Chapter 7

The Inverse and the Implicit Function Theorems

7.1 The Inverse Function Theorem（反函數定理）

反函數定理是用來探討一個函數的反函數是否存在的問題。只要一個函數不是一對一的，一般來說都不能定義其反函數，例如三角函數中，正弦、餘弦及正切函數都是周期函數，所以全域的反函數不存在。但是我們也知道有所謂的反三角函數 $\sin^{-1}$（或 arcsin），$\cos^{-1}$（或 arctan）及 $\tan^{-1}$（或 arctan），這是因為我們限制了原三角函數的定義域使其在新的定義域上是一對一的（因此反函數存在）。因此，要討論一個定在某一個（大範圍的）定義域的函數的反函數，常常我們最多只能說反函數只在某一小塊區域上存在。

如何知道一個函數在一小塊區域上的反函數存在，我們首先該問的是在定義域是一維（或是指單變數函數）的情況下發生什麼事？由一維的反函數定理 (Theorem 4.72) 我們知道首先應該要保留的條件是類似於微分不為零的這個條件。但是在多變數函數之下，微分不為零的條件該怎麼呈現，這是第一個問題。而當我們觀察 (4.6.1)，應該可以猜出在多變數版本裡面所該對應到的條件，即是 $(Df)(x)$ 這個 bounded linear map 的可逆性。

另外，假設 $f \in \mathscr{C}^1$，那麼由 Theorem 6.8 我們知道在一個點 x_0 如果 $(Df)(x_0)$ 可逆的話，那麼在一個鄰域裡 $(Df)(x)$ 都可逆。所以下面這個反函數定理的條件中只有 (Df) 在一個點可逆這個條件，因為我們目前想先知道小區域的反函數存不存在。

Theorem 7.1 (Inverse Function Theorem). *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be open, $x_0 \in \mathcal{D}$, $f : \mathcal{D} \to \mathbb{R}^n$ be of class $\mathscr{C}^1$, and $(Df)(x_0)$ be invertible. Then there exist an open neighborhood $\mathcal{U}$ of x_0 and an open neighborhood $\mathcal{V}$ of $f(x_0)$ such that*

1. *$f : \mathcal{U} \to \mathcal{V}$ is one-to-one and onto;*

2. *The inverse function* $f^{-1}: \mathcal{V} \to \mathcal{U}$ *is of class* $\mathscr{C}^1$*;*

3. *If* $x = f^{-1}(y)$*, then* $(Df^{-1})(y) = \big((Df)(x)\big)^{-1}$*;*

4. *If* f *is of class* $\mathscr{C}^r$ *for some* $r > 1$*, so is* f^{-1}*.*

Proof. Assume that $A = (Df)(x_0)$. Then $\|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} \neq 0$. Choose $\lambda > 0$ such that $2\lambda\|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} = 1$. Since $f \in \mathscr{C}^1$, there exists $\delta > 0$ such that

$$\big\|(Df)(x) - A\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} = \big\|(Df)(x) - (Df)(x_0)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} < \lambda \quad \text{whenever } x \in D(x_0,\delta) \cap \mathcal{D}\,.$$

By choosing δ even smaller if necessary, we can assume that $D(x_0,\delta) \subseteq \mathcal{D}$. Let $\mathcal{U} = D(x_0,\delta)$.
Claim: $f : \mathcal{U} \to \mathbb{R}^n$ is one-to-one (hence $f : \mathcal{U} \to f(\mathcal{U})$ is one-to-one and onto).
Proof of claim: For each $y \in \mathbb{R}^n$, define $\varphi_y(x) = x + A^{-1}\big(y - f(x)\big)$ (and we note that every fixed-point of φ_y corresponds to a solution to $f(x) = y$). Then

$$(D\varphi_y)(x) = \mathrm{Id} - A^{-1}(Df)(x) = A^{-1}\big(A - (Df)(x)\big)\,,$$

where Id is the identity map on $\mathbb{R}^n$. Therefore,

$$\big\|(D\varphi_y)(x)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} \leqslant \|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)}\big\|A - (Df)(x)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} < \frac{1}{2} \quad \forall\, x \in D(x_0,\delta)\,.$$

By the mean value theorem (Theorem 6.49),

$$\big\|\varphi_y(x_1) - \varphi_y(x_2)\big\|_{\mathbb{R}^n} \leqslant \frac{1}{2}\|x_1 - x_2\|_{\mathbb{R}^n} \qquad \forall\, x_1, x_2 \in D(x_0,\delta),\ x_1 \neq x_2\,; \tag{7.1.1}$$

thus at most one x satisfies $\varphi_y(x) = x$; that is, φ_y has at most one fixed-point. As a consequence, $f : D(x_0,\delta) \to \mathbb{R}^n$ is one-to-one.
Claim: The set $\mathcal{V} = f(\mathcal{U})$ is open.
Proof of claim: Let $b \in \mathcal{V}$. Then there is $a \in \mathcal{U}$ with $f(a) = b$. Choose $r > 0$ such that $\overline{D(a,r)} \subseteq \mathcal{U}$. We observe that if $y \in D(b,\lambda r)$, then

$$\|\varphi_y(a) - a\|_{\mathbb{R}^n} \leqslant \|A^{-1}\big(y - f(a)\big)\|_{\mathbb{R}^n} \leqslant \|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)}\|y - b\|_{\mathbb{R}^n} < \lambda\|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)}r = \frac{r}{2}\,;$$

thus if $y \in D(b,\lambda r)$ and $x \in D(a,r)$,

$$\|\varphi_y(x) - a\|_{\mathbb{R}^n} \leqslant \big\|\varphi_y(x) - \varphi_y(a)\big\|_{\mathbb{R}^n} + \|\varphi_y(a) - a\|_{\mathbb{R}^n} < \frac{1}{2}\|x - a\|_{\mathbb{R}^n} + \frac{r}{2} < r\,.$$

Therefore, if $y \in D(b,\lambda r)$, then $\varphi_y : D(a,r) \to D(a,r)$. By the continuity of φ_y,

$$\varphi_y : \overline{D(a,r)} \to \overline{D(a,r)}\,.$$

On the other hand, (7.1.1) implies that φ_y is a contraction mapping if $y \in D(b, \lambda r)$; thus by the contraction mapping principle 5.89 φ_y has a unique fixed-point $x \in D(a, r)$. As a result, every $y \in D(b, \lambda r)$ corresponds to a unique $x \in D(a, r)$ such that $\varphi_y(x) = x$ or equivalently, $f(x) = y$. Therefore,

$$D(b, \lambda r) \subseteq f\big(D(a, r)\big) \subseteq f(\mathcal{U}) = \mathcal{V}.$$

Next we show that $f^{-1} : \mathcal{V} \to \mathcal{U}$ is differentiable. We note that if $x \in D(x_0, \delta)$,

$$\|(Df)(x_0) - (Df)(x)\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)}\|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} < \lambda\|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} = \frac{1}{2};$$

thus Theorem 6.8 implies that $(Df)(x)$ is invertible if $x \in D(x_0, \delta)$.

