Chapter 5

Uniform Convergence and the Space of Continuous Functions

5.1 Pointwise and Uniform Convergence（逐點收斂與均勻收斂）

Definition 5.1. Let (M, d) and (N, ρ) be two metric spaces, $A \subseteq M$ be a set, and $f_k : A \to N$ be functions for $k = 1, 2, \cdots$. The sequence of functions $\{f_k\}_{k=1}^{\infty}$ is said to ***converge pointwise*** if $\{f_k(a)\}_{k=1}^{\infty}$ converges for all $a \in A$. In other words, $\{f_k\}_{k=1}^{\infty}$ converges pointwise if there exists a function $f : A \to N$ such that

$$\lim_{k\to\infty} \rho\big(f_k(a), f(a)\big) = 0 \qquad \forall\, a \in A.$$

In this case, $\{f_k\}_{k=1}^{\infty}$ is said to converge pointwise to f and is denoted by $f_k \to f$ p.w..

Let $B \subseteq A$ be a subset. The sequence of functions $\{f_k\}_{k=1}^{\infty}$ is said to ***converge uniformly*** on B if there exists $f : B \to N$ such that

$$\lim_{k\to\infty} \sup_{x\in B} \rho\big(f_k(x), f(x)\big) = 0\,.$$

In this case, $\{f_k\}_{k=1}^{\infty}$ is said to converge uniformly to f on B (or converge to f uniformly on B). In other words, $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on B if for every $\varepsilon > 0$, $\exists\, N > 0$ such that

$$\rho\big(f_k(x), f(x)\big) < \varepsilon \qquad \forall\, k \geqslant N \text{ and } x \in B\,.$$

Example 5.2. Let $f_k, f : [0,1] \to \mathbb{R}$ be given by

$$f_k(x) = \begin{cases} 0 & \text{if } \frac{1}{k} \leqslant x \leqslant 1\,, \\ -kx+1 & \text{if } 0 \leqslant x < \frac{1}{k}. \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 0 & \text{if } x \in (0,1], \\ 1 & \text{if } x = 0. \end{cases}$$

Then $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f on $[0,1]$. To see this, fix $x \in [0,1]$.

1. Case $x \neq 0$: Let $\varepsilon > 0$ be given, take $N > \frac{1}{x} \Leftrightarrow \frac{1}{N} < x$. If $k \geqslant N$,

$$\big|f_k(x) - f(x)\big| = \big|f_k(x) - 0\big| = |0-0| < \varepsilon\,.$$

2. Case $x = 0$: For any $\varepsilon > 0$, $k = 1,2,3,\dots$, $\big|f_k(0) - f(0)\big| = |1-1| = 0 < \varepsilon$.

However, $\{f_k\}_{k=1}^{\infty}$ does not converge uniformly to f on $[0,1]$ because

$$\sup_{x\in[0,1]} \big|f_k(x) - f(x)\big| = 1 \Rightarrow \lim_{k\to\infty} \sup_{x\in[0,1]} \big|f_k(x) - f(x)\big| = 1 \neq 0\,.$$

Example 5.3. Let $f_k : [0,1] \to \mathbb{R}$ be given by $f_k(x) = x^k$. Then for each $a \in [0,1)$, $f_k(a) \to 0$ as $k \to \infty$, while if $a = 1$, $f_k(a) = 1$ for all k. Therefore, if $f(x) = \begin{cases} 0 & \text{if } x \in [0,1)\,, \\ 1 & \text{if } x = 1\,, \end{cases}$ then $f_k \to f$ p.w.. However, since

$$\sup_{x\in[0,1]} \big|f_k(x) - f(x)\big| = \sup_{x\in[0,1)} \big|f_k(x)\big| = 1\,,$$

we must have

$$\lim_{k\to\infty} \sup_{x\in[0,1]} \big|f_k(x) - f(x)\big| = 1 \neq 0\,.$$

Therefore, $\{f_k\}_{k=1}^{\infty}$ does not converge uniformly to f on $[0,1]$.

On the other hand, if $0 < a < 1$, then

$$\sup_{x\in[0,a]} \big|f_k(x) - f(x)\big| \leqslant a^k\,;$$

thus by the Sandwich lemma,

$$\lim_{k\to\infty} \sup_{x\in[0,a]} \big|f_k(x) - f(x)\big| = 0\,.$$

Therefore, $\{f_k\}_{k=1}^{\infty}$ converges to uniformly f on $[0,a]$ if $0 < a < 1$.

Example 5.4. Let $f_k : \mathbb{R} \to \mathbb{R}$ be given by $f_k(x) = \dfrac{\sin x}{k}$. Then for each $x \in \mathbb{R}$, $|f_k(x)| \leqslant \dfrac{1}{k}$ which converges to 0 as $k \to \infty$. By the Sandwich lemma,

$$\lim_{k\to\infty} \big|f_k(x)\big| = 0 \qquad \forall\, x \in \mathbb{R}\,.$$

Therefore, $f_k \to 0$ p.w.. Moreover, since $\sup\limits_{x\in\mathbb{R}} \big|f_k(x)\big| \leqslant \dfrac{1}{k}$, $\lim\limits_{k\to\infty} \sup\limits_{x\in\mathbb{R}} \big|f_k(x)\big| = 0\,$. Therefore, $\{f_k\}_{k=1}^{\infty}$ converges uniformly to 0 on $\mathbb{R}$.

Proposition 5.5. *Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ be a set, and $f_k, f : A \to N$ be functions for $k = 1, 2, \cdots$. If $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on A, then $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f.*

Proof. For each $a \in A$,

$$\rho\big(f_k(a), f(a)\big) \leqslant \sup_{x\in A} \rho\big(f_k(x), f(x)\big)\,;$$

thus the Sandwich lemma shows that

$$\lim_{k\to\infty} \rho\big(f_k(a), f(a)\big) = 0$$

since $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on A. □

Proposition 5.6 (Cauchy criterion for uniform convergence)**.** *Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ be a set, and $f_k : A \to N$ be a sequence of functions. Suppose that (N,ρ) is complete. Then $\{f_k\}_{k=1}^{\infty}$ converges uniformly on $B \subseteq A$ if and only if for every $\varepsilon > 0$, $\exists\, N > 0$ such that*

$$\rho\big(f_k(x), f_\ell(x)\big) < \varepsilon \qquad \forall\, k, \ell \geqslant N \text{ and } x \in B\,.$$

Proof. "$\Rightarrow$" Suppose that $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on B. Let $\varepsilon > 0$ be given. Then $\exists\, N > 0$ such that

$$\rho\big(f_k(x), f(x)\big) < \frac{\varepsilon}{2} \qquad \forall\, k \geqslant N \text{ and } x \in B\,.$$

Then if $k, \ell \geqslant N$ and $x \in B$,

$$\rho\big(f_k(x), f_\ell(x)\big) \leqslant \rho\big(f_k(x), f(x)\big) + \rho\big(f(x), f_\ell(x)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

"$\Leftarrow$" Let $b \in B$. By assumption, $\{f_k(b)\}_{k=1}^{\infty}$ is a Cauchy sequence in (N,ρ); thus is convergent. Let $f(b)$ denote the limit of $\{f_k(b)\}_{k=1}^{\infty}$. Then $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f on B. We claim that the convergence is indeed uniform on B.

