基礎數學 MA-1015A

Chapter 5. Cardinality

§5.1 Equivalent Sets; Finite Sets

Definition

Two sets A and B are ***equivalent*** if there exists a one-to-one function from A onto B. The sets are also said to be ***in one-to-one correspondence***, and we write $A \approx B$. In notation,

$$A \approx B \Leftrightarrow (\exists f : A \to B)(f \text{ is a bijection}) .$$

If A and B are not equivalent, we write $A \not\approx B$.

Example

The set of even integers is equivalent to the set of odd integers: the function $f(x) = x + 1$ does the job.

Example

The set of even numbers is equivalent to the set of integers: the function $f(x) = \frac{x}{2}$ does the job.

§5.1 Equivalent Sets; Finite Sets

Example

The set of natural numbers is equivalent to the set of integers.

Example

For $a, b, c, d \in \mathbb{R}$, with $a < b$ and $c < d$, the open intervals (a, b) and (c, d) are equivalent. Therefore, any two open intervals are equivalent, even when the intervals have different length.

Example

Let $\mathcal{F}$ be the set of all binary sequences; that is, the set of all functions from $\mathbb{N} \to \{0, 1\}$. Then $\mathcal{F} \approx \mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$. To see this, we define $\phi : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ by $\phi(x) \equiv \left\{k \in \mathbb{N} \,\middle|\, x_k = 1\right\}$ for all $x \in \mathcal{F}$. Then ϕ is well-defined and $\phi : \mathcal{F} \xrightarrow[\textit{onto}]{1-1} \mathcal{P}(\mathbb{N})$.

§5.1 Equivalent Sets; Finite Sets

Theorem

Equivalence of sets is an equivalence relation on the class of all sets.

Proof.

1. **Reflexivity**: for all sets A, the identity map I_A is an one-to-one correspondence on A.
2. **Symmetry**: Suppose that $A \approx B$; that is, there exists a one-to-one correspondence ϕ from A to B. Then ϕ^{-1} is an one-to-one correspondence from B to A; thus $B \approx A$.
3. **Transitivity**: Suppose that $A \approx B$ and $B \approx C$. Then there exist one-to-one correspondences $\phi : A \xrightarrow[\text{onto}]{1-1} B$ and $\psi : B \xrightarrow[\text{onto}]{1-1} C$. Then $\psi \circ \phi : A \to C$ is an one-to-one correspondence; thus $A \approx C$.

□

§5.1 Equivalent Sets; Finite Sets

Lemma

Suppose that A, B, C and D are sets with $A \approx C$ and $B \approx D$.

1. *If A and B are disjoint and C and D are disjoint, then $A \cup B \approx C \cup D$.*
2. *$A \times B \approx C \times D$.*

Proof.

Suppose that $\phi : A \xrightarrow[\text{onto}]{1-1} C$ and $\psi : B \xrightarrow[\text{onto}]{1-1} D$.

1. Then $\phi \cup \psi : A \cup B \to C \cup D$ is an one-to-one correspondence.
2. Let $f : A \times B \to C \times D$ be given by
$$f(a,b) = \big(\phi(a), \psi(b)\big) .$$
Then f is an one-to-one correspondence from $A \times B$ to $C \times D$.

□

§5.1 Equivalent Sets; Finite Sets

Definition

For each natural number k, let $\mathbb{N}_k = \{1, 2, \cdots, k\}$. A set S is ***finite*** if $S = \varnothing$ or $S \approx \mathbb{N}_k$ for some $k \in \mathbb{N}$. A set S is ***infinite*** if S is not a finite set.

Theorem

For $k, j \in \mathbb{N}$, $\mathbb{N}_j \approx \mathbb{N}_k$ if and only if $k = j$.

Proof.