Let $b \in \mathcal{V}$ and $k \in \mathbb{R}^n$ such that $b + k \in \mathcal{V}$. Then there exists a unique $a \in \mathcal{U}$ and $h = h(k) \in \mathbb{R}^n$ such that $a + h \in \mathcal{U}$, $b = f(a)$ and $b + k = f(a + h)$. By the mean value theorem and (7.1.1),

$$\big\|\varphi_y(a + h) - \varphi_y(a)\big\|_{\mathbb{R}^n} < \frac{1}{2}\|h\|_{\mathbb{R}^n};$$

thus the fact that $f(a + h) - f(a) = k$ implies that

$$\|h - A^{-1}k\|_{\mathbb{R}^n} < \frac{1}{2}\|h\|_{\mathbb{R}^n}$$

which further shows that

$$\frac{1}{2}\|h\|_{\mathbb{R}^n} \leqslant \|A^{-1}k\|_{\mathbb{R}^n} \leqslant \|A^{-1}\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)}\|k\|_{\mathbb{R}^n} \leqslant \frac{1}{2\lambda}\|k\|_{\mathbb{R}^n}. \tag{7.1.2}$$

As a consequence, if k is such that $b + k \in \mathcal{V}$,

$$\begin{aligned}
\frac{\big\|f^{-1}(b + k) - f^{-1}(b) - \big((Df)(a)\big)^{-1}k\big\|_{\mathbb{R}^n}}{\|k\|_{\mathbb{R}^n}} &= \frac{\big\|a + h - a - \big((Df)(a)\big)^{-1}k\big\|_{\mathbb{R}^n}}{\|k\|_{\mathbb{R}^n}} \\
&\leqslant \big\|\big((Df)(a)\big)^{-1}\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} \frac{\big\|k - (Df)(a)(h)\big\|_{\mathbb{R}^n}}{\|k\|_{\mathbb{R}^n}} \\
&\leqslant \big\|\big((Df)(a)\big)^{-1}\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)} \frac{\big\|f(a + h) - f(a) - (Df)(a)(h)\big\|_{\mathbb{R}^n}}{\|h\|_{\mathbb{R}^n}} \frac{\|h\|_{\mathbb{R}^n}}{\|k\|_{\mathbb{R}^n}} \\
&\leqslant \frac{\big\|\big((Df)(a)\big)^{-1}\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^n)}}{\lambda} \frac{\big\|f(a + h) - f(a) - (Df)(a)(h)\big\|_{\mathbb{R}^n}}{\|h\|_{\mathbb{R}^n}}.
\end{aligned}$$

Using (7.1.2), $h \to 0$ as $k \to 0$; thus passing $k \to 0$ on the left-hand side of the inequality above, by the differentiability of f we conclude that

$$\lim_{k \to 0} \frac{\big\|f^{-1}(b + k) - f^{-1}(b) - \big((Df)(a)\big)^{-1}k\big\|_{\mathbb{R}^n}}{\|k\|_{\mathbb{R}^n}} = 0.$$

This proves 3.

To see 4, we note that the map $g : \mathrm{GL}(n) \to \mathrm{GL}(n)$ given by $g(L) = L^{-1}$ is infinitely many time differentiable; thus using the identity

$$(Df^{-1})(y) = \big((Df)(x)\big)^{-1} = \big(g \circ (Df) \circ f^{-1}\big)(y)\,,$$

by the chain rule we find that if $f \in \mathscr{C}^r$, then $Df^{-1} \in \mathscr{C}^{r-1}$ which is the same as saying that $f^{-1} \in \mathscr{C}^r$. ▫

Remark 7.2. Since $f^{-1} : \mathcal{V} \to \mathcal{U}$ is continuous, for any open subset $\mathcal{W}$ of $\mathcal{U}$ $f(\mathcal{W}) = (f^{-1})^{-1}(\mathcal{W})$ is open relative to $\mathcal{V}$, or $f(\mathcal{W}) = \mathcal{O} \cap \mathcal{V}$ for some open set $\mathcal{O} \subseteq \mathbb{R}^n$. In other words, if $\mathcal{U}$ is an open neighborhood of x_0 given by the inverse function theorem, then $f(\mathcal{W})$ is also open for all open subsets $\mathcal{W}$ of $\mathcal{U}$. We call this property as f is a ***local open mapping*** at x_0.

Remark 7.3. Since $(Df)(x_0) \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$, the condition that $(Df)(x_0)$ is invertible can be replaced by that the determinant of the Jacobian matrix of f at x_0 is not zero; that is,

$$\det\big(\big[(Df)(x_0)\big]\big) \neq 0\,.$$

The determinant of the Jacobian matrix of f at x_0 is called the ***Jacobian*** of f at x_0. The Jacobian of f at x sometimes is denoted by $\mathbf{J}_f(x)$ or $\dfrac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}$.

Example 7.4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x + 2x^2 \sin\dfrac{1}{x} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Let $0 \in (a,b)$ for some (small) open interval (a,b). Since $f'(x) = 1 - 2\cos\dfrac{1}{x} + 4x\sin\dfrac{1}{x}$ for $x \neq 0$, f has infinitely many critical points in (a,b), and (for whatever reasons) these critical points are local maximum points or local minimum points of f which implies that f is **not** locally invertible even though we have $f'(0) = 1 \neq 0$. One cannot apply the inverse function theorem in this case since f is not $\mathscr{C}^1$.

Corollary 7.5. *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, $f : \mathcal{U} \to \mathbb{R}^n$ be of class $\mathscr{C}^1$, and $(Df)(x)$ be invertible for all $x \in \mathcal{U}$. Then $f(\mathcal{W})$ is open for every open set $\mathcal{W} \subseteq \mathcal{U}$.*

在證明了小區域的（local）反函數定理 (Theorem 7.1) 之後，我們接下來要問的是全域的（global）反函數在什麼條件之下會存在。如果照一維的反函數定理，我們會猜測是不是只要 $(Df)(x)$ 在整個區域都可逆就能得到在全域的反函數都存在。以下給個反例說單單在這個條件之下，函數不一定會有一對一的性質。

Example 7.6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x,y) = (e^x \cos y, e^x \sin y)\,.$$

Then

$$\big[(Df)(x,y)\big] = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}.$$

It is easy to see that the Jacobian of f at any point is not zero (thus $(Df)(x)$ is invertible for all $x \in \mathbb{R}^2$), and f is not globally one-to-one (thus the inverse of f does not exist globally) since for example, $f(x,y) = f(x, y+2\pi)$.