Let $\varepsilon > 0$ be given. Then $\exists\, N > 0$ such that

$$\rho\big(f_k(x), f_\ell(x)\big) < \frac{\varepsilon}{2} \qquad \forall\, k,\ell \geqslant N \text{ and } x \in B\,.$$

Moreover, for each $x \in B$ there exists $N_x > 0$ such that

$$\rho\big(f_\ell(x), f(x)\big) < \frac{\varepsilon}{2} \qquad \forall\, \ell \geqslant N_x\,.$$

Then for all $k \geqslant N$ and $x \in B$,

$$\rho\big(f_k(x), f(x)\big) \leqslant \rho\big(f_k(x), f_\ell(x)\big) + \rho\big(f_\ell(x), f(x)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

in which we choose $\ell \geqslant \max\{N, N_x\}$ to conclude the inequality. □

Theorem 5.7. *Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ be a set, and $f_k : A \to N$ be a sequence of continuous functions converging to $f : A \to N$ uniformly on A. Then f is continuous on A; that is,*

$$\lim_{x\to a} f(x) = \lim_{x\to a}\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty}\lim_{x\to a} f_k(x) = f(a)\,.$$

Proof. Let $a \in A$ and $\varepsilon > 0$ be given. Since $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on A, $\exists\, N > 0$ such that

$$\rho\big(f_k(x), f(x)\big) < \frac{\varepsilon}{3} \qquad \forall\, k \geqslant N \text{ and } x \in A\,.$$

By the continuity of f_N, $\exists\, \delta > 0$ such that

$$\rho\big(f_N(x), f_N(a)\big) < \frac{\varepsilon}{3} \text{ whenever } x \in D(a,\delta) \cap A\,.$$

Therefore, if $x \in D(a,\delta) \cap A$, by the triangle inequality

$$\begin{aligned}\rho\big(f(x), f(a)\big) &\leqslant \rho\big(f(x), f_N(x)\big) + \rho\big(f_N(x), f_N(a)\big) + \rho\big(f_N(a), f(a)\big)\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,;\end{aligned}$$

thus f is continuous at a. □

Example 5.8. Let $f_k : [0,2] \to \mathbb{R}$ be given by $f_k(x) = \dfrac{x^k}{1+x^k}$. Then

1. For each $a \in [0,1)$, $f_k(a) \to 0$ as $k \to \infty$;

2. For each $a \in (1,2]$, $f_k(a) \to 1$ as $k \to \infty$;

3. If $a = 1$, then $f_k(a) = \dfrac{1}{2}$ for all k.

Let $f(x) = \begin{cases} 0 & \text{if } x \in [0,1)\,, \\ \dfrac{1}{2} & \text{if } x = 1\,, \\ 1 & \text{if } x \in (1,2]\,. \end{cases}$ Then $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f. However, $\{f_k\}_{k=1}^{\infty}$ does not converge uniformly to f on $[0,2]$ since f_k are continuous functions for all $k \in \mathbb{N}$ but f is not.

Remark 5.9. The uniform limit of sequence of continuous function might not be uniformly continuous. For example, let $A = (0,1)$ and $f_k(x) = \dfrac{1}{x}$ for all $k \in \mathbb{N}$. Then $\{f_k\}_{k=1}^{\infty}$ converges uniformly to $f(x) = \dfrac{1}{x}$, but the limit function is not uniformly continuous on A.

Theorem 5.10. *Let $I \subseteq \mathbb{R}$ be a finite interval, $f_k : I \to \mathbb{R}$ be a sequence of differentiable functions, and $g : I \to \mathbb{R}$ be a function. Suppose that $\big\{f_k(a)\big\}_{k=1}^{\infty}$ converges for some $a \in I$, and $\{f_k'\}_{k=1}^{\infty}$ converges uniformly to g on I. Then*

1. *$\{f_k\}_{k=1}^{\infty}$ converges uniformly to some function f on I.*

2. *The limit function f is differentiable on I, and $f'(x) = g(x)$ for all $x \in I$; that is,*
$$\lim_{k\to\infty} f_k'(x) = \lim_{k\to\infty} \frac{d}{dx} f_k(x) = \frac{d}{dx} \lim_{k\to\infty} f_k(x) = f'(x)\,.$$

Proof. 1. Let $\varepsilon > 0$ be given. Since $\big\{f_k(a)\big\}_{k=1}^{\infty}$ converges to $f(a)$, $\big\{f_k(a)\big\}_{k=1}^{\infty}$ is a Cauchy sequence. Therefore, $\exists\, N_1 > 0$ such that
$$\big|f_k(a) - f_\ell(a)\big| < \frac{\varepsilon}{2} \qquad \forall\, k, \ell \geqslant N_1\,.$$
By the uniform convergence of $\{f_k'\}_{k=1}^{\infty}$ on I and Proposition 5.6, $\exists\, N_2 > 0$ such that
$$\big|f_k'(x) - f_\ell'(x)\big| < \frac{\varepsilon}{2|I|} \qquad \forall\, k, \ell \geqslant N_2 \text{ and } x \in I\,,$$
where $|I|$ is the length of the interval.

Let $N = \max\{N_1, N_2\}$. By the mean value theorem, for all $k, \ell \geqslant N$ and $x \in I$, there exists ξ in between x and a such that
$$\big|f_k(x) - f_\ell(x) - f_k(a) + f_\ell(a)\big| = |f_k'(\xi) - f_\ell'(\xi)||x - a| < \frac{\varepsilon|x-a|}{2|I|} \leqslant \frac{\varepsilon}{2}\,;$$

thus for all $k, \ell \geqslant N$ and $x \in I$,

$$\big|f_k(x) - f_\ell(x)\big| \leqslant \big|f_k(a) - f_\ell(a)\big| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, Proposition 5.6 implies that $\{f_k\}_{k=1}^\infty$ converges uniformly on I.

2. Suppose that the uniform limit of $\{f_k\}_{k=1}^\infty$ is f. Let $x \in I$ be a fixed point, and define

$$\phi_k(t) = \begin{cases} \dfrac{f_k(t) - f_k(x)}{t - x} & \text{if } t \in I,\, t \neq x\,, \\ f_k'(x) & \text{if } t = x\,, \end{cases} \quad \text{and} \quad \phi(t) = \begin{cases} \dfrac{f(t) - f(x)}{t - x} & \text{if } t \in I,\, t \neq x\,, \\ g(x) & \text{if } t = x\,. \end{cases}$$

Then ϕ_k is continuous on I for all $k \in \mathbb{N}$, and $\{\phi_k\}_{k=1}^\infty$ converges pointwise to ϕ.

Claim: $\{\phi_k\}_{k=1}^\infty$ converges uniformly to ϕ on I.

Proof of claim: Let $\varepsilon > 0$ be given. Since $\{f_k'\}_{k=1}^\infty$ converges uniformly on I, there exists $N > 0$ such that

$$\sup_{t \in I} \big|f_k'(t) - f_\ell'(t)\big| < \varepsilon \qquad \forall\, k, \ell \geqslant N\,.$$

Since

$$\big|\phi_k(t) - \phi_\ell(t)\big| = \begin{cases} \dfrac{\big|f_k(t) - f_k(x) - f_\ell(t) + f_\ell(x)\big|}{|t - x|} & \text{if } t \neq x,\, t \in I\,, \\ \big|f_k'(x) - f_\ell'(x)\big| & \text{if } t = x\,, \end{cases}$$

by the mean value theorem we obtain that

$$\big|\phi_k(t) - \phi_\ell(t)\big| \leqslant \sup_{s \in I} \big|f_k'(s) - f_\ell'(s)\big| < \varepsilon \qquad \forall\, k, \ell \geqslant N \text{ and } t \in I\,.$$

Finally, by Theorem 5.7, ϕ is continuous on I; thus

$$f'(x) = \lim_{t \to x} \phi(t) = \phi(x) = g(x)\,.$$

□

Example 5.11. Assume that $f_k : I \to \mathbb{R}$ is differentiable for all $k \in \mathbb{N}$, and $\{f_k'\}_{k=1}^\infty$ converges uniformly to g on I. Then $\{f_k\}_{k=1}^\infty$ might **NOT** converge. For example, consider $f_k(x) = k$. Then $f_k' \equiv 0$ but $\{f_k\}_{k=1}^\infty$ does not converge.