It suffices to prove the $\Rightarrow$ direction. Suppose that $\phi : \mathbb{N}_k \to \mathbb{N}_j$ is a one-to-one correspondence. W.L.O.G. we can assume that $k \leqslant j$. If $k < j$, then $\phi(\mathbb{N}_k) = \{\phi(1), \phi(2), \cdots, \phi(k)\} \neq \mathbb{N}_j$ since the number of elements in $\phi(\mathbb{N}_k)$ and $\mathbb{N}_j$ are different. In other words, if $k < j$, $\phi : \mathbb{N}_k \to \mathbb{N}_j$ cannot be surjective. This implies that $\mathbb{N}_k \approx \mathbb{N}_j$ if and only if $k = j$. □

§5.1 Equivalent Sets; Finite Sets

Definition

Let S be a finite set. If $S = \varnothing$, then S has ***cardinal number*** 0 (or ***cardinality*** 0), and we write $\#S = 0$. If $S \approx \mathbb{N}_k$ for some natural number k, then S has ***cardinal number*** k (or ***cardinality*** k), and we write $\#S = k$.

Remark: The cardinality of a set S can also be denoted by $n(S)$, $\overline{\overline{S}}$, $\text{card}(S)$ as well.

Theorem

If A is finite and $B \approx A$, then B is finite.

Lemma

If S is a finite set with cardinality k and x is any object not in S, then $S \cup \{x\}$ is finite and has cardinality $k+1$.

§5.1 Equivalent Sets; Finite Sets

Lemma

For every $k \in \mathbb{N}$, every subset of $\mathbb{N}_k$ is finite.

Proof.

Let $S = \{k \in \mathbb{N} \mid \text{the statement "every subset of } \mathbb{N}_k \text{ is finite" holds}\}$.

1. There are only two subsets of $\mathbb{N}_1$, namely $\varnothing$ and $\mathbb{N}_1$. Since $\varnothing$ and $\mathbb{N}_1$ are both finite, we have $1 \in S$.
2. Suppose that $k \in S$. Then every subset of $\mathbb{N}_k$ is finite. Since $\mathbb{N}_{k+1} = \mathbb{N}_k \cup \{k+1\}$, every subset of $\mathbb{N}_{k+1}$ is either a subset of $\mathbb{N}_k$, or the union of a subset of $\mathbb{N}_k$ and $\{k+1\}$. By the fact that $k \in S$, we conclude from the previous lemma that every subset of $\mathbb{N}_{k+1}$ is finite.

Therefore, **PMI** implies that $S = \mathbb{N}$. □

§5.1 Equivalent Sets; Finite Sets

Theorem

Every subset of a finite set is finite.

Proof.

Let $A \subseteq B$ and B is a finite set.

1. If $A = \varnothing$, then A is a finite set (and $\#A = 0$).
2. If $A \neq \varnothing$, then $B \neq \varnothing$. Since B is finite, there exists $k \in \mathbb{N}$ such that $B \approx N_k$; thus there exists a one-to-one correspondence $\phi : \mathbb{N}_k \to B$. Therefore, $\phi^{-1}(A)$ is a non-empty subset of $\mathbb{N}_k$, and the previous lemma implies that $\phi^{-1}(A)$ is finite. Since $A \approx \phi^{-1}(A)$, we conclude that A is a finite set. □

§5.1 Equivalent Sets; Finite Sets

Theorem

1. *If A and B are disjoint finite sets, then $A \cup B$ is finite, and*
$$\#(A \cup B) = \#A + \#B\,.$$
2. *If A and B are finite sets, then $A \cup B$ is finite, and*
$$\#(A \cup B) = \#A + \#B - \#(A \cap B)\,.$$
3. *If $A_1, A_2, \cdots, A_n$ are finite sets, then $\bigcup_{k=1}^{n} A_k$ is finite.*

Proof.