要再加什麼條件進來才能得到反函數在全域都存在是個不容易的問題。在一維的情況下，導數是 sign definite 就表示函數在全域是嚴格單調的。在高維度的情況，即使是 $(Df)(x)$ 到處都可逆，仍然有很多情況可能發生（如上例）。下面這個定理（全域的反函數存在定理），從某種角度來說並沒有真的加了什麼條件以確保全域的反函數存在，只是多要求了在所考慮的區域邊界上函數是一對一的。這個條件在一維的情況之下是自動成立的：因為如果一單變數函數的導數是 sign definite，那麼函數在邊界上必定是一對一的（因為嚴格單調的關係）。

Theorem 7.7 (Global Existence of Inverse Function)**.** *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be open, $f : \mathcal{D} \to \mathbb{R}^n$ be of class $\mathscr{C}^1$, and $(Df)(x)$ be invertible for all $x \in \mathcal{D}$. Suppose that K is a connected compact subset of $\mathcal{D}$, and $f : \partial K \to \mathbb{R}^n$ is one-to-one. Then $f : K \to \mathbb{R}^n$ is one-to-one.*

Proof. Define $E = \big\{x \in K \,\big|\, \exists\, y \in K, y \neq x \ni f(x) = f(y)\big\}$. Our goal is to show that $E = \varnothing$.
Claim 1: E is closed.
Proof of claim 1: Suppose the contrary that E is not closed. Then there exists $\{x_k\}_{k=1}^\infty \subseteq E$, $x_k \to x$ as $k \to \infty$ but $x \in K \backslash E$. Since $x_k \in E$, by the definition of E there exists $y_k \in E$ such that $y_k \neq x_k$ and $f(x_k) = f(y_k)$. By the compactness of K, there exists a convergent subsequence $\{y_{k_j}\}_{j=1}^\infty$ of $\{y_k\}_{k=1}^\infty$ with limit $y \in K$. Since $x \notin E$ and $f(x_{k_j}) = f(y_{k_j}) \to f(y)$ as $j \to \infty$, we must have $x = y$; thus $y_{k_j} \to x$ as $j \to \infty$.

Since $(Df)(x)$ is invertible, by the inverse function theorem there exists $\delta > 0$ such that $f : D(x, \delta) \to \mathbb{R}^n$ is one-to-one. By the convergence of sequences $\{x_{k_j}\}_{j=1}^{\infty}$ and $\{y_{k_j}\}_{j=1}^{\infty}$, there exists $N > 0$ such that

$$x_{k_j}, y_{k_j} \in D(x, \delta) \qquad \forall\, j \geqslant N\,.$$

This implies that $f : D(x, \delta) \to \mathbb{R}^n$ cannot be one-to-one $\big($since $x_{k_j} \neq y_{k_j}$ but $f(x_{k_j}) = f(y_{k_j})\big)$, a contradiction. Therefore, E is closed.

Claim 2: E is open relative to K; that is, for every $x \in E$, there exists an open set $\mathcal{U}$ such that $x \in \mathcal{U}$ and $\mathcal{U} \cap K \subseteq E$.

Proof of claim 2:

1. $x \in \partial K \cap E$: By the injectivity of f on ∂K, there exists $x' \in E \cap \text{int}(K)$, $x' \neq x$, such that $f(x) = f(x')$. Since $(Df)(x)$ and $(Df)(x')$ are invertible, by the inverse function theorem there exist open neighborhoods $\mathcal{U}_1$ of x and $\mathcal{U}_2$ of x', as well as open neighborhoods $\mathcal{V}_1$, $\mathcal{V}_2$ of $f(x)$, such that $f : \mathcal{U}_1 \to \mathcal{V}_1$ and $f : \mathcal{U}_2 \to \mathcal{V}_2$ are both one-to-one and onto. Since $x \neq x'$, W.L.O.G. we can assume that $\mathcal{U}_2 \subseteq \text{int}(K)$ and $\mathcal{U}_1 \cap \mathcal{U}_2 = \varnothing$. Since $\mathcal{V}_1 \cap \mathcal{V}_2$ is open, the continuity of f implies that $f^{-1}(\mathcal{V}_1 \cap \mathcal{V}_2) = \mathcal{O} \cap \mathcal{D}$ for some open set $\mathcal{O}$; thus
$$f : \mathcal{U}_1 \cap \mathcal{O} \cap K \to \mathcal{V}_1 \cap \mathcal{V}_2 \cap f(K) \text{ is one-to-one}\,,$$
$$f : \mathcal{U}_2 \cap \mathcal{O} \cap K \to \mathcal{V}_1 \cap \mathcal{V}_2 \cap f(K) \text{ is one-to-one and onto}\,.$$
Let $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{O}$. Then every $\bar{x} \in \mathcal{U} \cap K$ corresponds to a unique $\tilde{x} \in \mathcal{U}_2 \cap \mathcal{O} \cap K$ such that $f(\bar{x}) = f(\tilde{x})$. Since $\mathcal{U}_1 \cap \mathcal{U}_2 = \varnothing$, we must have $\bar{x} \neq \tilde{x}$. Therefore, $\bar{x} \in E$, or equivalently, $\mathcal{U} \cap K \subseteq E$.

2. $x \in \text{int}(K) \cap E$: It suffices to show that there exists $r > 0$ such that $B(x, r) \subseteq E$. Suppose

Now we show that $E = \varnothing$. Since K is connected, E is open relative to K and E is closed, Remark 3.46 implies that $E = K$ or $E = \varnothing$. Suppose the case that $E = K$. Let $x \in \partial K \subseteq E$. Then there exists $y \in E$ such that $y \neq x$ and $f(x) = f(y)$. Since $f : \partial K \to \mathbb{R}^n$ is one-to-one, $y \notin \partial K$. Therefore, we have shown that if $E = K$, then $f(\partial K) \subseteq f\big(\text{int}(K)\big)$.

By Theorem 4.21, the compactness of K implies that $f(K)$ is compact; thus there is $b \in \mathbb{R}^n$ such that $b \notin f(K)$. Consider the function $\varphi : K \to \mathbb{R}$ defined by

$$\varphi(x) = \frac{1}{2}\|f(x) - b\|_{\mathbb{R}^n}^2 = \frac{1}{2}\sum_{j=1}^{n}|f_i(x) - b_i|^2\,.$$

Then φ is a continuous function on K; thus φ attains its maximum at $x_0 \in K$. Since $f(\partial K) \subseteq f\big(\text{int}(K)\big)$, we can assume that $x_0 \in \text{int}(K)$; thus Theorem 6.94 implies that $(D\varphi)(x_0) = 0$. As a consequence,

$$\big[(Df)(x_0)\big]^{\mathrm{T}}\big[f(x_0) - b\big] = 0\,.$$

By the choice of b, $f(x_0) - b \neq 0$; thus we must have that $(Df)(x_0)$ is not invertible, a contradiction. □

Example 7.8. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given as in Example 7.6, and $\mathcal{D} = \big\{(x, y)\,\big|\, x \in \mathbb{R}, 0 < y < 2\pi\big\}$. Then $f : \mathcal{D} \to \mathbb{R}^2$ is one-to-one. If K is a compact subset of $\mathcal{D}$, then $f : K \to \mathbb{R}^2$ is also one-to-one (thus $f : \partial K \to \mathbb{R}^2$ must be one-to-one as well).

Corollary 7.9. *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a bounded open convex set, and $f : \mathcal{D} \to \mathbb{R}^n$ be of class $\mathscr{C}^1$ such that*

1. *f and Df are continuous on $\bar{\mathcal{D}}$;*
2. *the Jacobian* $\det\big(\big[(Df)(x)\big]\big) \neq 0$ *for all $x \in \bar{\mathcal{D}}$;*
3. *$f : \partial\mathcal{D} \to \mathbb{R}^n$ is one-to-one.*

Then $f : \bar{\mathcal{D}} \to \mathbb{R}^n$ is one-to-one. Moreover, $f^{-1} : f(\bar{\mathcal{D}}) \to \mathbb{R}^n$ is continuous, and $f^{-1} : f(\mathcal{D}) \to \mathcal{D}$ is of class $\mathscr{C}^1$.