Example 5.12. Assume that $f_k : I \to \mathbb{R}$ is differentiable for all $k \in \mathbb{N}$, and $\{f_k\}_{k=1}^\infty$ converges uniformly to f on I. Then f might **NOT** be differentiable. In fact, there are differentiable functions $f_k : [a,b] \to \mathbb{R}$ such that f_k converges uniformly to f on $[a,b]$ but f is not differentiable. For example, consider

$$f_k(x) = \begin{cases} \dfrac{k}{2}x^2 & \text{if } |x| \leqslant \dfrac{1}{k}, \\ |x| - \dfrac{1}{2k} & \text{if } \dfrac{1}{k} \leqslant |x| \leqslant 1. \end{cases}$$

Observe that $f_k(-x) = f_k(x)$, so it suffices to consider $x \geqslant 0$.

1. Let $f(x) = |x|$. Then $f_k \to f$ uniformly:

$$\begin{aligned} \sup_{x\in[-1,1]} \big|f_k(x) - f(x)\big| &= \sup_{x\in[0,1]} \big|f_k(x) - x\big| = \max\Big\{ \sup_{x\in[0,\frac{1}{k}]} \big|f_k(x) - x\big|, \sup_{x\in[\frac{1}{k},1]} \big|f_k(x) - x\big| \Big\} \\ &= \max\Big\{ \sup_{x\in[0,\frac{1}{k}]} \Big|\frac{kx^2}{2} - x\Big|, \sup_{x\in[\frac{1}{k},1]} \Big|x - \frac{1}{2k} - x\Big| \Big\} \\ &\leqslant \sup_{x\in[0,\frac{1}{k}]} \Big|\frac{kx^2}{2}\Big| + |x| \leqslant \frac{k}{2}\Big(\frac{1}{k}\Big)^2 + \frac{1}{k} = \frac{3}{2k} \to 0 \text{ as } k \to \infty\,. \end{aligned}$$

2. To see if f_k are differentiable, it suffices to show $f_k'(\frac{1}{k})$ exists.

$$f_k'\Big(\frac{1}{k}\Big) = \lim_{h\to 0} \frac{f_k(\frac{1}{k}+h) - f_k(\frac{1}{k})}{h} = \lim_{h\to 0} \frac{1}{h} \begin{cases} \Big(\dfrac{1}{k} + h\Big) - \dfrac{1}{2k} - \dfrac{1}{2k} & \text{if } h > 0 \\ \dfrac{k}{2}\Big(\dfrac{1}{k} + h\Big)^2 - \dfrac{1}{2k} & \text{if } h < 0 \end{cases}$$

$$= \lim_{h\to 0} \frac{1}{h} \begin{cases} h & \text{if } h > 0 \\ h + \dfrac{k}{2}h^2 & \text{if } h < 0 \end{cases} = 1\,.$$

Example 5.13. Assume that $f_k : [-1,1] \to \mathbb{R}$ be given by

$$f_k(x) = \begin{cases} 0 & \text{if } x \in [-1,0]\,, \\ \dfrac{k^2}{2}x^2 & \text{if } x \in \big(0, \dfrac{1}{k}\big]\,, \\ 1 - \dfrac{k^2}{2}\Big(x - \dfrac{2}{k}\Big)^2 & \text{if } x \in \big(\dfrac{1}{k}, \dfrac{2}{k}\big]\,, \\ 1 & \text{if } x \in \big(\dfrac{2}{k}, 1\big]\,. \end{cases}$$

Then $f_k'(x) = \begin{cases} 0 & \text{if } x \in [-1,0]\,, \\ k^2 x & \text{if } x \in \big(0, \frac{1}{k}\big]\,, \\ -k^2\big(x - \frac{2}{k}\big) & \text{if } x \in \big(\frac{1}{k}, \frac{2}{k}\big]\,, \\ 0 & \text{if } x \in \big(\frac{2}{k}, 1\big]\,, \end{cases}$ and $\{f_k'\}_{k=1}^{\infty}$ converges pointwise to 0 but not uniformly on $[-1,1]$. We note that $\{f_k\}_{k=1}^{\infty}$ converges to a discontinuous function

$$f(x) = \begin{cases} 0 & \text{if } x \in [-1,0]\,, \\ 1 & \text{if } x \in (0,1]\,, \end{cases}$$

so the convergence of $\{f_k\}_{k=1}^{\infty}$ cannot be uniform on $[-1,1]$.

Example 5.14. Suppose $f_k : [0,1] \to \mathbb{R}$ are differentiable on $(0,1)$ and f_k converges uniformly to f on $[0,1]$ for some $f : [0,1] \to \mathbb{R}$. Does f_k' converge uniformly?
Answer: No! Take $f_k = \dfrac{\sin(k^2x)}{k}$, $k = 1, 2, \cdots$, then $f_k \to 0$ uniformly on $[0,1]$ since

$$\sup_{x\in[0,1]} \big|f_k(x) - 0\big| = \sup_{x\in[0,1]} \Big|\frac{\sin(k^2x)}{k}\Big| \leqslant \frac{1}{k} \Rightarrow \lim_{k\to\infty} \sup_{x\in[0,1]} \big|f_k(x) - 0\big| = 0\,.$$

Now $f_k'(x) = k\cos(k^2x)$ and $f_k'(0) = k \to \infty$ as $k \to \infty$.

Example 5.15. There are differentiable functions $f_k : [a,b] \to \mathbb{R}$ such that f_k converges uniformly to f on $[a,b]$ but $\lim\limits_{k\to\infty} f_k' \neq (\lim\limits_{k\to\infty} f_k)'$. For example, take $f_k(x) = \dfrac{x}{1+k^2x^2}$ on $[-1,1]$. Then $f_k'(x) = \dfrac{1-k^2x^2}{(1+k^2x^2)^2}$.

1. Since $\lim\limits_{k\to\infty} \sup\limits_{x\in[-1,1]} \Big|\dfrac{x}{1+k^2x^2} - 0\Big| = \lim\limits_{k\to\infty} \dfrac{1}{2k} = 0$, f_k converges uniformly to 0 on $[-1,1]$.
2. $(\lim\limits_{k\to\infty} f_k(x))' = 0' = 0$.
3. $\lim\limits_{k\to\infty} f_k'(x) = \lim\limits_{k\to\infty} \dfrac{1-k^2x^2}{(1+k^2x^2)^2} = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, |x| < 1. \end{cases}$ Note that f_k' does not converge uniformly.