1. W.L.O.G., we assume that $A \approx \mathbb{N}_k$ and $B \approx \mathbb{N}_\ell$ for some $k, \ell \in \mathbb{N}$. Let $H = \{k+1, k+2, \cdots, k+\ell\}$. Then $\mathbb{N}_\ell \approx H$ since $\phi(x) = k + x$ is a one-to-one correspondence from $\mathbb{N}_\ell \to \{k+1, k+2, \cdots, k+\ell\}$. Therefore, $A \cup B \approx \mathbb{N}_k \cup H = \mathbb{N}_{k+\ell}$; thus $\#(A \cup B) = \#A + \#B$. □

§5.1 Equivalent Sets; Finite Sets

Proof of $\#(A \cup B) = \#A + \#B - \#(A \cap B)$.

2. Note that $A \cup B$ is the disjoint union of A and $B - A$, where $B - A$ is a subset of a finite set B which makes $B - A$ a finite set. Therefore, $A \cup B$ is finite.

   To see $\#(A \cup B) = \#A + \#B - \#(A \cap B)$, using ① it suffices to show that $\#(B - A) = \#B - \#(A \cap B)$. Nevertheless, note that $B = (B - A) \cup (A \cap B)$ in which the union is in fact a disjoint union; thus ① implies that

$$\#B = \#(B - A) + \#(A \cap B)$$

   or equivalently,

$$\#(B - A) = \#B - \#(A \cap B)\,.$$

□

§5.1 Equivalent Sets; Finite Sets

Proof.

3. Let $A_1, A_2, \cdots$ be finite sets, and

$$S = \left\{ n \in \mathbb{N} \,\middle|\, \bigcup_{k=1}^{n} A_k \text{ is finite} \right\}.$$

Then $1 \in S$ by assumption. Suppose that $n \in S$. Then $n+1 \in S$ because of ②. **PMI** then implies that $S = \mathbb{N}$. □

§5.1 Equivalent Sets; Finite Sets

Lemma

Let $k \geqslant 2$ be a natural number. For $x \in \mathbb{N}_k$, $\mathbb{N}_k \setminus \{x\} \approx \mathbb{N}_{k-1}$.

Theorem (Pigeonhole Principle - 鴿籠原理)

Let $n, r \in \mathbb{N}$ and $f : \mathbb{N}_n \to \mathbb{N}_r$ be a function. If $n > r$, then f is not injective.

Corollary

If $\#A = n$, $\#B = r$ and $r < n$, then there is no one-to-one function from A to B.

Corollary

If A is finite, then A is not equivalent to any of its proper subsets.

§5.2 Infinite Sets

Recall that a set A is infinite if A is not finite. By the last corollary in the previous section, if a set is equivalent to one of its proper subset, then that set cannot be finite. Therefore, $\mathbb{N}$ is not finite since there is a one-to-one correspondence from $\mathbb{N}$ to the set of even numbers.

The set of natural numbers $\mathbb{N}$ is a set with infinite cardinality. The standard symbol for the cardinality of $\mathbb{N}$ is $\aleph$. There are two kinds of infinite sets, denumerable（無窮可數）sets and uncountable（不可數）sets.

Definition

A set S is said to be ***denumerable*** if $S \approx \mathbb{N}$. For a denumerable set S, we say S has cardinal number $\aleph_0$ (or cardinality $\aleph_0$) and write $\#S = \aleph_0$.

§5.2 Infinite Sets

Example

The set of even numbers and the set of odd numbers are denumerable.

Example

The set $\{p, q, r\} \cup \{n \in \mathbb{N} \mid n \neq 5\}$ is denumerable.

Theorem

The set $\mathbb{Z}$ is denumerable.

Proof.

Consider the function $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even}, \\ \dfrac{1-x}{2} & \text{if } x \text{ is odd}. \end{cases}$$

□

§5.2 Infinite Sets

Theorem

1. *The set $\mathbb{N} \times \mathbb{N}$ is denumerable.*
2. *If A and B are denumerable sets, then $A \times B$ is denumerable.*

Proof.

1. Consider the function $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $F(m, n) = 2^{m-1}(2n - 1)$. Then $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is bijective.
2. If A and B are denumerable sets, then $A \approx \mathbb{N}$ and $B \approx \mathbb{N}$. Then $A \times B \approx \mathbb{N} \times \mathbb{N}$; thus $A \times B \approx \mathbb{N}$ since $\approx$ is an equivalence relation. $\square$

Definition

A set S is said to be ***countable*** if S is finite or denumerable. We say S is ***uncountable*** if S is not countable.