Proof. We first claim that there exists a small $\varepsilon > 0$ such that $f : \mathcal{D}_\varepsilon \to \mathbb{R}^n$ is one-to-one, where

$$\mathcal{D}_\varepsilon \equiv \big\{x \in \mathcal{D}\,\big|\, d(x, \partial\mathcal{D}) < \varepsilon\big\}.$$

Assume the contrary that for every $k > 0$, there exists $x_k, y_k \in \mathcal{D}$ such that

(a) $x_k \neq y_k$; (b) $d(x_k, \partial\Omega) < \dfrac{1}{k}$ and $d(y_k, \partial\Omega) < \dfrac{1}{k}$; (c) $f(x_k) = f(y_k)$.

Since $\{x_k\}_{k=1}^\infty$ and $\{y_k\}_{k=1}^\infty$ are bounded (due to the boundedness of $\mathcal{D}$), by the Bolzano-Weierstrass Theorem (or Corollary 3.29) there exist $\big\{x_{k_j}\big\}_{j=1}^\infty$ and $\big\{y_{k_j}\big\}_{j=1}^\infty$ such that $x_{k_j} \to x \in \bar{\mathcal{D}}$ and $y_{k_j} \to y \in \bar{\mathcal{D}}$. By (b), $x, y \in \partial\mathcal{D}$; thus the fact that $f : \partial\mathcal{D} \to \mathbb{R}^n$ is one-to-one implies that $x = y$. Therefore, $x_{k_j} \to x$ and $y_{k_j} \to x$ as $j \to \infty$.

Let $u_j = \dfrac{x_{k_j} - y_{k_j}}{\|x_{k_j} - y_{k_j}\|_{\mathbb{R}^n}}$. Since $\{u_j\}_{j=1}^\infty$ is bounded in $\mathbb{R}^n$, by the Bolzano-Weierstrass Theorem again there is a convergent subsequence $\big\{u_{j_\ell}\big\}_{\ell=1}^\infty$ with limit $u \neq 0$. Moreover, by the convexity of $\mathcal{D}$, the mean value theorem implies that for each $i = 1, \cdots, n$, there exists $c_{i\ell}$ on the line segment joining $x_{k_{j_\ell}}$ and $y_{k_{j_\ell}}$ such that

$$0 = f_i(x_{k_{j_\ell}}) - f_i(y_{k_{j_\ell}}) = (Df_i)(c_{i\ell})(x_{k_{j_\ell}} - y_{k_{j_\ell}}) = \|x_{k_{j_\ell}} - y_{k_{j_\ell}}\|_{\mathbb{R}^n}(Df_i)(c_{i\ell})\big(u_{j_\ell}\big)$$

which by (a) further shows that $(Df_i)(c_{i\ell})\big(u_{j_\ell}\big) = 0$ for all $i = 1, \cdots, n$ and $\ell \in \mathbb{N}$. Since $c_{i\ell} \to x$ as $\ell \to \infty$, passing $\ell \to \infty$ we conclude that $(Df_i)(x)(u) = 0$. This holds for each $i = 1, \cdots, n$; thus $(Df)(x)(u) = 0$. Therefore, $\det\big([(Df)(x)]\big) = 0$, a contradiction.

Now suppose that there exists $x, y \in \mathcal{D}$ such that $f(x) = f(y)$. Choose a compact set $K \subseteq \mathcal{D}$ such that $x, y \in K$ and $\partial K \subseteq \mathcal{D}_\varepsilon$ (this can be done, for example, by choosing that $K = \bar{\mathcal{D}} \backslash \mathcal{D}_\delta$ for some small $\delta > 0$). Since $f : \mathcal{D}_\varepsilon \to \mathbb{R}^n$ is one-to-one, $f : \partial K \to \mathbb{R}^n$ is one-to-one. By Theorem 7.7, $f : K \to \mathbb{R}^n$ is one-to-one. Then $x = y$; thus $f : \mathcal{D} \to \mathbb{R}^n$ is one-to-one.

Next, we show that $f : \bar{\mathcal{D}} \to \mathbb{R}^n$ is one-to-one. Assume the contrary that there exists $x \in \mathcal{D}$ and $y \in \partial\mathcal{D}$ such that $f(x) = f(y)$. By the inverse function theorem there exists open neighborhood $\mathcal{U}$ of x and $\mathcal{V}$ of $f(x)$ such that $f : \mathcal{U} \to \mathcal{V}$ is one-to-one and onto. By choosing $\mathcal{U}$ even smaller if necessary, we can assume that there exists $\{y_k\}_{k=1}^\infty \subseteq \mathcal{D} \backslash \mathcal{U}$ and $y_k \to y$ as $k \to \infty$. By the continuity of f, $f(y_k) \to f(y)$ as $k \to \infty$. However, since $f : \mathcal{D} \to \mathbb{R}^n$ is one-to-one, $\big\{f(y_k)\big\}_{k=1}^\infty \notin \mathcal{V}$; thus $\big\{f(y_k)\big\}_{k=1}^\infty$ cannot converge to $f(y)$ as $k \to \infty$ (since $f(y) \in \mathcal{V}$), a contradiction.

Finally, the inverse function theorem implies that $f^{-1} : f(\mathcal{D}) \to \mathcal{D}$ is of class $\mathscr{C}^1$, and the continuity of f^{-1} on $f(\bar{\mathcal{D}})$ follows from the fact that $(f^{-1})^{-1}(F) = f(F)$ is closed in $\bar{\mathcal{D}}$ for all closed subset F of $\mathbb{R}^n$. □

Remark 7.10. Suppose that $\mathcal{D} \subseteq \mathbb{R}^n$ in Corollary 7.9 is open, bounded, connected but **not** convex. The Whitney extension theorem (which is not covered in this text) implies that there exists a function $F : \mathbb{R}^n \to \mathbb{R}^n$ so that $F = f$ and $DF = Df$ on $\bar{\mathcal{D}}$. Then Theorem 7.7 can be applied to guarantee that F is one-to-one on $\bar{\mathcal{D}}$.

7.2 The Implicit Function Theorem（隱函數定理）

Theorem 7.11 (Implicit Function Theorem)**.** *Let $\mathcal{D} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be open, and $F : \mathcal{D} \to \mathbb{R}^m$ be a function of class $\mathscr{C}^1$. Suppose that for some $(x_0, y_0) \in \mathcal{D}$, where $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^m$,*

$F(x_0, y_0) = 0$ and

$$\big[(D_yF)(x_0, y_0)\big] = \begin{bmatrix} \dfrac{\partial F_1}{\partial y_1} & \cdots & \dfrac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_m}{\partial y_1} & \cdots & \dfrac{\partial F_m}{\partial y_m} \end{bmatrix} (x_0, y_0)$$

is invertible. Then there exists an open neighborhood $\mathcal{U} \subseteq \mathbb{R}^n$ of x_0, an open neighborhood $\mathcal{V} \subseteq \mathbb{R}^m$ of y_0, and $f : \mathcal{U} \to \mathcal{V}$ such that

