基礎數學 MA-1015A

Chapter 4. Functions

§4.1 Functions as Relations

Recall the usual definition of functions from A to B:

Definition

Let A and B be sets. A ***function*** $f : A \to B$ consists of two sets A and B together with a "rule" that assigns to each $x \in A$ a special element of B denoted by $f(x)$. One writes $x \mapsto f(x)$ to denote that x is mapped to the element $f(x)$. A is called the ***domain*** of f, and B is called the ***target*** or ***co-domain*** of f. The ***range*** of f or the ***image*** of f, is the subset of B defined by $f(A) = \{f(x) \,|\, x \in A\}$.

Each function is associated with a collection of ordered pairs

$$\{(x, f(x)) \,|\, x \in A\} \subseteq A \times B .$$

Since a collection of ordered pairs is a relation, we can say that a function is a relation from one set to another.

§4.1 Functions as Relations

However, not every relation can serve as a function. A function is a relation with additional special properties and we have the following

Definition (Alternative Definition of Functions)

A ***function*** (or ***mapping***) from A to B is a relation f from A to B such that

1. the domain of f is A; that is, $(\forall x \in A)(\exists y \in B)((x, y) \in f)$, and
2. if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

We write $f : A \to B$, and this is read "f is a function from A to B" or "f maps A to B". The set B is called the ***co-domain*** of f. In the case where $B = A$, we say f is a function on A.

When $(x, y) \in f$, we write $y = f(x)$ instead of xfy. We say that y is the ***image*** of f at x (or value of f at x) and that x is a ***pre-image*** of y.

§4.1 Functions as Relations

Remark:

1. A function has only one domain and one range but many possible co-domains.
2. A function on $\mathbb{R}$ is usually called a real-valued function or simply real function. The domain of a real function is usually understood to be the largest possible subset of $\mathbb{R}$ on which the function takes values.

Definition

A function x with domain $\mathbb{N}$ is called an ***infinite sequence***, or simply a ***sequence***. The image of the natural number n is usually written as x_n instead of $x(n)$ and is called the n-***th term of the sequence***.

§4.1 Functions as Relations

Definition

Let A, B be sets, and $A \subseteq B$.

1. The the ***identity function/map*** on A is the function $I_A : A \to A$ given by $I_A(x) = x$ for all $x \in A$.
2. The ***inclusion function/map*** from A to B is the function $\iota : A \to B$ given by $\iota(x) = x$ for all $x \in A$.
3. The ***characteristic/indicator function*** of A (defined on B) is the map $\mathbf{1}_A : B \to \mathbb{R}$ given by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A\,, \\ 0 & \text{if } x \in B \backslash A\,. \end{cases}$$

§4.1 Functions as Relations

Definition (Cont'd)

4. The ***greatest integer function*** on $\mathbb{R}$ is the function $[\cdot] : \mathbb{R} \to \mathbb{Z}$ given by

$$[x] = \text{the largest integer which is not greater than } x.$$

The function $[\cdot]$ is also called the ***floor function*** or the ***Gauss function***.

5. Let R be an equivalence relation on A. The ***canonical map*** for the equivalence relation R is the map from A to A/R which maps $x \in A$ to $\bar{x}$, the equivalence class of x modulo R.

§4.1 Functions as Relations

Theorem

Two functions f and g are equal if and only if

1. $\mathrm{Dom}(f) = \mathrm{Dom}(g)$, *and*
2. *for all* $x \in \mathrm{Dom}(f)$, $f(x) = g(x)$.

Example

The identity map of A and the inclusion map from A to B are identical functions.

Example

$f(x) = \dfrac{x}{x}$ and $g(x) = 1$ are different functions since they have different domains.

§4.1 Functions as Relations

Remark:

When a rule of correspondence assigns more than one values to an object in the domain, we say "the function is not well-defined", meaning that it is not really a function. A proof that a function is well-defined is nothing more than a proof that the relation defined by a given rule is single valued.

Example

Let $\bar{x}$ denote the equivalence class of x modulo the congruence relation modulo 4 and $\tilde{y}$ denote the equivalence class of y modulo the congruence relation modulo 10. Define $f(\bar{x}) = \widetilde{2 \cdot x}$. Then this "function" is not really a function since $\bar{0} = \bar{4}$ but $\widetilde{2 \cdot 0} = \tilde{0}$ while $\widetilde{2 \cdot 4} = \tilde{8} \neq \tilde{0}$. In other words, the way f assigns value to $\bar{x}$ is not well-defined.