Theorem 5.16. *Let $f_k : [a,b] \to \mathbb{R}$ be a sequence of Riemann integrable functions which converges uniformly to f on $[a,b]$. Then f is Riemann integrable, and*

$$\lim_{k\to\infty} \int_a^b f_k(x)dx = \int_a^b \lim_{k\to\infty} f_k(x)dx = \int_a^b f(x)dx\,. \tag{5.1.1}$$

Proof. Let $\varepsilon > 0$ be given. Since $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on $[a,b]$, $\exists\, N > 0$ such that

$$\big|f_k(x) - f(x)\big| < \frac{\varepsilon}{4(b-a)} \qquad \forall\, k \geqslant N \text{ and } x \in [a,b]\,. \tag{5.1.2}$$

Since f_N is Riemann integrable on $[a, b]$, by Riemann's condition there exists a partition $\mathcal{P}$ of $[a, b]$ such that

$$U(f_N, \mathcal{P}) - L(f_N, \mathcal{P}) < \frac{\varepsilon}{2}\,.$$

Using (4.7.3), we find that

$$\begin{aligned} U(f, \mathcal{P}) - L(f, \mathcal{P}) &= U(f - f_N + f_N, \mathcal{P}) - L(f - f_N + f_N, \mathcal{P}) \\ &\leqslant U(f - f_N, \mathcal{P}) + U(f_N, \mathcal{P}) - L(f - f_N, \mathcal{P}) - L(f_N, \mathcal{P}) \\ &\leqslant \frac{\varepsilon}{4(b-a)}(b-a) + \frac{\varepsilon}{4(b-a)}(b-a) + U(f_N, \mathcal{P}) - L(f_N, \mathcal{P}) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon\,; \end{aligned}$$

thus by Riemann's condition f is Riemann integrable on $[a, b]$.

Now, if $k \geqslant N$, (5.1.2) implies that

$$\begin{aligned} \Big|\int_a^b f_k(x)dx - \int_a^b f(x)dx\Big| = \Big|\int_a^b \big(f_k(x) - f(x)\big)dx\Big| &\leqslant \int_a^b \big|f_k(x) - f(x)\big|dx \\ &\leqslant \frac{\varepsilon}{4(b-a)}(b-a) = \frac{\varepsilon}{4} < \varepsilon \end{aligned}$$

which shows (5.1.1). □

Example 5.17. Let $\{q_k\}_{k=1}^{\infty}$ be the rational numbers in $[0, 1]$, and

$$f_k(x) = \begin{cases} 0 & \text{if } x \in \{q_1, q_2, \cdots, q_k\}\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

Then f_k converges pointwise to the Dirichlet function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0, 1]\,, \\ 1 & \text{if } x \in [0, 1]\backslash\mathbb{Q}\,. \end{cases}$$

However, $\{f_k\}_{k=1}^{\infty}$ does not converge uniformly to f since f_k are Riemann integrable on $[0, 1]$ for all $k \in \mathbb{N}$ but f is not.

Example 5.18. Let $f_k : [0, 1] \to \mathbb{R}$ be functions given in Example 5.13, and let $g_k = f_k'$. Then $\{g_k\}_{k=1}^{\infty}$ converges pointwise to 0, but $\displaystyle\int_0^1 g_k(x)dx = 1$ for all $k \in \mathbb{N}$.

5.2 Series of Functions and The Weierstrass M-Test

Definition 5.19. Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a norm space, $A\subseteq M$ be a subset, and $g_k, g : A\to\mathcal{V}$ be functions. We say that the series $\sum\limits_{k=1}^{\infty} g_k$ converges pointwise if the sequence of partitial sum $\{s_n\}_{n=1}^{\infty}$ given by

$$s_n=\sum_{k=1}^{n} g_k$$

converges pointwise. We use $\sum\limits_{k=1}^{\infty} g_k = g$ p.w. to denote that the series $\sum\limits_{k=1}^{\infty} g_k$ converges pointwise to g. We say that $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly on $B\subseteq A$ if $\{s_n\}_{n=1}^{\infty}$ converges uniformly on B.

Example 5.20. Consider the geometric series $\sum\limits_{k=0}^{\infty} x^k$. The partial sum s_n is given by

$$s_n(x)=\begin{cases}\dfrac{1-x^{n+1}}{1-x} & \text{if } x\neq 1\,,\\ n+1 & \text{if } x=1\,.\end{cases}$$

Then

1. $\sum\limits_{k=0}^{\infty} x^k$ converges pointwise to $g(x)=\dfrac{1}{1-x}$ in $(-1,1)$.

2. $\sum\limits_{k=0}^{\infty} x^k$ does not converge pointwise in $(-\infty,-1]\cup[1,\infty)$.

3. $\sum\limits_{k=0}^{\infty} x^k$ converges uniformly on $(-a,a)$ if $0<a<1$ since

$$\sup_{x\in(-a,a)}|s_n(x)-g(x)|=\sup_{x\in(-a,a)}\frac{|x|^{n+1}}{1-x}\leqslant\frac{|a|^{n+1}}{1-a}\to 0 \text{ as } n\to\infty.$$

4. $\sum\limits_{k=0}^{\infty} x^k$ does not converge uniformly on $(-1,1)$ since $\sup\limits_{x\in(-1,1)}|s_n(x)-g(x)|=\infty$.

The following two corollaries are direct consequences of Proposition 5.6 and Theorem 5.7.

Corollary 5.21. *Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a complete normed vector space, $A \subseteq M$ be a subset, and $g_k : A \to \mathcal{V}$ be functions. Then $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly on A if and only if*

$$\forall\, \varepsilon > 0, \exists\, N > 0 \ni \Big\| \sum_{k=m+1}^{n} g_k(x) \Big\| < \varepsilon \qquad \forall\, n > m \geqslant N \text{ and } x \in A\,.$$

Corollary 5.22. *Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, $A \subseteq M$ be a subset, and $g_k, g : A \to \mathcal{V}$ be functions. If $g_k : A \to \mathcal{V}$ are continuous and $\sum\limits_{k=1}^{\infty} g_k(x)$ converges to g uniformly on A, then g is continuous.*

Theorem 5.23. *Let $f : (a,b) \to \mathbb{R}$ be an infinitely differentiable functions; that is, $f^{(k)}(x)$ exists for all $k \in \mathbb{N}$ and $x \in (a,b)$. Let $c \in (a,b)$ and suppose that for some $0 < h < \infty$, $\big|f^{(k)}(x)\big| \leqslant M$ for all $x \in (c-h, c+h) \subseteq (a,b)$. Then*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \qquad \forall\, x \in (c-h, c+h)\,.$$

Proof. First, we claim that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^n \int_c^x \frac{(y-x)^n}{n!} f^{(n+1)}(y)dy \qquad \forall\, x \in (a,b)\,. \tag{5.2.1}$$

By the fundamental theorem or Calculus (Theorem 4.90) it is clear that (5.2.1) holds for $n = 0$. Suppose that (5.2.1) holds for $n = m$. Then

$$\begin{aligned} f(x) &= \sum_{k=0}^{m} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^m \Big[\frac{(y-x)^{m+1}}{(m+1)!} f^{(m+1)}(y)\Big|_{y=c}^{y=x} - \int_c^x \frac{(y-x)^{m+1}}{(m+1)!} f^{(m+2)}(y)dy \Big] \\ &= \sum_{k=0}^{m+1} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^{m+1} \int_c^x \frac{(y-x)^{m+1}}{(m+1)!} f^{(m+2)}(y)dy \end{aligned}$$

which implies that (5.2.1) also holds for $n = m+1$. By induction (5.2.1) holds for all $n \in \mathbb{N}$.

Letting $s_n(x) = \sum\limits_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$, then if $x \in (c-h, c+h)$,

$$\big|s_n(x) - f(x)\big| \leqslant \Big| \int_c^x \frac{h^n}{n!} M dy \Big| \leqslant \frac{h^{n+1}}{n!} M\,.$$

Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} \frac{h^{n+1}}{n!} M = 0$, $\exists\, N > 0$ such that $\Big|\frac{h^{n+1}}{n!}\Big| M < \varepsilon$ if $n \geqslant N$. As a consequence, if $n \geqslant N$,

$$\big|s_n(x) - f(x)\big| < \varepsilon \quad \text{whenever } n \geqslant N\,.$$

□

Example 5.24. The series $\sum_{k=0}^{\infty}(-1)^k\dfrac{x^{2k+1}}{(2k+1)!}$ converges to $\sin x$ uniformly on any bounded subset of $\mathbb{R}$.