1. *$F\big(x, f(x)\big) = 0$ for all $x \in \mathcal{U}$;*

2. *$y_0 = f(x_0)$;*

3. *$(Df)(x) = -\big((D_yF)(x, f(x))\big)^{-1}(D_xF)\big(x, f(x)\big)$ for all $x \in \mathcal{U}$, where the matrix representation of $D_xF(x, f(x)) \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$ is given by*

$$\big[(D_xF)(x, y)\big] = \begin{bmatrix} \dfrac{\partial F_1}{\partial x_1} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_m}{\partial x_1} & \cdots & \dfrac{\partial F_m}{\partial x_n} \end{bmatrix} (x, y)\,.$$

4. *f is of class $\mathscr{C}^1$;*

5. *If F is of class $\mathscr{C}^r$ for some $r > 1$, so is f.*

Proof. Let $z = (x, y)$ and $w = (u, v)$, where $x, u \in \mathbb{R}^n$ and $y, v \in \mathbb{R}^m$. Define G by $G(x, y) = \big(x, F(x, y)\big)$, and write $w = G(z)$. Then $G : \mathcal{D} \to \mathbb{R}^{n+m}$, and

$$\big[(DG)(x, y)\big] = \begin{bmatrix} \mathbb{I}_n & 0 \\ (D_xF)(x, y) & (D_yF)(x, y) \end{bmatrix},$$

where $\mathbb{I}_n$ is the $n \times n$ identity matrix. We note that the Jacobian of G at (x_0, y_0) is $\det\big([(D_yF)(x_0, y_0)]\big)$ which does not vanish since $(D_yF)(x_0, y_0)$ is invertible, so the inverse function theorem implies that there exists open neighborhoods $\mathcal{O}$ of (x_0, y_0) and $\mathcal{W}$ of $\big(x_0, F(x_0, y_0)\big) = (x_0, 0)$ such that

(a) $G : \mathcal{O} \to \mathcal{W}$ is one-to-one and onto;

(b) the inverse function $G^{-1} : \mathcal{W} \to \mathcal{O}$ is of class $\mathscr{C}^r$;

(c) $(DG^{-1})\big(x, F(x,y)\big) = \big((DG)(x,y)\big)^{-1}$.

By Remark 7.2, W.L.O.G. we can assume that $\mathcal{O} = \mathcal{U} \times \mathcal{V}$, where $\mathcal{U} \subseteq \mathbb{R}^n$ and $\mathcal{V} \subseteq \mathbb{R}^m$ are open, and $x_0 \in \mathcal{U}$, $y_0 \in \mathcal{V}$.

Write $G^{-1}(u,v) = \big(\varphi(u,v), \psi(u,v)\big)$, where $\varphi : \mathcal{W} \to \mathcal{U}$ and $\psi : \mathcal{W} \to \mathcal{V}$. Then

$$(u,v) = G\big(\varphi(u,v), \psi(u,v)\big) = \big(\varphi(u,v), F(u, \psi(u,v))\big)$$

which implies that $\varphi(u,v) = u$ and $v = F(u, \psi(u,v))$. Let $f(x) = \psi(x, 0)$. Then $\big(u, f(u)\big) \in \mathcal{U} \times \mathcal{V}$ is the unique point satisfying $F\big(u, f(u)\big) = 0$ if $u \in \mathcal{U}$. Therefore, $f : \mathcal{U} \to \mathcal{V}$, and

$$F\big(x, f(x)\big) = 0 \qquad \forall\, x \in \mathcal{U}\,.$$

Since $G(x_0, y_0) = (x_0, 0) = G\big(x_0, f(x_0)\big)$, $(x_0, y_0), \big(x_0, f(x_0)\big) \in \mathcal{O}$, and $G : \mathcal{O} \to \mathcal{W}$ is one-to-one, we must have $y_0 = f(x_0)$.

By (b) and (c), we have G^{-1} is of class $\mathscr{C}^1$, and

$$(DG^{-1})(u,v) = \big((DG)(x,y)\big)^{-1}\,.$$

As a consequence, $\psi \in \mathscr{C}^1$, and

$$\begin{aligned}\begin{bmatrix}(D_u\varphi)(u,v) & (D_v\varphi)(u,v)\\ (D_u\psi)(u,v) & (D_v\psi)(u,v)\end{bmatrix} &= \begin{bmatrix}\mathbb{I}_n & 0\\ (D_xF)(x,y) & (D_yF)(x,y)\end{bmatrix}^{-1}\\ &= \begin{bmatrix}\mathbb{I}_n & 0\\ -\big((D_yF)(x,y)\big)^{-1}(D_xF)(x,y) & \big((D_yF)(x,y)\big)^{-1}\end{bmatrix}.\end{aligned}$$

Evaluating the equation above at $v = 0$, we conclude that

$$(Df)(u) = (D_u\psi)(u,0) = -\big((D_yF)(u,f(u))\big)^{-1}(D_xF)\big(u, f(u)\big)$$

which implies 3. We also note that 4 follows from (b) and 5 follows from 3. ◻

Alternative proof of Theorem 7.11 ***without*** *applying the inverse function theorem.* Let $z = (x,y)$, $z_0 = (x_0, y_0)$, $A = (D_xF)(x_0,y_0)$ and $B = (D_yF)(x_0,y_0)$. Define

$$r(x,y) = F(x,y) - A(x - x_0) - B(y - y_0)\,.$$

Our goal is to solve the equation

$$0 = A(x - x_0) + B(y - y_0) + r(x,y)$$

for y. By the invertibility of B, this is equivalent of finding a fixed-point of the map

$$\Phi_x(y) = y_0 - B^{-1}\big[A(x-x_0) + r(x,y)\big]\,.$$

Since r is of class $\mathscr{C}^1$ and $(Dr)(x_0,y_0) = 0$,

$$\exists\,\delta > 0 \ni \|(Dr)(x,y)\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)} < \min\Big\{\frac{1}{4m\|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}}, \frac{1}{2m}\Big\} \quad \forall\, z \in D(z_0,\delta)\,.$$

Therefore, the mean value theorem (Theorem 6.49) implies that

$$\begin{aligned}\big\|r(x,y) - r(x_0,y_0)\big\|_{\mathbb{R}^m} &\leqslant \sum_{i=1}^m \big|r_i(x,y) - r_i(x_0,y_0)\big| = \sum_{i=1}^m \big|(Dr_i)(c_i)(z-z_0)\big| \\ &\leqslant \frac{\|z-z_0\|_{\mathbb{R}^{n+m}}}{4\|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}} < \frac{\delta}{4\|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}}\end{aligned}$$

for all $z = (x,y) \in D\big(x_0,\frac{\delta}{2}\big) \times D\big(y_0,\frac{\delta}{2}\big) \subseteq D(z_0,\delta)$, and

$$\big\|\Phi_x(y_1) - \Phi_x(y_2)\big\|_{\mathbb{R}^m} \leqslant \|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}\|r(x,y_1) - r(x,y_2)\|_{\mathbb{R}^m} < \frac{1}{4}\|y_1-y_2\|_{\mathbb{R}^m} \tag{7.2.1}$$

if $x \in D\big(x_0,\frac{\delta}{2}\big)$, $y_1,y_2 \in D\big(y_0,\frac{\delta}{2}\big)$ and $y_1 \neq y_2$. As a consequence, for each (fixed) x satisfying $\|x-x_0\|_{\mathbb{R}^n} < r \equiv \min\Big\{\frac{\delta}{4\big(1+\|A\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}\big)\|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}}, \frac{\delta}{2}\Big\}$, if $\|y-y_0\|_{\mathbb{R}^m} < \frac{\delta}{2}$ we have

$$\begin{aligned}\|\Phi_x(y) - y_0\|_{\mathbb{R}^n} &\leqslant \|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}\big\|A(x-x_0) + r(x,y)\big\|_{\mathbb{R}^m} \\ &\leqslant \|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}\big[\|A\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}\|x-x_0\|_{\mathbb{R}^n} + \|r(x,y)\|_{\mathbb{R}^m}\big] < \frac{\delta}{2}\,.\end{aligned} \tag{7.2.2}$$