§4.1 Functions as Relations

Example

Let $\bar{x}$ denote the equivalence class of x modulo the congruence relation modulo 8 and $\tilde{y}$ denote the equivalence class of y modulo the congruence relation modulo 4. The function $f: \mathbb{Z}_8 \to \mathbb{Z}_4$ given by $f(\bar{x}) = \widetilde{x+2}$ is well-defined. To see this, suppose that $\bar{x} = \bar{z}$ in $\mathbb{Z}_8$. Then 8 divides $x-z$ which implies that 4 divides $x-z$; thus 4 divides $(x+2)-(z+2)$. Therefore, $x+2 = z+2 \pmod 4$ or equivalently, $\widetilde{x+2} = \widetilde{z+2}$. So f is well-defined.

§4.2 Construction of Functions

Definition

Let $f: A \to B$. The ***inverse*** of f is the relation from B to A:

$$f^{-1} = \left\{(y,x) \in B \times A \,\middle|\, y = f(x)\right\} = \left\{(y,x) \in B \times A \,\middle|\, (x,y) \in f\right\}.$$

When f^{-1} describes a function, f^{-1} is called the ***inverse function/map*** of f.

Definition

Let $f: A \to B$ and $g: B \to C$ be functions. The ***composite*** of f and g is the relation from A to C:

$$g \circ f = \Big\{(x,z) \in A \times C \,\Big|\, \text{there exists (a unique) } y \in B \text{ such that } (x,y) \in f \text{ and } (y,z) \in g\Big\}.$$

§4.2 Construction of Functions

Remark: Using the notation in the definition of functions, if $(x, z) \in g \circ f$, then $z = (g \circ f)(x)$. On the other hand, if $(x, z) \in g \circ f$, there exists (a unique) $y \in B$ such that $(x, y) \in f$ and $(y, z) \in g$. Then $y = f(x)$ and $z = g(y)$. Therefore, we also have $z = g(f(x))$; thus $(g \circ f)(x) = g(f(x))$.

Theorem

Let A, B and C be sets, and $f : A \to B$ and $g : B \to C$ be functions. Then $g \circ f$ is a function from A to C.

§4.2 Construction of Functions

Proof of $g \circ f$ is a function from A to C.

By the definition of composition of relations, $g \circ f$ is a relation from A to C.

1. First, we show that $\text{Dom}(g \circ f) = A$. Clearly $\text{Dom}(g \circ f) \subseteq A$, so it suffices to show that $A \subseteq \text{Dom}(g \circ f)$. Let $x \in A$. Since $f : A \to B$ is a function, there exists $y \in B$ such that $(x, y) \in f$. Since $g : B \to C$ is a function, there exists $z \in C$ such that $(y, z) \in g$. This shows that for every $x \in A$, there exists $z \in C$ such that $(x, z) \in g \circ f$; thus $\text{Dom}(g \circ f) = A$.
2. Next, we show that if $(x, z_1) \in g \circ f$ and $(x, z_2) \in g \circ f$, then $z_1 = z_2$. Suppose that $(x, z_1) \in g \circ f$ and $(x, z_2) \in g \circ f$. Then there exists $y_1, y_2 \in B$ such that $(x, y_1) \in f$ and $(y_1, z_1) \in g$, while $(x, y_2) \in f$ and $(y_2, z_2) \in g$. Since f is a function, $y_1 = y_2$; thus that g is a function implies that $z_1 = z_2$. □

§4.2 Construction of Functions

Recall that if A, B, C, D are sets, R be a relation from A to B, S be a relation from B to C, and T be a relation from C to D. Then

1. $T \circ (S \circ R) = (T \circ S) \circ R$.
2. $I_B \circ R = R$ and $R \circ I_A = R$.

Theorem

Let A, B, C, D be sets, and $f : A \to B$, $g : B \to C$, $h : C \to D$ be functions. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

Theorem

Let $f : A \to B$ be a function. Then $f \circ I_A = f$ and $I_B \circ f = f$.

Theorem

Let $f : A \to B$ be a function, and $C = \mathrm{Rng}(f)$. If $f^{-1} : C \to A$ is a function, then $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_C$.