Theorem 5.25 (Weierstrass M-test)**.** *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a complete normed vector space, $A\subseteq M$ be a subset, and $g_k : A\to\mathcal{V}$ be a sequence of functions. Suppose that there exists $M_k>0$ such that $\sup\limits_{x\in A}\|g_k(x)\|\leqslant M_k$ for all $k\in\mathbb{N}$ and $\sum_{k=1}^{\infty}M_k$ converges. Then $\sum_{k=1}^{\infty}g_k$ converges uniformly and absolutely (that is, $\sum_{k=1}^{\infty}\|g_k\|$ converges uniformly) on A.*

Proof. We show that the partial sum $s_n=\sum_{k=1}^{n}g_k$ satisfies the Cauchy criterion. Let $\varepsilon>0$ be given. Since $\sum_{k=1}^{\infty}M_k$ converges (which means $\sum_{k=1}^{n}M_k$ converges as $n\to\infty$), there exists $N>0$ such that

$$\sum_{k=m+1}^{n}M_k=\Big|\sum_{k=m+1}^{n}M_k\Big|<\varepsilon\quad\forall\, n>m\geqslant N\,.$$

Therefore,

$$\Big\|\sum_{k=m+1}^{n}g_k(x)\Big\|\leqslant\sum_{k=m+1}^{n}\|g_k(x)\|\leqslant\sum_{k=m+1}^{n}M_k<\varepsilon\quad\forall\, n>m\geqslant N \text{ and } x\in A\,.$$

Apply Proposition 5.6 to the sequence $\{s_n\}_{n=1}^{\infty}$, we conclude the theorem. ▫

Theorem 5.7 and 5.25 together imply the following

Corollary 5.26. *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a complete normed vector space, $A\subseteq M$ be a subset, and $g_k : A\to\mathcal{V}$ be a sequence of continuous functions. Suppose that there exists $M_k>0$ such that $\sup\limits_{x\in A}\|g_k(x)\|\leqslant M_k$ for all $k\in\mathbb{N}$ and $\sum_{k=1}^{\infty}M_k$ converges. Then $\sum_{k=1}^{\infty}g_k$ is continuous on A.*

Example 5.27. Consider the series $f(x)=\sum_{k=0}^{\infty}\Big(\dfrac{x^k}{k!}\Big)^2$. For all $x\in[-R,R]$, $\Big(\dfrac{x^k}{k!}\Big)^2\leqslant\dfrac{R^{2k}}{(k!)^2}$. Moreover,

$$\limsup_{k\to\infty}\frac{R^{2(k+1)}}{((k+1)!)^2}\Big/\frac{R^{2k}}{(k!)^2}=\limsup_{k\to\infty}\frac{R^2}{(k+1)^2}=0\,;$$

thus the ratio test and the Weierstrass M-test imply that the series $\sum_{k=0}^{\infty}\Big(\dfrac{x^k}{k!}\Big)^2$ converges uniformly on $[-R,R]$. Theorem 5.7 then shows that f is continuous on $[-R,R]$. Since R is arbitrary, we find that f is continuous on $\mathbb{R}$.

Example 5.28. Let $\{a_k\}_{k=0}^{\infty}$ be a bounded sequence. Then $\sum\limits_{k=0}^{\infty} \frac{a_k}{k!}x^k$ converges to a continuous function.

Example 5.29. Consider the function $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum\limits_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2}$. We can in fact show (much later) that $f(x) = |x|$ for all $x \in [-\pi, \pi]$, and by the Weierstrass M-test it is easy to see that the convergence is uniform on $\mathbb{R}$.

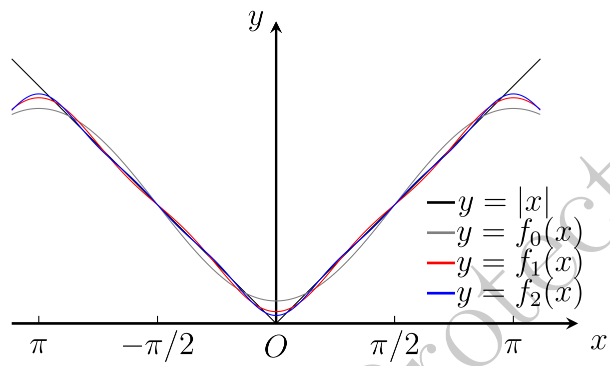

Figure 5.1: The graph of some partial sums

5.3 Integration and Differentiation of Series

The following two theorems are direct consequences of Theorem 5.10 and 5.16.

Theorem 5.30. *Let $g_k : [a, b] \to \mathbb{R}$ be a sequence of Riemann integrable functions. If $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly on $[a, b]$, then*

$$\int_a^b \sum_{k=1}^{\infty} g_k(x)dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)dx\,.$$

Theorem 5.31. *Let $g_k : (a, b) \to \mathbb{R}$ be a sequence of differentiable functions. Suppose that $\sum\limits_{k=1}^{\infty} g_k$ converges for some $c \in (a, b)$, and $\sum\limits_{k=1}^{\infty} g_k'$ converges uniformly on (a, b). Then*

$$\sum_{k=1}^{\infty} g_k'(x) = \frac{d}{dx}\sum_{k=1}^{\infty} g_k(x)\,.$$

Definition 5.32. A series is called a ***power series about*** c or ***centered at*** c if it is of the form $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ for some sequence $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$ (or $\mathbb{C}$) and $c \in \mathbb{R}$ (or $\mathbb{C}$).

Proposition 5.33. *If a power series centered at c is convergent at some point $b \neq c$, then the power series converges pointwise on $D(c,|b-c|)$, and converges uniformly on any compact subsets of $D(c,|b-c|)$.*

Proof. Since the series $\sum\limits_{k=0}^{\infty} a_k(b-c)^k$ converges, $|a_k||b-c|^k \to 0$ as $k \to \infty$; thus there exists $M > 0$ such that $|a_k||b-c|^k \leqslant M$ for all k.

1. $x \in D(c,|b-c|)$, the series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges absolutely since
$$\sum_{k=0}^{\infty} |a_k(x-c)^k| \leqslant \sum_{k=0}^{\infty} |a_k||x-c|^k = \sum_{k=0}^{\infty} |a_k||b-c|^k \frac{|x-c|^k}{|b-c|^k} \leqslant M \sum_{k=0}^{\infty} \Big(\frac{|x-c|}{|b-c|}\Big)^k$$
which converges (because of the geometric series test or ratio test).

2. Let $K \subseteq D(c,|b-c|)$ be a compact set. Then
$$\operatorname{dist}(K, \partial D(c,|b-c|)) \equiv \inf\big\{|x-y| \,\big|\, x \in K, |y-c| = |b-c|\big\} > 0\,.$$
Define $r = \dfrac{|b-c| - \operatorname{dist}(K, \partial D(c,|b-c|))}{|b-c|}$. Then $0 \leqslant r < 1$, and $|x-c| \leqslant r|b-c|$ for all $x \in K$. Therefore, $|a_k(x-c)^k| \leqslant Mr^k$ if $x \in K$; thus the Weierstrass M-test implies that the series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges uniformly on K. □

By the proposition above, we immediately conclude that the collection of all x at which the power series converges must be connected and symmetric; thus is a disc or a point. This observation induce the following

Definition 5.34. A non-negative number R is called the ***radius of convergence*** of the power series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ if the series converges for all $x \in D(c,R)$ but diverges if $x \notin \overline{D(c,R)}$. In other words,
$$R = \sup\Big\{r \geqslant 0 \,\Big|\, \sum_{k=0}^{\infty} a_k(x-c)^k \text{ converges in } \overline{D(c,R)}\Big\}\,.$$
The ***interval of convergence*** or ***convergence interval*** of a power series is the collection of all x at which the power series converges.