§5.2 Infinite Sets

Theorem

The open interval $(0,1)$ *is uncountable.*

Proof.

Assume the contrary that there exists a bijection $f:\mathbb{N}\to(0,1)$. Write $f(k)$ in decimal expansion (十進位展開); that is,

$$\begin{aligned} f(1) &= 0.d_{11}d_{21}d_{31}\cdots \\ f(2) &= 0.d_{12}d_{22}d_{32}\cdots \\ &\vdots \\ f(k) &= 0.d_{1k}d_{2k}d_{3k}\cdots \\ &\vdots \end{aligned}$$

Here we note that repeated 9's are chosen by preference over terminating decimals; that is, for example, we write $\frac{1}{4}=0.249999\cdots$ instead of $\frac{1}{4}=0.250000\cdots$. □

§5.2 Infinite Sets

Proof. (Cont'd).

Let $x \in (0,1)$ be such that $x = 0.d_1 d_2 \cdots$, where

$$d_k = \begin{cases} 5 & \text{if} \quad d_{kk} \neq 5\,, \\ 3 & \text{if} \quad d_{kk} = 5\,. \end{cases}$$

（建構一個 x 使其小數點下第 k 位數與 $f(k)$ 的小數點下第 k 位數不相等）. Then $x \neq f(k)$ for all $k \in \mathbb{N}$, a contradiction; thus $(0,1)$ is uncountable. □

Definition

A set S has cardinal number $\mathbf{c}$ (or cardinality $\mathbf{c}$) if S is equivalent to $(0,1)$. We write $\#S = \mathbf{c}$, which stands for ***continuum***.

§5.2 Infinite Sets

Theorem

1. *Even open interval (a, b) is uncountable and has cardinality $\mathbf{c}$.*
2. *The set $\mathbb{R}$ of all real numbers is uncountable and has cardinality $\mathbf{c}$.*

Proof.

1. The function $f(x) = a + (b - a)x$ maps from $(0, 1)$ to (a, b) and is a one-to-one correspondence.
2. Using ①, $(0, 1) \approx \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Moreover, the function $f(x) = \tan x$ maps from $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to $\mathbb{R}$ and is a one-to-one correspondence; thus $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \approx \mathbb{R}$. Since $\approx$ is an equivalence relation, $(0, 1) \approx \mathbb{R}$. □

§5.2 Infinite Sets

Example

The circle with the north pole removed is equivalent to the real line.

Example

The set $A = (0,2) \cup [5,6)$ has cardinality $\mathbf{c}$ since the function $f : (0,1) \to A$ given by

$$f(x) = \begin{cases} 4x & \text{if } 0 < x < \frac{1}{2}, \\ 2x+4 & \text{if } \frac{1}{2} \leqslant x < 1 \end{cases}$$

is a one-to-one correspondence from $(0,1)$ to A.

§5.3 Countable Sets

Theorem

Let S be a non-empty set. The following statements are equivalent:

1. *S is countable;*
2. *there exists a surjection $f : \mathbb{N} \to S$;*
3. *there exists an injection $f : S \to \mathbb{N}$.*

Proof.

"① ⇒ ②" First suppose that $S = \{x_1, \cdots, x_n\}$ is finite. Define $f : \mathbb{N} \to S$ by

$$f(k) = \begin{cases} x_k & \text{if } k < n\,, \\ x_n & \text{if } k \geqslant n\,. \end{cases}$$

Then $f : \mathbb{N} \to S$ is a surjection. Now suppose that S is denumerable. Then by definition of countability, there exists $f : \mathbb{N} \xrightarrow[\textit{onto}]{1-1} S$. □

§5.3 Countable Sets

1. S is countable;
2. there exists a surjection $f\colon \mathbb{N} \to S$;

Proof. (Cont'd).