§4.2 Construction of Functions

Definition

Let $f: A \to B$ be a function, and $D \subseteq A$. The ***restriction*** of f to D, denoted by $f|_D$, is the function

$$f|_D = \{(x, y) \mid y = f(x) \text{ and } x \in D\}.$$

If g and h are functions and g is a restriction of h, the h is called an ***extension*** of g.

Example

Let F and G be functions

$$F = \{(1,2),(2,6),(3,-9),(5,7)\},$$
$$G = \{(1,8),(2,6),(4,8),(5,7),(8,3)\}.$$

Then $F \cap G = \{(2,6),(5,7)\}$ is a function with domain $\{2,5\}$ which is a proper subset of $\text{Dom}(F) \cap \text{Dom}(G) = \{1,2,5\}$.

On the other hand, $\{(1,2),(1,8)\} \subseteq F \cup G$; thus $F \cup G$ cannot be a function.

§4.2 Construction of Functions

Theorem

Suppose that f and g are functions. Then $f \cap g$ is a function with domain $A = \{x \mid f(x) = g(x)\}$, and $f \cap g = f|_A = g|_A$.

Proof.

Let $(x, y) \in f \cap g$. Then $y = f(x) = g(x)$; thus

$$\mathrm{Dom}(f \cap g) = \{x \mid f(x) = g(x)\} (\equiv A) .$$

If $(x, y_1), (x, y_2) \in f \cap g$, $(x, y_1), (x, y_2) \in f$ which, by the fact that f is a function, implies that $y_1 = y_2$. Therefore, $f \cap g$ is a function. Moreover,

$$f \cap g = \{(x, y) \mid \exists\, x \in A, y = f(x)\}$$

which implies that $f \cap g = f|_A$. □

§4.2 Construction of Functions

For $f \cup g$ being a function, it is (sufficient and) necessary that if $x \in \text{Dom}(f) \cap \text{Dom}(g)$, then $f(x) = g(x)$. Moreover, if $f \cup g$ is a function, then $f = (f \cup g)|_{\text{Dom}(f)}$ and $g = (f \cup g)|_{\text{Dom}(g)}$. In particular, we have the following

Theorem

Let f and g be functions with $\mathrm{Dom}(f) = A$ and $\mathrm{Dom}(g) = B$. If $A \cap B = \emptyset$, then $f \cup g$ is a function with domain $A \cup B$. Moreover,

$$(f \cup g)(x) = \begin{cases} f(x) & \text{if } x \in A\,, \\ g(x) & \text{if } x \in B\,. \end{cases}$$

§4.2 Construction of Functions

Theorem

Let f and g be functions with $\mathrm{Dom}(f) = A$ and $\mathrm{Dom}(g) = B$. If $A \cap B = \varnothing$, then $f \cup g$ is a function with domain $A \cup B$. Moreover,

$$(f \cup g)(x) = \begin{cases} f(x) & \text{if } x \in A\,, \\ g(x) & \text{if } x \in B\,. \end{cases} \qquad (\star)$$

Proof.

Clearly $\mathrm{Dom}(f \cup g) = A \cup B$. Suppose that $(x, y_1), (x, y_2) \in f \cup g$. If $(x, y_1) \in f$, then $x \in \mathrm{Dom}(f)$; thus by the fact that $A \cap B = \varnothing$, we must have $(x, y_2) \in f$. Since f is a function, $y_1 = f(x) = y_2$. Similarly, if $(x, y_1) \in g$, then $(x, y_2) \in g$ which also implies that $y_1 = g(x) = y_2$. Therefore, $f \cup g$ is a function and $(\star)$ is valid. □

§4.2 Construction of Functions

Definition

Let f be a real-valued function defined on an interval $I \subseteq \mathbb{R}$.

1. The function f is said to be ***increasing*** / ***decreasing*** on I if $x \leqslant y$ implies that $f(x) \leqslant f(y)$ / $f(x) \geqslant f(y)$ for all $x, y \in I$.
2. The function f is said to be ***strictly increasing*** / ***strictly decreasing*** on I if $x < y$ implies that $f(x) < f(y)$ / $f(x) > f(y)$ for all $x, y \in I$.

§4.3 Functions that are Onto; One-to-One Functions

Definition

Let $f: A \to B$ be a function.