Remark 5.35. A power series converges pointwise on its interval of convergence.

Theorem 5.36. *Let $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{C}$, $c \in \mathbb{C}$, $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ be a power series with radius of convergence $R > 0$, and $K \subseteq D(c,R)$ be a compact set. Then*

1. *The power series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges uniformly on K.*

2. *The power series $\sum\limits_{k=0}^{\infty}(k+1)a_{k+1}(x-c)^k$ converges pointwise on $D(c,R)$, and converges uniformly on K.*

Proof. 1. It is simply a restatement of Proposition 5.33.

2. By 1, it suffices to show that the power series $\sum\limits_{k=0}^{\infty}(k+1)a_{k+1}(x-c)^k$ converges pointwise on $D(c,R)$. Clearly the series converges at $x = c$. Let $x \in D(c,R)$ and $x \neq c$. Since $|x-c| < R$, there exists $b \in D(c,R)$ such that
$$|b-c| = \frac{R + |x-c|}{2}.$$
Then if $r = \dfrac{|x-c|}{|b-c|}$, $0 < r < 1$ and
$$\sum_{k=0}^{\infty}(k+1)|a_{k+1}||x-c|^k \leqslant \sum_{k=0}^{\infty}(k+1)|a_{k+1}||b-c|^k\Big(\frac{|x-c|}{|b-c|}\Big)^k \leqslant M\sum_{k=0}^{\infty}(k+1)r^k$$
for some $M > 0$. Note that the ratio test implies that the series $\sum\limits_{k=0}^{\infty}(k+1)r^k$ converges if $0 < r < 1$; thus the power series $\sum\limits_{k=0}^{\infty}(k+1)|a_{k+1}||x-c|^k$ converges by the comparison test. □

Corollary 5.37. *Let $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, and $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ be a power series with radius of convergence $R > 0$. Then $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ is differentiable in $(c-R, c+R)$ and is Riemann integrable over any closed intervals $[\alpha,\beta] \subseteq (c-R,c+R)$. Moreover,*
$$\frac{d}{dx}\sum_{k=0}^{\infty} a_k(x-c)^k = \sum_{k=1}^{\infty} k a_k(x-c)^{k-1} \qquad \forall\, x \in (c-R, c+R)$$
and
$$\int_{\alpha}^{\beta}\sum_{k=0}^{\infty} a_k(x-c)^k dx = \sum_{k=0}^{\infty} a_k\int_{\alpha}^{\beta}(x-c)^k dx\,.$$

Example 5.38. Let $\{a_k\}_{k=0}^{\infty}$ be a bounded sequence. Then

$$\frac{d}{dx}\Big(\sum_{k=0}^{\infty}\frac{a_k}{k!}x^k\Big) = \sum_{k=1}^{\infty}\frac{a_k}{(k-1)!}x^{k-1} = \sum_{k=0}^{\infty}\frac{a_{k+1}}{k!}x^k .$$

Example 5.39. We show $\int_0^t e^x dx = e^t - 1$ as follows. By Theorem 5.23, $e^x = \sum\limits_{k=0}^{\infty}\frac{x^k}{k!}$ and the convergence is uniform on any bounded sets of $\mathbb{R}$; thus Corollary 5.37 implies that

$$\int_0^t e^x dx = \int_0^t \sum_{k=0}^{\infty}\frac{x^k}{k!}dx = \sum_{k=0}^{\infty}\int_0^t \frac{x^k}{k!}dx = \sum_{k=0}^{\infty}\frac{t^{k+1}}{(k+1)!} = \sum_{k=1}^{\infty}\frac{t^k}{k!} = e^t - 1 .$$

Example 5.40. $\frac{d}{dx}\Big(\sum\limits_{k=1}^{\infty}\frac{x^k}{k}\Big) = \sum\limits_{k=1}^{\infty}x^{k-1} = \sum\limits_{k=0}^{\infty}x^k$ for all $x \in (-1,1)$; thus

$$\frac{d}{dx}\Big(\sum_{k=1}^{\infty}\frac{x^k}{k}\Big) = \frac{1}{1-x} \qquad \forall\, x \in (-1,1) .$$

As a consequence,

$$\sum_{k=1}^{\infty}\frac{t^k}{k} = \int_0^t \frac{d}{dx}\Big(\sum_{k=1}^{\infty}\frac{x^k}{k}\Big)dx = -\log(1-t) \qquad \forall\, t \in (-1,1) . \tag{5.3.1}$$

Using the alternating series test, it is clear that the left-hand side of (5.3.1) converges at $t = -1$. What is the value of

$$-\sum_{k=1}^{\infty}\frac{(-1)^k}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots ?$$

Consider the partial sum $\frac{d}{dx}\Big(\sum\limits_{k=1}^{n}\frac{x^k}{k}\Big) = \sum\limits_{k=0}^{n-1}x^k = \frac{1-x^n}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x}$. Integrating both sides over $[-1,0]$,

$$\Big|\sum_{k=1}^{n}\frac{(-1)^k}{k} + \log 2\Big| \leqslant \int_{-1}^{0}\frac{|x|^n}{1-x}dx \leqslant \int_{-1}^{0}(-x)^n dx = \frac{1}{n+1} \to 0 \text{ as } n \to \infty;$$

thus

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \log 2 .$$

In other words,

$$\sum_{k=1}^{\infty}\frac{t^k}{k} = -\log(1-t) \qquad \forall\, t \in [-1,1) .$$

Example 5.41. It is clear that $\dfrac{1}{1+x^2} = \sum\limits_{k=0}^{\infty} (-x^2)^k = \sum\limits_{k=0}^{\infty} (-1)^k x^{2k}$ for all $x \in (-1,1)$. So if $x \in (-1,1)$,

$$\begin{aligned}\tan^{-1} x &= \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{k=0}^{\infty} (-1)^k t^{2k} dt = \sum_{k=0}^{\infty} \int_0^x (-1)^k t^{2k} dt \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} t^{2k+1}\Big|_{t=0}^{t=x} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .\end{aligned}$$

The right-hand side of the identity above converges at $x = 1$. What is the value of

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots ?$$

Mimic the previous example, we consider

$$\begin{aligned}\tan^{-1} x &= \int_0^x \frac{dt}{1+t^2} = \int_0^x \frac{1-(-t^2)^{n+1}}{1+t^2} dt + \int_0^x \frac{(-t^2)^{n+1}}{1+t^2} dt \\ &= \int_0^x \sum_{k=0}^{n} (-1)^k t^{2k} dt + \int_0^x \frac{(-t^2)^{n+1}}{1+t^2} dt \\ &= \sum_{k=0}^{n} \int_0^x (-1)^k t^{2k} dt + \int_0^x \frac{(-t^2)^{n+1}}{1+t^2} dt = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} x^{2k+1} + \int_0^x \frac{(-t^2)^{n+1}}{1+t^2} dt \,;\end{aligned}$$

thus plugging $x = 1$,

$$\Big|\tan^{-1} 1 - \sum_{k=0}^{n} \frac{(-1)^k}{2k+1}\Big| \leqslant \int_0^1 \frac{t^{2(n+1)}}{1+t^2} dt \leqslant \int_0^1 t^{2(n+1)} dt = \frac{1}{2n+3} \to 0 \text{ as } n \to \infty.$$

