4.1}$$

We remark that condition 2 above implies that $\boldsymbol{r}(a)$ is the "leftmost" point of the curve, and $\boldsymbol{r}(c)$ is the "rightmost" point of the curve.

Since x is strictly increasing on $[a, c]$ and x is continuous, by the Intermediate Value Theorem (Theorem 1.58) we find that for each point $p \in \big[x(a), x(c)\big]$ there exists a unique $t \in [a, c]$ such that $x(t) = p$. Define $q = y(t)$. Then the map $p \mapsto q$ is a function. This implies that the curve $\boldsymbol{r}\big([a, c]\big)$ is the graph of a continuous function $f : \big[x(a), x(c)\big] \to \mathbb{R}$. Moreover, $y(t) = f(x(t))$ for all $t \in [a, c]$. Similarly, the curve $\boldsymbol{r}\big([c, b]\big)$, the "upper part of C", is the graph of a continuous function $g : \big[x(b), x(c)\big] \to \mathbb{R}$ and $y(t) = g(x(t))$ for all $t \in [c, b]$. Since $x(a) = x(b)$, the substitution of variable $x = x(t)$ implies that

$$\begin{aligned}
&\int_{x(a)}^{x(c)} \big[g(x) - f(x)\big]\,dx \\
&\quad = \int_{x(b)}^{x(c)} g(x)\,dx - \int_{x(a)}^{x(c)} f(x)\,dx = \int_b^c g(x(t))x'(t)\,dt - \int_a^c f(x(t))x'(t)\,dt \\
&\quad = \int_b^c y(t)x'(t)\,dt - \int_a^c y(t)x'(t)\,dt = -\int_a^b x'(t)y(t)\,dt\,;
\end{aligned}$$

thus (12.4.1) is concluded since the area of the region enclosed by C is given by the left-hand side of the equality above.

Similar argument can be applied to conclude that

$$\text{the area of the region enclosed by } C \text{ is } \int_a^b x(t)y'(t)\,dt\,. \tag{12.4.2}$$

if xy' is Riemann integrable on $[a,b]$ and every horizontal line intersects with the curve C at at most two points. Combining (12.4.1) and (12.4.2), we obtain that

$$\text{the area of the region enclosed by } C \text{ is } \frac{1}{2}\int_a^b \Big[x(t)y'(t) - x'(t)y(t)\Big]\,dt \tag{12.4.3}$$

provided that $x'y$ and xy' are Riemann integrable on $[a,b]$ and every vertical line and horizontal line intersects with the curve C at at most two points.

Remark 12.34. In general, **the restriction that every vertical line or horizontal line intersects with curve C at at most two points can be removed** from the condition for the use of (12.4.1), (12.4.2) and (12.4.3). We will see this later in Chapter 15 (but for now we will treat this as a fact for we have proved a special case).

Remark 12.35. Using the convention that $\mathbf{u}\times\mathbf{v} = u_1v_2 - u_2v_1$ when $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ are vectors in the plane, (12.4.3) can be rewritten as

$$\text{the area of the region enclosed by } C \text{ is } \frac{1}{2}\int_a^b \boldsymbol{r}(t)\times\boldsymbol{r}'(t)\,dt\,. \tag{12.4.3'}$$

Without confusion, the area can also be written as $\frac{1}{2}\int_a^b \boldsymbol{r}(t)\times d\boldsymbol{r}(t)$.