1. The function f is said to be ***surjective*** or ***onto*** B if $\text{Rng}(f) = B$. When f is surjective, f is called a surjection, and we write $f: A \xrightarrow{\text{onto}} B$.
2. The function f is said to be ***injective*** or ***one-to-one*** if it holds that "$f(x) = f(y) \Rightarrow x = y$". When f is injective, f is called a injection, and we write $f: A \xrightarrow{1-1} B$.
3. The function f is called a ***bijection*** if it is both injective and surjective. When f is a bijection, we write $f: A \xrightarrow[\text{onto}]{1-1} B$.

§4.3 Functions that are Onto; One-to-One Functions

Remark:

1. It is always true that $\text{Rng}(f) \subseteq B$; thus $f: A \to B$ is onto if and only if $B \subseteq \text{Rng}(f)$. In other words, $f: A \to B$ is onto if and only if every $b \in B$ has a pre-image. Therefore, to prove that $f: A \to B$ is onto B, it is sufficient to show that for every $b \in B$ there exists $a \in A$ such that $f(a) = b$.
2. The direct proof of that $f: A \to B$ is injective is to verify the property that "$f(x) = f(y) \Rightarrow x = y$". A proof of the injectivity of f by contraposition assumes that $x \neq y$ and one needs to show that $f(x) \neq f(y)$.

§4.3 Functions that are Onto; One-to-One Functions

Theorem

1. *If $f : A \to B$ is onto B and $g : B \to C$ is onto C, then $g \circ f$ is onto C.*
2. *If $f : A \to B$ is one-to-one and $g : B \to C$ is one-to-one, then $g \circ f$ is one-to-one.*

Proof.

1. Let $c \in C$. By the surjectivity of g, there exists $b \in B$ such that $g(b) = c$. The surjectivity of f then implies the existence of $a \in A$ such that $f(a) = b$. Therefore, $(g \circ f)(a) = g(f(a)) = g(b) = c$ which concludes ①.
2. Assume that $(g \circ f)(x) = (g \circ f)(y)$. Then $g(f(x)) = g(f(y))$; thus by the injectivity of g, $f(x) = f(y)$. Therefore, the injectivity of f implies that $x = y$ which concludes ②. □

§4.3 Functions that are Onto; One-to-One Functions

Theorem

If $f : A \to B$, $g : B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection.

Theorem

Let $f : A \to B$ and $g : B \to C$ be functions.

1. *If $g \circ f$ is onto C, then g is onto C.*
2. *If $g \circ f$ is one-to-one, then f is one-to-one.*

Proof.

1. Let $c \in C$. Since $g \circ f$ is onto C, there exists $a \in A$ such that $(g \circ f)(a) = c$. Let $b = f(a)$. Then $g(b) = g(f(a)) = (g \circ f)(a) = c$.
2. Suppose that $f(x) = f(y)$. Then $(g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y)$, and the injectivity of $g \circ f$ implies that $x = y$. □

§4.3 Functions that are Onto; One-to-One Functions

Remark:

1. In part ① of the theorem above, we cannot conclude that f is also onto B since there might be a proper subset $\widetilde{B} \subsetneq B$ such that $f : A \to \widetilde{B}$, $g : \widetilde{B} \to C$ and $g \circ f$ is onto C. For example, Let $A = B = \mathbb{R}$, $C = \mathbb{R}^+ \cup \{0\}$, and $f(x) = g(x) = x^2$. Then clearly f is not onto B but $g \circ f$ is onto C.
2. In part ② of the theorem above, we cannot conclue that g is one-to-one since it might happen that g is one-to-one on $\text{Rng}(f) \subsetneq B$ but g is not one-to-one on B. For example, let $A = C = \mathbb{R}^+ \cup \{0\}$, $B = \mathbb{R}$, and $f(x) = x^2$, $g(x) = \log(1 + |x|)$. Then clearly g is not one-to-one, but $g \circ f$ is one-to-one.

§4.3 Functions that are Onto; One-to-One Functions

Theorem

If $f: A \to B$ is one-to-one, then every restriction of f is one-to-one.

In the following we consider the function $f \cup g$. Recall that if $\text{Dom}(f) \cap \text{Dom}(g) = \varnothing$, then $(f \cup g)(x) \overset{(\star)}{=} \begin{cases} f(x) & \text{if } x \in \text{Dom}(f)\,, \\ g(x) & \text{if } x \in \text{Dom}(g)\,. \end{cases}$

Theorem

Let $f: A \to C$ and $g: B \to D$ be functions. Suppose that A and B are disjoint sets.