Let $M = \big\{y \in \mathbb{R}^m \,\big|\, \|y-y_0\|_{\mathbb{R}^m} \leqslant \frac{\delta}{2}\big\}$. Then for each $x \in \mathcal{U} \equiv D(x_0,r)$, (7.2.1) and (7.2.2) imply that $\Phi_x : M \to M$ is a contraction mapping; thus there is a unique fixed-point $y \in M$. Denote this unique fixed-point as $f(x)$. Then $f : \mathcal{U} \to \mathcal{V} \equiv D\big(y_0,\frac{\sqrt{3}\delta}{2}\big)$ (the choice of this $\mathcal{V}$ guarantees that $\mathcal{U}\times\mathcal{V} \subseteq D(z_0,\delta)$) and

$$F\big(x,f(x)\big) = A(x-x_0) + B\big(f(x)-y_0\big) + r\big(x,f(x)\big) = 0\,.$$

Moreover, since $F(x,y) = 0$ if and only if y is a fixed-point of Φ_x, and the contraction mapping principle provides the uniqueness of the fixed-point if $(x,y) \in \mathcal{U}\times M$. Since $(x_0,y_0),(x_0,f(x_0)) \in \mathcal{U}\times M$, we must have $y_0 = f(x_0)$.

To see the differentiability of f, we first claim that $f:\mathcal{U}\to\mathcal{V}$ is continuous. Since $f(x)$ is the fixed-point of Φ_x,

$$f(x)=y_0-B^{-1}\big(A(x-x_0)+r(x,f(x))\big)\,.$$

If $x_1,x_2\in\mathcal{U}$, then $(x_1,f(x_1)),(x_2,f(x_2))\in D(z_0,\delta)$; thus (7.2.1) implies that

$$\begin{aligned}\big\|f(x_1)-f(x_2)\big\|_{\mathbb{R}^m}&=\big\|B^{-1}A(x_1-x_2)\big\|_{\mathbb{R}^m}+\big\|r(x_1,f(x_1))-r(x_2,f(x_2))\big\|_{\mathbb{R}^m}\\&\leqslant\|B^{-1}A\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}\|x_1-x_2\|_{\mathbb{R}^n}+\frac{1}{2}\sqrt{\|x_1-x_2\|^2_{\mathbb{R}^n}+\|f(x_1)-f(x_2)\|^2_{\mathbb{R}^m}}\\&\leqslant\|B^{-1}A\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}\|x_1-x_2\|_{\mathbb{R}^n}+\frac{1}{2}\|x_1-x_2\|_{\mathbb{R}^n}+\frac{1}{2}\|f(x_1)-f(x_2)\|_{\mathbb{R}^m}\,.\end{aligned}$$

Therefore,

$$\big\|f(x_1)-f(x_2)\big\|_{\mathbb{R}^m}\leqslant\big(2\|B^{-1}A\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}+1\big)\|x_1-x_2\|_{\mathbb{R}^n}\tag{7.2.3}$$

which implies that $f:\mathcal{U}\to\mathcal{V}$ is (Lipschitz) continuous.

Now let $a\in\mathcal{U}$ and $\varepsilon>0$ be given. Define $b=(a,f(a))$, and $\widetilde{A}=(D_xF)(b)$, $\widetilde{B}=(D_yF)(b)$. We would like to show that there exists $\delta_1>0$ such that

$$\big\|f(x)-f(a)+\widetilde{B}^{-1}\widetilde{A}(x-a)\big\|_{\mathbb{R}^m}\leqslant\varepsilon\|x-a\|_{\mathbb{R}^n}\qquad\forall\,x\in D(a,\delta_1)\,.$$

Since $F\in\mathscr{C}^1$ and the map $L\mapsto L^{-1}$ is continuous, there exists $\delta_2>0$ such that

$$\big\|(D_yF)(z)^{-1}(D_xF)(z)-(D_yF)(z_0)^{-1}(D_xF)(z_0)\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}\leqslant\frac{\varepsilon}{4}\qquad\forall\,z\in D(z_0,\delta_2)\,.$$

Moreover, since $r\in\mathscr{C}^1$, there exists $\delta_3>0$ such that

$$\big\|r(z)-r(b)-(Dr)(b)(z-b)\big\|_{\mathbb{R}^m}\leqslant\frac{\varepsilon}{2\|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}}\|z-b\|_{\mathbb{R}^{n+m}}\quad\forall\,z\in D(b,\delta_3)$$

and

$$\big\|(Dr)(z)-(Dr)(b)\big\|_{\mathscr{B}(\mathbb{R}^{n+m},\mathbb{R}^m)}\leqslant\frac{\varepsilon}{2\|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}}\|z-b\|_{\mathbb{R}^{n+m}}\quad\forall\,z\in D(b,\delta_3)\,.$$

Choose $\delta_1=\min\Big\{\dfrac{\delta_2}{2},\dfrac{\delta_3}{2},\dfrac{\delta_3}{2(2\|B^{-1}A\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}+1)}\Big\}$. Then if $\|x-a\|_{\mathbb{R}^n}<\delta_1$, using (7.2.3) we find that

$$\big\|(x,f(x))-(a,f(a))\big\|_{\mathbb{R}^{n+m}}\leqslant\|x-a\|_{\mathbb{R}^n}+\|f(x)-f(a)\|_{\mathbb{R}^m}<\min\{\delta_2,\delta_3\}\,;$$

thus if $\|x-a\|_{\mathbb{R}^n} < \delta_1$,

$$\begin{aligned}
&\big\|f(x) - f(a) + \widetilde{B}^{-1}\widetilde{A}(x-a)\big\|_{\mathbb{R}^m} \\
&\quad = \big\|\big(\widetilde{B}^{-1}\widetilde{A} - B^{-1}A\big)(x-a) + B^{-1}\big(r(x,f(x)) - r(a,f(a))\big)\big\|_{\mathbb{R}^m} \\
&\quad \leqslant \big\|\widetilde{B}^{-1}\widetilde{A} - B^{-1}A\big\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)}\|x-a\|_{\mathbb{R}^n} \\
&\qquad + \|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}\big\|r(x,f(x)) - r(a,f(a)) - (Dr)(b)\big(x-a, f(x)-f(a)\big)\big\|_{\mathbb{R}^m} \\
&\qquad + \|B^{-1}\|_{\mathscr{B}(\mathbb{R}^m,\mathbb{R}^m)}\big\|(Dr)(b)\big((x-a, f(x)-f(a)\big)\big\|_{\mathbb{R}^m} \leqslant \varepsilon\|x-a\|_{\mathbb{R}^n}\,.
\end{aligned}$$

Therefore, f is differentiable on $\mathcal{U}$, and

$$(Df)(x) = -\big((D_yF)(x,f(x))\big)^{-1}(D_xF)\big(x,f(x)\big) \qquad \forall\, x \in \mathcal{U}\,. \tag{7.2.4}$$

Since F is of class $\mathscr{C}^1$ and f is continuous on $\mathcal{U}$, we find that Df is continuous; thus f is of class $\mathscr{C}^1$. ▫

Example 7.12. Let $F(x,y) = x^2 + y^2 - 1$.