Therefore,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \tan^{-1} 1 = \frac{\pi}{4} .$$

5.4 The Space of Continuous Functions

Definition 5.42. Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, and $A \subseteq M$ be a subset. We define $\mathscr{C}(A;\mathcal{V})$ as the collection of all continuous functions on A with value in $\mathcal{V}$; that is,

$$\mathscr{C}(A;\mathcal{V}) = \big\{ f : A \to \mathcal{V} \,\big|\, f \text{ is continuous on } A \big\} .$$

Let $\mathscr{C}_b(A;\mathcal{V})$ be the subspace of $\mathscr{C}(A;\mathcal{V})$ which consists of all bounded continuous functions on A; that is,

$$\mathscr{C}_b(A;\mathcal{V}) = \big\{f \in \mathscr{C}(A;\mathcal{V}) \,\big|\, f \text{ is bounded}\big\}\,.$$

Every $f \in \mathscr{C}_b(A;\mathcal{V})$ is associated with a non-negative real number $\|f\|_\infty$ given by

$$\|f\|_\infty = \sup\big\{\|f(x)\| \,\big|\, x \in A\big\} = \sup_{x\in A}\|f(x)\|$$

The number $\|f\|_\infty$ is called the ***sup-norm*** of f.

Proposition 5.43. *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a normed vector space, $A \subseteq M$ be a subset.*

1. *$\mathscr{C}(A;\mathcal{V})$ and $\mathscr{C}_b(A;\mathcal{V})$ are vector spaces.*
2. *$\big(\mathscr{C}_b(A;\mathcal{V}),\|\cdot\|_\infty\big)$ is a normed vector space.*
3. *If $K \subseteq M$ is compact, then $\mathscr{C}(K;\mathcal{V}) = \mathscr{C}_b(K;\mathcal{V})$.*

Proof. 1 and 2 are trivial, and 3 is concluded by Theorem 4.21. □

Remark 5.44. In general $\|\cdot\|_\infty$ is not a "norm" on $\mathscr{C}(A;\mathcal{V})$. For example, the function $f(x) = \dfrac{1}{x}$ belongs to $\mathscr{C}((0,1);\mathbb{R})$ and $\|f\|_\infty = \infty$. Note that to be a norm $\|f\|_\infty$ has to take values in $\mathbb{R}$, and $\infty \notin \mathbb{R}$.

Proposition 5.45. *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a normed vector space, $A \subseteq M$ be a subset, and $f_k \in \mathscr{C}_b(A;\mathcal{V})$ for all $k \in \mathbb{N}$. Then $\{f_k\}_{k=1}^\infty$ converges uniformly on A if and only if $\{f_k\}_{k=1}^\infty$ converges in $\big(\mathscr{C}_b(A;\mathcal{V}),\|\cdot\|_\infty\big)$.*

Proof. ($\Leftarrow$) Suppose that $\{f_k\}_{k=1}^\infty$ converges in $\big(\mathscr{C}_b(A;\mathcal{V}),\|\cdot\|_\infty\big)$. Then there exists $f \in \big(\mathscr{C}_b(A;\mathcal{V}),\|\cdot\|_\infty\big)$ such that $\lim\limits_{k\to\infty}\|f_k - f\|_\infty = 0$, and by the definition of the sup-norm,

$$\lim_{k\to\infty}\sup_{x\in A}\|f_k(x) - f(x)\| = 0\,.$$

Therefore, $\{f_k\}_{k=1}^\infty$ converges to f uniformly on A.

($\Rightarrow$) Suppose that $\{f_k\}_{k=1}^\infty$ converges uniformly on A. Then there exists a function $f : A \to \mathcal{V}$ such that

$$\lim_{k\to\infty}\sup_{x\in A}\|f_k(x) - f(x)\| = 0\,.$$

By the definition of the sup-norm, it suffices to show that $f \in \mathscr{C}_b(A;\mathcal{V})$ in order to conclude the proposition. By Theorem 5.7, we obtain that $f \in \mathscr{C}(A;\mathcal{V})$. Moreover, the uniform convergence implies that there exists $N > 0$ such that

$$\|f_k(x) - f(x)\| < 1 \qquad \forall\, k \geqslant N \text{ and } x \in A\,.$$

In particular, the boundedness of f_N provides $M > 0$ such that $\|f_N(x)\| \leqslant M$ for all $x \in A$; thus

$$\|f(x)\| \leqslant \|f_N(x)\| + \|f(x) - f_N(x)\| \leqslant M + 1 \qquad \forall\, x \in A\,.$$

This implies that f is bounded; thus $f \in \mathscr{C}_b(A;\mathcal{V})$. ▫

Theorem 5.46. *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a normed vector space, and $A \subseteq M$ be a subset. If $(\mathcal{V},\|\cdot\|)$ is complete, so is $\big(\mathscr{C}_b(A;\mathcal{V}),\|\cdot\|_\infty\big)$.*

Proof. Let $\{f_k\}_{k=1}^\infty$ be a Cauchy sequence in $\big(\mathscr{C}_b(A;\mathcal{V}),\|\cdot\|_\infty\big)$. Then

$$\forall\,\varepsilon > 0, \exists\, N > 0 \ni \|f_k - f_\ell\|_\infty < \varepsilon \text{ if } k,\ell \geqslant N\,.$$

By the definition of the sup-norm, the statement above implies that

$$\forall\,\varepsilon > 0, \exists\, N > 0 \ni \|f_k(x) - f_\ell(x)\| < \varepsilon \quad \text{if } k,\ell \geqslant N \text{ and } x \in A$$

which implies that $\{f_k\}_{k=1}^\infty$ satisfies the Cauchy criterion. By Proposition 5.6, $\{f_k\}_{k=1}^\infty$ converges uniformly on A, and Proposition 5.45 shows that $\{f_k\}_{k=1}^\infty$ converges in $\big(\mathscr{C}_b(A;\mathcal{V}),\|\cdot\|_\infty\big)$. ▫

Example 5.47. The set $B = \big\{f \in \mathscr{C}([0,1];\mathbb{R}) \,\big|\, f(x) > 0 \text{ for all } x \in [0,1]\big\}$ is open in $\big(\mathscr{C}([0,1];\mathbb{R}),\|\cdot\|_\infty\big)$.

Reason: Let $f \in B$ be given. Since $[0,1]$ is compact and f is continuous, by the extreme value theorem there exists $x_0 \in [0,1]$ so that $\inf\limits_{x\in[0,1]} f(x) = f(x_0) > 0$. Take $\varepsilon = \dfrac{f(x_0)}{2}$. Now if g is such that $\|g - f\|_\infty = \sup\limits_{x\in[0,1]} \big|g(x) - f(x)\big| < \varepsilon = \dfrac{f(x_0)}{2}$, we have for any $y \in [0,1]$,

$$\begin{aligned}
&\big|g(y) - f(y)\big| \leqslant \sup_{x\in[0,1]} \big|g(x) - f(x)\big| < \frac{f(x_0)}{2} \\
&\Rightarrow f(y) - \frac{f(x_0)}{2} \leqslant g(y) \leqslant f(y) + \frac{f(x_0)}{2} \\
&\Rightarrow g(y) \geqslant f(y) - \frac{f(x_0)}{2} \geqslant f(x_0) - \frac{f(x_0)}{2} = \frac{f(x_0)}{2} > 0\,.
\end{aligned}$$

Therefore, $g \in B$; thus $D(f, \varepsilon) \subseteq B$.