1. *If f is onto C and g is onto D, then $f \cup g: A \cup B \to C \cup D$ is onto $C \cup D$.*
2. *If f is one-to-one, g is one-to-one, and C and D are disjoint, then $f \cup g: A \cup B \to C \cup D$ is one-to-one.*

§4.3 Functions that are Onto; One-to-One Functions

Proof.

We note that $f \cup g : A \cup B \to C \cup D$ is a function.

1. Let $y \in C \cup D$. Then $y \in C$ or $y \in D$. W.L.O.G., we can assume that $y \in C$. Since $f : A \to C$ is onto C, there exists $x \in A$ such that $(x, y) \in f$. Using $(\star)$, $(f \cup g)(x) = f(x) = y$. Therefore, $f \cup g$ is onto $C \cup D$.
2. Suppose that $(x_1, y), (x_2, y) \in f \cup g \subseteq (A \times C) \cup (B \times D)$. Then $(x_1, y) \in f$ or $(x_1, y) \in g$. W.L.O.G., we can assume that $(x_1, y) \in f$. Since $f \subseteq A \times C$ and $g \subseteq B \times D$, by the fact that $C \cap D = \emptyset$ we must have $(x_2, y) \in f$ for otherwise $y \in C \cap D$, a contradiction. Now, since $(x_1, y), (x_2, y) \in f$, the injectivity of f then implies that $x_1 = x_2$. □

§4.4 Inverse Functions

Recall that the inverse of a relation $f : A \to B$ is the relation f^{-1} satisfying

$$y f^{-1} x \quad \Leftrightarrow \quad x f y \quad \Leftrightarrow \quad (x, y) \in f \quad \Leftrightarrow \quad y = f(x) \,.$$

This relation is a function, called the inverse function of f, if the relation itself is a function with certain domain.

Definition

A function $f : A \to B$ is said to be a ***one-to-one correspondence*** if f is a bijection.

§4.4 Inverse Functions

Theorem

Let $f: A \to B$ be a function.

1. *f^{-1} is a function from $\mathrm{Rng}(f)$ to A if and only if f is one-to-one.*
2. *If f^{-1} is a function, then f^{-1} is one-to-one.*

Proof.

1. "$\Rightarrow$" If $(x_1, y), (x_2, y) \in f$, then $(y, x_1), (y, x_2) \in f^{-1}$. Since f^{-1} is a function, we must have $x_1 = x_2$. Therefore, f is one-to-one.
   "$\Leftarrow$" If $(y, x_1), (y, x_2) \in f^{-1}$, then $(x_1, y), (x_2, y) \in f$, and the injectivity of f implies that $x_1 = x_2$. Therefore, by the fact that $\mathrm{Rng}(f) = \mathrm{Dom}(f^{-1})$, f^{-1} is a function with domain $\mathrm{Rng}(f)$.
2. Suppose that f^{-1} is a function, and $(y_1, x), (y_2, x) \in f^{-1}$. Then $(x, y_1), (x, y_2) \in f$ which, by the fact that f is a function, implies that $y_1 = y_2$. Therefore, f^{-1} is one-to-one. □

§4.4 Inverse Functions

Corollary

The inverse of a one-to-one correspondence is a one-to-one correspondence.

Theorem

Let $f : A \to B$, $g : B \to A$ be functions. Then

1. $g = f^{-1}$ *if and only if* $g \circ f = I_A$ *and* $f \circ g = I_B$ (*if and only if* $f = g^{-1}$).
2. *If f is surjective, and $g \circ f = I_A$, then $g = f^{-1}$.*
3. *If f is injective, and $f \circ g = I_B$, then $g = f^{-1}$.*

Recall that "If $C = \mathrm{Rng}(f)$ and $f^{-1} : C \to A$ is a function, then $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_C$". Therefore, the $\Rightarrow$ direction in ① has already been proved.

§4.4 Inverse Functions

Proof.

We first prove the following two claims:

(a) If $g \circ f = I_A$, then $f^{-1} \subseteq g$. (b) If $f \circ g = I_B$, then $g \subseteq f^{-1}$.