1. If $(x_0,y_0) = (1,0)$, then $F_x(x_0,y_0) = 2 \neq 0$; thus the implicit function theorem implies that locally x can be expressed as a function of y.

2. If $(x_0,y_0) = (0,-1)$, then $F_y(x_0,y_0) = -2 \neq 0$; thus the implicit function theorem implies that locally y can be expressed as a function of x.

3. If $(x_0,y_0) = \big(-\frac{1}{2}, \frac{\sqrt{3}}{2}\big)$, then $F_x(x_0,y_0) = -1 \neq 0$ and $F_y(x_0,y_0) = \sqrt{3} \neq 0$; thus the implicit function theorem implies that locally x can be expressed as a function of y and locally y can be expressed as a function of x.

Example 7.13. Suppose that (x,y,u,v) satisfies the equation

$$\left\{\begin{aligned} xu + yv^2 &= 0 \\ xv^3 + y^2u^6 &= 0 \end{aligned}\right.$$

and $(x_0,y_0,u_0,v_0) = (1,-1,1,-1)$. Let $F(x,y,u,v) = (xu + yv^2, xv^3 + y^2u^6)$. Then $F(x_0,y_0,u_0,v_0) = 0$.

1. Since $(D_{x,y}F)(x_0,y_0,u_0,v_0) = \begin{bmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} \end{bmatrix}(x_0,y_0,u_0,v_0) = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ is invertible, locally (x,y) can be expressed in terms of u,v; that is, locally $x = x(u,v)$ and $y = y(u,v)$.

2. Since $(D_{y,u}F)(x_0,y_0,u_0,v_0) = \begin{bmatrix} \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial u} \\ \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial u} \end{bmatrix}(x_0,y_0,u_0,v_0) = \begin{bmatrix} 1 & 1 \\ -2 & 6 \end{bmatrix}$ is invertible, locally (y,u) can be expressed in terms of x, v.

Example 7.14. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$f(x,y,z) = (xe^y + ye^z, xe^z + ze^y)\,.$$

Then f is of class $\mathscr{C}^1$, $f(-1,1,1) = (0,0)$ and

$$\big[(Df)(x,y,z)\big] = \begin{bmatrix} e^y & xe^y + e^z & ye^z \\ e^z & ze^y & xe^z + e^y \end{bmatrix}.$$

Since $(D_{y,z}f)(-1,1,1) = \begin{bmatrix} 0 & e \\ e & 0 \end{bmatrix}$ is invertible, the implicit function theorem implies that the system

$$\left\{ \begin{array}{l} xe^y + ye^z = 0 \\ xe^z + ze^y = 0 \end{array} \right.$$

can be solved for y and z as continuously differentiable function of x for x near -1 and (y,z) near $(1,1)$. Furthermore, if we write $(y,z) = g(x)$ for x near -1, then

$$g'(x) = \begin{bmatrix} xe^y + e^z & ye^z \\ ze^y & xe^z + e^y \end{bmatrix}^{-1} \begin{bmatrix} e^y \\ e^z \end{bmatrix}.$$

7.3 The Lagrange Multipliers

In this section we are concerned with the optimization problem

"find the extreme value of function $y = f(x)$ subject to the constraint $g(x) = 0$", (7.3.1)

where $f : \mathbb{R}^n \to \mathbb{R}$ is a real-value differentiable function, and $g : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued $\mathscr{C}^1$-function for some $m < n$. We assume that the feasible set $\{x \,|\, g(x) = 0\}$ is non-empty and does not contain isolated points (or the extreme value of f is not attained at isolated points of the feasible set), and all the local extreme points of f do not belong to the feasible set (otherwise we can ignore the constraint and find the local extreme value of f directly).

Suppose that f attains its extreme value, subject to the constraint $g = 0$, at point a, and the Jacobian matrix of g at a has rank m (that is, $[(Dg)(a)]$ has full rank). Without loss of generality we can assume that the square matrix

$$\begin{bmatrix} \dfrac{\partial g_1}{\partial x_1}(a) & \cdots & \dfrac{\partial g_1}{\partial x_m}(a) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_m}{\partial x_1}(a) & \cdots & \dfrac{\partial g_m}{\partial x_m}(a) \end{bmatrix}$$

is invertible (thus has rank m). Then the implicit function theorem implies there exist an open neighborhood $\mathcal{V}$ of $(a_{m+1}, \cdots, a_n)$ and a $\mathscr{C}^1$-function $\varphi : \mathcal{V} \to \mathbb{R}^m$ such that in a neighborhood of a, the feasible set can be expressed as

$$(x_1, \cdots, x_m) = \varphi(x_{m+1}, \cdots, x_n) \qquad (x_{m+1}, \cdots, x_n) \in \mathcal{V}.$$

Under these settings, the original optimization problem (7.3.1) is transformed as "the function $y = f\big(\varphi(x_{m+1}, \cdots, x_n), x_{m+1}, \cdots, x_n\big)$ attains its extreme value at $(a_{m+1}, \cdots, a_n)$"; thus

$$\begin{aligned} \frac{\partial}{\partial x_{m+1}}\Big|_{(x_{m+1},\cdots,x_n)=(a_{m+1},\cdots,a_n)} f\big(\varphi(x_{m+1}, \cdots, x_n), x_{m+1}, \cdots, x_n\big) &= 0\,, \\ \frac{\partial}{\partial x_{m+2}}\Big|_{(x_{m+1},\cdots,x_n)=(a_{m+1},\cdots,a_n)} f\big(\varphi(x_{m+1}, \cdots, x_n), x_{m+1}, \cdots, x_n\big) &= 0\,, \\ &\vdots \\ \frac{\partial}{\partial x_{n}}\Big|_{(x_{m+1},\cdots,x_n)=(a_{m+1},\cdots,a_n)} f\big(\varphi(x_{m+1}, \cdots, x_n), x_{m+1}, \cdots, x_n\big) &= 0\,, \end{aligned}$$

or using the chain rule,

$$\begin{bmatrix} \dfrac{\partial f}{\partial x_1}(a) & \cdots & \dfrac{\partial f}{\partial x_m}(a) \end{bmatrix} \frac{\partial \varphi}{\partial x_j}(a_{m+1}, \cdots, a_n) + \frac{\partial f}{\partial x_j}(a) = 0 \quad \text{for } m+1 \leqslant j \leqslant n\,. \tag{7.3.2}$$