Example 12.36. Let C be the curve parameterized by $\boldsymbol{r}(t) = (\cos t, \sin t)$, $t\in[0,2\pi]$. Then clearly $\boldsymbol{r}$ satisfies condition 1-3. Therefore, the area of the region enclosed by C can be computed by the following three ways:

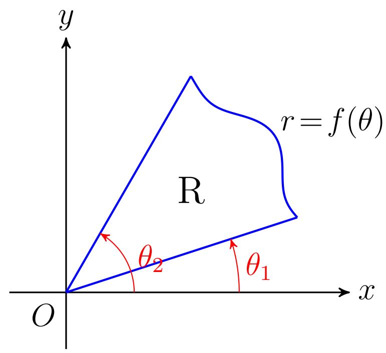
1. Using (12.4.1),
$$-\int_0^{2\pi}\frac{d\cos t}{dt}\sin t\,dt = \int_0^{2\pi}\sin^2 t\,dt = \int_0^{2\pi}\frac{1-\cos(2t)}{2}\,dt = \frac{1}{2}\Big(t - \frac{\sin(2t)}{2}\Big)\Big|_{t=0}^{t=2\pi} = \pi\,.$$
2. Using (12.4.2),
$$\int_0^{2\pi}\cos t\frac{d\sin t}{dt}\,dt = \int_0^{2\pi}\cos^2 t\,dt = \int_0^{2\pi}\frac{1+\cos(2t)}{2}\,dt = \frac{1}{2}\Big(t + \frac{\sin(2t)}{2}\Big)\Big|_{t=0}^{t=2\pi} = \pi\,.$$
3. Using (12.4.3),
$$\frac{1}{2}\int_0^{2\pi}\Big(\cos t\frac{d\sin t}{dt} - \frac{d\cos t}{dt}\sin t\Big)\,dt = \frac{1}{2}\int_0^{2\pi}\big(\cos^2 t + \sin^2 t\big)\,dt = \frac{1}{2}\int_0^{2\pi} 1dt = \pi\,.$$

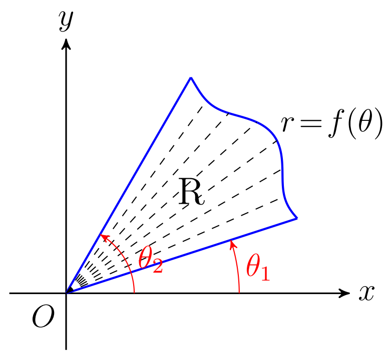
12.4.3 Area and arc length in polar coordinates

Now we consider the area of the region given by the polar representation

$$\big\{(r,\theta)\,\big|\,0 \leqslant r \leqslant f(\theta), \theta_1 \leqslant \theta \leqslant \theta_2\big\}\,, \tag{12.4.4}$$

where $f : [\theta_1, \theta_2] \to \mathbb{R}$ is non-negative and continuous.





Remark 12.37. Note that the region given in (12.4.4) is enclosed by the curve C parameterized by

$$\boldsymbol{r}(t) = \big(x(t), y(t)\big) = \begin{cases} \big(t-\theta_1+f(\theta_1)\big)\big(\cos\theta_1, \sin\theta_1\big) & \text{if } \theta_1 - f(\theta_1) \leqslant t \leqslant \theta_1\,, \\ f(t)\big(\cos t, \sin t\big) & \text{if } \theta_1 \leqslant t \leqslant \theta_2\,, \\ \big(\theta_2+f(\theta_2)-t\big)\big(\cos\theta_2, \sin\theta_2\big) & \text{if } \theta_2 \leqslant t \leqslant \theta_2+f(\theta_2)\,. \end{cases}$$

Then

$$x(t)y'(t) - x'(t)y(t) = \big(x'(t), y'(t)\big)\cdot\big(-y(t), x(t)\big) = \begin{cases} 0 & \text{if } \theta_1 - f(\theta_1) \leqslant t \leqslant \theta_1\,, \\ f(t)^2 & \text{if } \theta_1 \leqslant t \leqslant \theta_2\,, \\ 0 & \text{if } \theta_2 \leqslant t \leqslant \theta_2+f(\theta_2)\,; \end{cases}$$

thus using (12.4.3) we find that

$$\text{the area given in (12.4.4) is } \frac{1}{2}\int_{\theta_1}^{\theta_2} f(\theta)^2\, d\theta\,.$$