"① ⇐ ②" W.L.O.G. we assume that S is an infinite set. Let $k_1 = 1$. Since $\#(S) = \infty$, $S_1 \equiv S - \{f(k_1)\} \neq \varnothing$; thus $N_1 \equiv f^{-1}(S_1)$ is a non-empty subset of $\mathbb{N}$. By the well-ordered principle (**WOP**) of $\mathbb{N}$, N_1 has a smallest element denoted by k_2. Since $\#(S) = \infty$, $S_2 = S - \{f(k_1), f(k_2)\} \neq \varnothing$; thus $N_2 \equiv f^{-1}(S_2)$ is a non-empty subset of $\mathbb{N}$ and possesses a smallest element denoted by k_3. We continue this process and obtain a set $\{k_1, k_2, \cdots\} \subseteq \mathbb{N}$, where $k_1 < k_2 < \cdots$, and k_j is the smallest element of $N_{j-1} \equiv f^{-1}(S - \{f(k_1), f(k_2), \cdots, f(k_{j-1})\})$. □

§5.3 Countable Sets

Proof. (Cont'd).

Claim: $f : \{k_1, k_2, \cdots\} \to S$ is one-to-one and onto.

Proof of claim: The injectivity of f is easy to see since $f(k_j) \notin \{f(k_1), f(k_2), \cdots, f(k_{j-1})\}$ for all $j \geqslant 2$. For surjectivity, assume the contrary that there is $s \in S$ such that $s \notin f(\{k_1, k_2, \cdots\})$. Since $f : \mathbb{N} \to \mathbb{S}$ is onto, $f^{-1}(\{s\})$ is a non-empty subset of $\mathbb{N}$; thus possesses a smallest element k. Since $s \notin f(\{k_1, k_2, \cdots\})$, there exists $\ell \in \mathbb{N}$ such that $k_\ell < k < k_{\ell+1}$. Therefore, $k \in N_\ell$ and $k < k_{\ell+1}$ which contradicts to the fact that $k_{\ell+1}$ is the smallest element of N_ℓ. □

Let $g : \mathbb{N} \to \{k_1, k_2, \cdots\}$ be defined by $g(j) = k_j$. Then g is one-to-one and onto; thus $h = g \circ f : \mathbb{N} \xrightarrow[onto]{1-1} S$. □

§5.3 Countable Sets

1. S is countable;
3. there exists an injection $f\colon S \to \mathbb{N}$.

Proof. (Cont'd).

"① ⇒ ③" If $S = \{x_1, \cdots, x_n\}$ is finite, we simply let $f\colon S \to \mathbb{N}$ be $f(x_n) = n$. Then f is clearly an injection. If S is denumerable, by definition there exists $g : \mathbb{N} \xrightarrow[onto]{1-1} S$ which implies that $f = g^{-1} : S \to \mathbb{N}$ is an injection. □

§5.3 Countable Sets

1. S is countable;
3. there exists an injection $f\colon S \to \mathbb{N}$.

Proof. (Cont'd).

"①$\Leftarrow$③" Let $f\colon S \to \mathbb{N}$ be an injection. If f is also surjective, then $f\colon S \xrightarrow[\text{onto}]{1-1} \mathbb{N}$ which implies that S is denumerable. Now suppose that $f(S) \subsetneq \mathbb{N}$. Since S is non-empty, there exists $s \in S$. Let $g : \mathbb{N} \to S$ be defined by

$$g(n) = \begin{cases} f^{-1}(n) & \text{if } n \in f(S)\,, \\ s & \text{if } n \notin f(S)\,. \end{cases}$$

Then clearly $g : \mathbb{N} \to S$ is surjective; thus the equivalence between ① and ② implies that S is countable. □

§5.3 Countable Sets

Example

We have seen that the set $\mathbb{N} \times \mathbb{N}$ is countable. Now consider the map $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(m, n) = 2^m 3^n$. This map is not a bijection; however, it is an injection; thus the theorem above implies that $\mathbb{N} \times \mathbb{N}$ is countable.