Noting that the implicit function theorem implies that

$$\big[(D\varphi)(a_{m+1}, \cdots, a_n)\big] = -\begin{bmatrix} \dfrac{\partial g_1}{\partial x_1}(a) & \cdots & \dfrac{\partial g_1}{\partial x_m}(a) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_m}{\partial x_1}(a) & \cdots & \dfrac{\partial g_m}{\partial x_m}(a) \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial g_1}{\partial x_{m+1}}(a) & \cdots & \dfrac{\partial g_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \dfrac{\partial g_m}{\partial x_{m+1}}(a) & \cdots & \dfrac{\partial g_m}{\partial x_n}(a) \end{bmatrix},$$

we find that for $m+1 \leqslant j \leqslant n$,

$$\frac{\partial \varphi}{\partial x_j}(a_{m+1}, \cdots, a_n) = -\begin{bmatrix} \dfrac{\partial g_1}{\partial x_1}(a) & \cdots & \dfrac{\partial g_1}{\partial x_m}(a) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_m}{\partial x_1}(a) & \cdots & \dfrac{\partial g_m}{\partial x_m}(a) \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial g_1}{\partial x_j}(a) \\ \vdots \\ \dfrac{\partial g_m}{\partial x_j}(a) \end{bmatrix}.$$

Define

$$\begin{bmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_m\end{bmatrix} = -\begin{bmatrix}\dfrac{\partial f}{\partial x_1}(a) & \cdots & \dfrac{\partial f}{\partial x_m}(a)\end{bmatrix}\begin{bmatrix}\dfrac{\partial g_1}{\partial x_1}(a) & \cdots & \dfrac{\partial g_1}{\partial x_m}(a)\\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_m}{\partial x_1}(a) & \cdots & \dfrac{\partial g_m}{\partial x_m}(a)\end{bmatrix}^{-1}.$$

Then (7.3.2) implies that

$$\frac{\partial f}{\partial x_j}(a) + \lambda_1\frac{\partial g_1}{\partial x_j}(a) + \lambda_2\frac{\partial g_2}{\partial x_j}(a) + \cdots + \lambda_m\frac{\partial g_m}{\partial x_j}(a) = 0 \qquad \text{for } m+1 \leqslant j \leqslant n. \tag{7.3.3}$$

On the other hand, by the fact that

$$\begin{bmatrix}\dfrac{\partial g_1}{\partial x_1}(a) & \cdots & \dfrac{\partial g_1}{\partial x_m}(a)\\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_m}{\partial x_1}(a) & \cdots & \dfrac{\partial g_m}{\partial x_m}(a)\end{bmatrix}^{-1}\begin{bmatrix}\dfrac{\partial g_1}{\partial x_1}(a) & \cdots & \dfrac{\partial g_1}{\partial x_m}(a)\\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_m}{\partial x_1}(a) & \cdots & \dfrac{\partial g_m}{\partial x_m}(a)\end{bmatrix} = \mathrm{I}_{m\times m},$$

for $1 \leqslant j \leqslant m$,

$$\begin{bmatrix}\dfrac{\partial g_1}{\partial x_1}(a) & \cdots & \dfrac{\partial g_1}{\partial x_m}(a)\\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_m}{\partial x_1}(a) & \cdots & \dfrac{\partial g_m}{\partial x_m}(a)\end{bmatrix}^{-1}\begin{bmatrix}\dfrac{\partial g_1}{\partial x_j}(a)\\ \vdots \\ \dfrac{\partial g_m}{\partial x_j}(a)\end{bmatrix} = \mathrm{e}_j,$$

where $\{\mathrm{e}_j\}_{j=1}^m$ is the standard basis of $\mathbb{R}^m$. Therefore, left multiplying the equation above by the row vector $-\begin{bmatrix}\dfrac{\partial f}{\partial x_1}(a) & \cdots & \dfrac{\partial f}{\partial x_m}(a)\end{bmatrix}$, we obtain that

$$\lambda_1\frac{\partial g_1}{\partial x_j}(a) + \lambda_2\frac{\partial g_2}{\partial x_j}(a) + \cdots + \lambda_m\frac{\partial g_m}{\partial x_j}(a) = -\frac{\partial f}{\partial x_j}(a) \qquad \text{for } 1 \leqslant j \leqslant m. \tag{7.3.4}$$

Combining (7.3.3) and (7.3.4), we conclude that

$$\frac{\partial f}{\partial x_j}(a) + \lambda_1\frac{\partial g_1}{\partial x_j}(a) + \lambda_2\frac{\partial g_2}{\partial x_j}(a) + \cdots + \lambda_m\frac{\partial g_m}{\partial x_j}(a) = 0 \qquad \text{for } 1 \leqslant j \leqslant n$$

or equivalently,

$$\big[(Df)(a)\big] + \begin{bmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_m\end{bmatrix}\big[(Dg)(a)\big] = 0\,.$$

We then establish the following

Theorem 7.15 (Lagrange Multipliers)**.** *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set, $f : \mathcal{U} \to \mathbb{R}$ be differentiable, $g : \mathcal{U} \to \mathbb{R}^m$ be of class $\mathscr{C}^1$ for some $m < n$, and $\mathcal{S} = \{x \in \mathcal{U} \,|\, g(x) = 0\}$. Suppose that $\mathcal{S} \neq \varnothing$ and $f|_{\mathcal{S}}$, the restriction of f on $\mathcal{S}$, attains its extreme value at $a \in \mathcal{S}$. If $[(Dg)(a)]$ has full rank, then there exist $\lambda_1, \lambda_2, \cdots, \lambda_m \in \mathbb{R}$ such that*

$$[(Df)(a)] + \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \end{bmatrix} [(Dg)(a)] = 0\,.$$

7.4 Exercises

§7.1 The Inverse Function Theorem

Problem 7.1. Prove Corollary 7.4; that is, show that if $\mathcal{U} \subseteq \mathbb{R}^n$ is open, $f : \mathcal{U} \to \mathbb{R}^n$ is of class $\mathscr{C}^1$, and $(Df)(x)$ is invertible for all $x \in \mathcal{U}$, then $f(\mathcal{W})$ is open for every open set $\mathcal{W} \subseteq \mathcal{U}$.

§7.2 The Implicit Function Theorem

Problem 7.2. Assume that one proves the implicit function theorem without applying the inverse theorem. Show the inverse function using the implicit function theorem.

Problem 7.3. Suppose that $F(x, y, z) = 0$ is such that the functions $z = f(x, y)$, $x = g(y, z)$, and $y = h(z, x)$ all exist by the implicit function theorem. Show that $f_x \cdot g_y \cdot h_z = -1$.

Problem 7.4. Suppose that the implicit function theorem applies to $F(x, y) = 0$ so that $y = f(x)$. Find a formula for f'' in terms of F and its partial derivatives. Similarly, suppose that the implicit function theorem applies to $F(x_1, x_2, y) = 0$ so that $y = f(x_1, x_2)$. Find formulas for $f_{x_1x_1}$, $f_{x_1x_2}$ and $f_{x_2x_2}$ in terms of F and its partial derivatives.

Problem 7.5. Given $F : \mathbb{R}^4 \to \mathbb{R}^2$ and suppose that $F(x, y) = 0$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. State conditions which guarantees that the equation

$$\frac{\partial y_1}{\partial x_1}\frac{\partial x_2}{\partial y_1} + \frac{\partial y_2}{\partial x_1}\frac{\partial x_2}{\partial y_2} = 0$$

holds. Prove or justify your answer.

§7.3 The Lagranege Multipliers

Problem 7.6 (**True or False**)**.** Determine whether the following statements are true or false. If it is true, prove it. Otherwise, give a counter-example.

1.

2.

3.