To see (a), let $(y, x) \in f^{-1}$ be given. Then $(x, y) \in f$ or $y = f(x)$. Since $(g \circ f) = I_A$, we must have

$$g(y) = g(f(x)) = (g \circ f)(x) = I_A(x) = x$$

or equivalently, $(y, x) \in g$. Therefore, $f^{-1} \subseteq g$.

To see (b), let $(y, x) \in g$ be given. Then $x = g(y)$; thus the fact that $(f \circ g) = I_B$ implies that

$$f(x) = f(g(y)) = (f \circ g)(y) = I_B(y) = y$$

or equivalently, $(x, y) \in f$. Therefore, $(y, x) \in f^{-1}$; thus $g \subseteq f^{-1}$.

1. "$\Rightarrow$" Done.

   "$\Leftarrow$" This direction is a direct consequence of the claims. □

§4.4 Inverse Functions

Proof. (Cont'd).

2. Suppose that $f: A \to B$ is surjective and $g \circ f = I_A$. Then claim (a) implies that $f^{-1} \subseteq g$; thus it suffices to show that $g \subseteq f^{-1}$. Let $(y, x) \in g$. Then by the surjectivity of f there exists $x_1 \in A$ such that $y = f(x_1)$ or equivalently, $(y, x_1) \in f^{-1}$. On the other hand,
$$x = g(y) = g(f(x_1)) = (g \circ f)(x_1) = I_A(x_1) = x_1 \,.$$
Therefore, $g \subseteq f^{-1}$.
3. Now suppose that $f: A \to B$ is injective and $f \circ g = I_B$. Then claim (b) implies that $g \subseteq f^{-1}$; thus it suffices to show that $f^{-1} \subseteq g$. Let $(y, x) \in f^{-1}$ or equivalently, $(x, y) \in f$ or $y = f(x)$. By the fact that $f \circ g = I_B$, we have $f(g(y)) = y$; thus the injectivity of f implies that $g(y) = x$ or $(y, x) \in g$. Therefore, $f^{-1} \subseteq g$ which completes the proof. □

§4.4 Inverse Functions

Since we have shown in the previous theorem that for functions $f : A \to B$ and $g : B \to A$,

1. $g = f^{-1}$ if and only if $g \circ f = I_A$ and $f \circ g = I_B$,
2. If f is surjective, and $g \circ f = I_A$, then $g = f^{-1}$,
3. If f is injective, and $f \circ g = I_B$, then $g = f^{-1}$,

we can conclude the following

Corollary

If $f : A \to B$ is an one-to-one correspondence, and $g : B \to A$ be a function. Then $g = f^{-1}$ if and only if $g \circ f = I_A$ or $f \circ g = I_B$.

Example

Let $A = \mathbb{R}$ and $B = \{x \mid x \geqslant 0\}$. Define $f : A \to B$ by $f(x) = x^2$ and $g : B \to A$ by $g(y) = \sqrt{y}$. Then $f \circ g = I_B$ but g is not inverse function of f since $(g \circ f)(x) = |x|$ for all $x \in A$.

§4.4 Inverse Functions

Definition

Let A be a non-empty set. A ***permutation*** of A is a one-to-one correspondence from A onto A.

Theorem

Let A be a non-empty set. Then

1. *the identity map I_A is a permutation of A.*
2. *the composite of permutations of A is a permutation of A.*
3. *the inverse of a permutation of A is a permutation of A.*
4. *if f is a permutation of A, then $f \circ I_A = I_A \circ f = f$.*
5. *if f is a permutation of A, then $f \circ f^{-1} = f^{-1} \circ f = I_A$.*
6. *if f and g are permutations of A, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

§4.5 Set Images

Definition

Let $f: A \to B$ be a function, and $X \subseteq A$, $Y \subseteq B$. The ***image*** of X (under f) or ***image set*** of X, denoted by $f(X)$, is the set

$$f(X) = \{y \in B \,|\, y = f(x) \text{ for some } x \in X\} = \{f(x) \,|\, x \in X\},$$

and the ***pre-image*** of Y (under f) or the ***inverse image*** of Y, denoted by $f^{-1}(Y)$, is the set

$$f^{-1}(Y) = \{x \in A \,|\, f(x) \in Y\}.$$

Remark: Here are some facts about images of sets that follow from the definitions:

(a) If $a \in D$, then $f(a) \in f(D)$.

(b) If $a \in f^{-1}(E)$, then $f(a) \in E$.

(c) If $f(a) \in E$, then $a \in f^{-1}(E)$.