Example

The set $\mathbb{Q}^+$ of positive rational numbers is denumerable. Since $\mathbb{Q}^+$ is infinite, it suffice to check the countability of $\mathbb{Q}^+$. Consider the map $f : \mathbb{N}^2 \to \mathbb{Q}^+$ defined by $f(m, n) = \frac{m}{n}$. Then f is onto $\mathbb{Q}^+$; thus the theorem above implies that $\mathbb{Q}^+$ is countable.

§5.3 Countable Sets

Theorem

Any non-empty subset of a countable set is countable.

Proof.

Let S be a countable set, and A be a non-empty subset of S. Since S is countable, by the previous theorem there exists a surjection $f : \mathbb{N} \to S$. On the other hand, since A is a non-empty subset of S, there exists $a \in A$. Define

$$g(x) = \begin{cases} x & \text{if } x \in A\,, \\ a & \text{if } x \notin A\,. \end{cases}$$

Then $g : S \to A$ is a surjection; thus $h = g \circ f : \mathbb{N} \to A$ is also a surjection. The previous theorem shows that A is countable. □

Corollary

A set A is countable if and only if $A \approx S$ for some $S \subseteq \mathbb{N}$.

§5.3 Countable Sets

Theorem

The union of denumerable denumerable sets is denumerable. In other words, if $\mathcal{F}$ is a denumerable collection of denumerable sets, then $\bigcup_{A\in\mathcal{F}} A$ is denumerable.

Proof.

Let $\mathcal{F} = \{A_i \,|\, i \in \mathbb{N}, A_i \text{ is denumerable}\}$ be an indexed family of denumerable sets, and define $A = \bigcup_{i=1}^{\infty} A_i$. Since A_i is denumerable, we write $A_i = \{x_{i1}, x_{i2}, x_{i3}, \cdots\}$. Then $A = \{x_{ij} \,|\, i, j \in \mathbb{N}\}$. Let $f : \mathbb{N} \times \mathbb{N} \to A$ be defined by $f(i,j) = x_{ij}$. Then $f : \mathbb{N} \times \mathbb{N} \to A$ is a surjection. Moreover, since $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$, there exists a bijection $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$; thus $h = f \circ g : \mathbb{N} \to A$ is a surjection which implies that A is countable. Since $A_1 \subseteq A$, A is infinite; thus A is denumerable. □

§5.3 Countable Sets

Corollary

The union of countable countable sets is countable（可數個可數集的聯集是可數的）.

Proof.

By adding empty sets into the family or adding $\mathbb{N}$ into a finite set if necessary, we find that the union of countable countable sets is a subset of the union of denumerable denumerable sets. Since a (non-empty) subset of a countable set is countable, we find that the union of countable countable sets is countable. □

§5.3 Countable Sets

Corollary

The set of rational numbers $\mathbb{Q}$ is countable.

Proof.

Let $\mathbb{Q}^+$ and $\mathbb{Q}^-$ denote the collection of positive and negative rational numbers, respectively. We have shown that the set $\mathbb{Q}^+$ is countable. Since $\mathbb{Q}^+ \approx \mathbb{Q}^-$ (between them there exists a one-to-one correspondence $f(x) = -x$), $\mathbb{Q}^-$ is also countable. Therefore, the previous theorem $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$ is countable. □

§5.3 Countable Sets

Corollary

1. *If* $\mathcal{F}$ *is a finite pairwise disjoint family of denumerable sets, then* $\bigcup_{A\in\mathcal{F}} A$ *is countable.*
2. *If* A *and* B *are countable sets, then* $A \cup B$ *is countable.*
3. *If* $\mathcal{F}$ *is a finite collection of countable sets, then* $\bigcup_{A\in\mathcal{F}} A$ *is countable.*
4. *If* $\mathcal{F}$ *is a denumerable family of countable sets, then* $\bigcup_{A\in\mathcal{F}} A$ *is countable.*