(d) If $f(a) \in f(D)$ and f is one-to-one, then $a \in D$.

§4.5 Set Images

Theorem

Let $f : A \to B$ be a function. Suppose that C, D are subsets of A, and E, F are subsets of B. Then

1. $f(C \cap D) \subseteq f(C) \cap f(D)$. *In particular, if $C \subseteq D$, then $f(C) \subseteq f(D)$.*
2. $f(C \cup D) = f(C) \cup f(D)$.
3. $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$. *In particular, if $E \subseteq F$, then $f^{-1}(E) \subseteq f^{-1}(F)$.*
4. $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$.
5. $C \subseteq f^{-1}(f(C))$.
6. $f(f^{-1}(E)) \subseteq E$.

§4.5 Set Images

Proof of $f(C \cap D) \subseteq f(C) \cap f(D)$.

Let $y \in f(C \cap D)$. Then there exists $x \in C \cap D$ such that $y = f(x)$. Therefore, $y \in f(C)$ and $y \in f(D)$; thus $y \in f(C) \cap f(D)$. □

Remark: It is possible that $f(C \cap D) \subsetneq f(C) \cap f(D)$. For example, $f(x) = x^2$, $C = (-\infty, 0)$ and $D = (0, \infty)$. Then $C \cap D = \varnothing$ which implies that $f(C \cap D) = \varnothing$; however, $f(C) = f(D) = (0, \infty)$.

Proof of $f(C \cup D) = f(C) \cup f(D)$.

Let $y \in B$ be given. Then

$$\begin{aligned} y \in f(C \cup D) &\Leftrightarrow (\exists\, x \in C \cup D)\big(y = f(x)\big) \\ &\Leftrightarrow (\exists\, x \in C)\big(y = f(x)\big) \vee (\exists\, x \in D)\big(y = f(x)\big) \\ &\Leftrightarrow \big(y \in f(C)\big) \vee \big(y \in f(D)\big) \\ &\Leftrightarrow y \in f(C) \cup f(D)\,. \end{aligned}$$

□

§4.5 Set Images

Proof of $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$.

Let $x \in A$ be given. Then

$$\begin{aligned} x \in f^{-1}(E \cap F) &\Leftrightarrow f(x) \in E \cap F \\ &\Leftrightarrow \big(f(x) \in E\big) \wedge \big(f(x) \in F\big) \\ &\Leftrightarrow \big(x \in f^{-1}(E)\big) \wedge \big(x \in f^{-1}(F)\big) \\ &\Leftrightarrow x \in f^{-1}(E) \cap f^{-1}(F) . \end{aligned}$$

□

Proof of $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$.

Let $x \in A$ be given. Then

$$\begin{aligned} x \in f^{-1}(E \cup F) &\Leftrightarrow f(x) \in E \cup F \\ &\Leftrightarrow \big(f(x) \in E\big) \vee \big(f(x) \in F\big) \\ &\Leftrightarrow \big(x \in f^{-1}(E)\big) \vee \big(x \in f^{-1}(F)\big) \\ &\Leftrightarrow x \in f^{-1}(E) \cup f^{-1}(F) . \end{aligned}$$

□

§4.5 Set Images

Proof of $C \subseteq f^{-1}(f(C))$.

Let $x \in C$. Then $f(x) \in f(C)$; thus $x \in f^{-1}(f(C))$. Therefore, $C \subseteq f^{-1}(f(C))$. □

Remark: It is possible that $C \subsetneq f^{-1}(f(C))$. For example, if $f(x) = x^2$ and $C = [0,1]$, then $f^{-1}(f(C)) = f^{-1}([0,1]) = [-1,1] \supsetneq [0,1]$.

Proof of $f(f^{-1}(E)) \subseteq E$.

Suppose that $y \in f(f^{-1}(E))$. Then there exists $x \in f^{-1}(E)$ such that $f(x) = y$. Since $x \in f^{-1}(E)$, there exists $z \in E$ such that $f(x) = z$. Then $y = z$ which implies that $y \in E$. Therefore, $f(f^{-1}(E)) \subseteq E$. □

Remark: It is possible that $f(f^{-1}(E)) \subsetneq E$. For example, if $f(x) = x^2$ and $E = [-1,1]$, then $f(f^{-1}(E)) = f([0,1]) = [0,1] \subsetneq [-1,1]$.