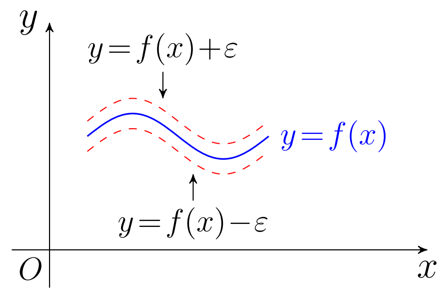

Figure 5.2: $g \in D(f, \varepsilon)$ if the graph of g lies in between the two red dash lines

Example 5.48. Find the closure of B given in the previous example.

Proof. *Claim*: $\text{cl}(B) = \{f \in \mathscr{C}([0,1], \mathbb{R}) \,|\, f(x) \geqslant 0\}$.
Proof of claim: We show $\forall f \in \{f \in \mathscr{C}([0,1], \mathbb{R}) \,|\, f(x) \geqslant 0\}, \exists f_k \in B \ni \|f_k - f\|_\infty \to 0$ as $k \to \infty$. Take $f_k(x) = f(x) + \dfrac{1}{k}$, then $f_k \in B$ ($\because f_k(x) > 0$), and

$$\|f_k - f\|_\infty = \sup_{x \in [0,1]} |f_k(x) - f(x)| \leqslant \sup_{x \in [0,1]} \frac{1}{k} = \frac{1}{k} \to 0 \text{ as } k \to \infty\,. \qquad \square$$

5.5 The Arzelà-Ascoli Theorem

在這一節中，我們將研究一般情況下，連續函數列的逐點收斂與均勻收斂之間的具體差異為何。更具體地說，我們希望能找到一個條件，使得逐點收斂的連續函數列，其均勻收斂性等價於該條件成立。這個條件，刻劃了均勻收斂與逐點收斂的真實差異，而這個特別的條件，也將提供額外（且有效）的判斷法，幫助我們判斷在連續函數空間裡面的集合是否緊緻。

5.5.1 Equi-continuous family of functions

The first part of this section is devoted to the investigation of the difference between the pointwise convergence and the uniform convergence of sequence of continuous functions.

Definition 5.49. Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, and $A \subseteq M$ be a subset. A subset $B \subseteq \mathscr{C}_b(A; \mathcal{V})$ is said to be ***equi-continuous***（等度連續）if

$$\forall \varepsilon > 0, \exists \delta > 0 \ni \|f(x_1) - f(x_2)\| < \varepsilon \quad \text{whenever } d(x_1, x_2) < \delta,\ x_1, x_2 \in A, \text{ and } f \in B\,.$$

Remark 5.50. 1. If $B \subseteq \mathscr{C}_b(A;\mathcal{V})$ is equi-continuous, and C is a subset of B, then C is also equi-continuous.

2. In an equi-continuous set of functions B, every $f \in B$ is uniformly continuous.

Remark 5.51. For a uniformly continuous function f, let $\delta_f(\varepsilon)$ (we have defined this number in Remark 4.51) denote the largest δ that can be used in the definition of the uniform continuity; that is, $\delta_f(\varepsilon)$ has the property that

$$\|f(x) - f(y)\| < \varepsilon \text{ whenever } d(x,y) < \delta, x, y \in A \quad \Leftrightarrow \quad 0 < \delta \leqslant \delta_f(\varepsilon)\,.$$

Suppose that every element in $B \subseteq \mathscr{C}_b(A;\mathcal{V})$ is uniformly continuous on A. Then B is equi-continuous if and only if $\inf\limits_{f \in B} \delta_f(\varepsilon) > 0$.

Example 5.52. Let $B = \big\{ f \in \mathscr{C}_b((0,1);\mathcal{V}) \,\big|\, |f'(x)| \leqslant 1 \text{ for all } x \in (0,1) \big\}$. Then B is equi-continuous (by choosing $\delta = \epsilon$ for any given ϵ, and applying the mean value theorem).

Example 5.53. Let $f_k : [0,1] \to \mathbb{R}$ be a sequence of functions given by

$$f_k(x) = \begin{cases} kx & \text{if } 0 \leqslant x \leqslant \dfrac{1}{k}\,, \\ 2 - kx & \text{if } \dfrac{1}{k} \leqslant x \leqslant \dfrac{2}{k}\,, \\ 0 & \text{if } x \geqslant \dfrac{2}{k}\,, \end{cases}$$

and $B = \{f_k\}_{k=1}^\infty$. Then B is not equi-continuous since the largest δ for each k is $\dfrac{\varepsilon}{k}$ which converges to 0.

Lemma 5.54. *Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, and $K \subseteq M$ be a compact subset. If $B \subseteq \mathscr{C}(K;\mathcal{V})$ is pre-compact, then B is equi-continuous.*

Proof. Suppose the contrary that B is not equi-continuous. Then $\exists\, \varepsilon > 0$ such that

$$\forall\, k \in \mathbb{N}, \exists\, x_k, y_k \in K \text{ and } f_k \in B \ni d(x_k, y_k) < \frac{1}{k} \text{ but } \|f_k(x_k) - f_k(y_k)\| \geqslant \varepsilon\,.$$

Since B is pre-compact in $\big(\mathscr{C}(K;\mathcal{V}), \|\cdot\|_\infty\big)$ and K is compact in (M,d), there exists a subsequence $\big\{f_{k_j}\big\}_{j=1}^\infty$ and $\{x_{k_j}\}_{j=1}^\infty$ such that $\big\{f_{k_j}\big\}_{j=1}^\infty$ converges uniformly to some function $f \in \big(\mathscr{C}(K;\mathcal{V}), \|\cdot\|_\infty\big)$ and $\{x_{k_j}\}_{j=1}^\infty$ converges to some $a \in K$. We must also have $\{y_{k_j}\}_{j=1}^\infty$ converges to a since $d(x_{k_j}, y_{k_j}) < \dfrac{1}{k_j}$.

Since f is continuous at a,

$$\exists\,\delta>0 \ni \|f(x)-f(a)\|<\frac{\varepsilon}{5} \qquad \text{if } x\in D(a,\delta)\cap K.$$

Moreover, since $\{f_{k_j}\}_{j=1}^{\infty}$ converges to f uniformly on K and $x_{k_j}, y_{k_j}\to a$ as $j\to\infty$, $\exists\,N>0$ such that

$$\|f_{k_j}(x)-f(x)\|<\frac{\varepsilon}{5} \qquad \text{if } j\geqslant N \text{ and } x\in K$$

and

$$d(x_{k_j},a)<\delta \quad\text{and}\quad d(y_{k_j},a)<\delta \qquad \text{if } j\geqslant N\,.$$

As a consequence, for all $j\geqslant N$,

$$\begin{aligned}\varepsilon\leqslant\|f_{k_j}(x_{k_j})-f_{k_j}(y_{k_j})\| &\leqslant \|f_{k_j}(x_{k_j})-f(x_{k_j})\|+\|f(x_{k_j})-f(a)\|\\ &\quad+\|f(y_{k_j})-f(a)\|+\|f(y_{k_j})-f_{k_j}(y_{k_j})\|<\frac{4\varepsilon}{5}\end{aligned}$$

which is a contradiction. □

Alternative proof of Lemma 5.54. Suppose the contrary that B is not equi-continuous. Then $\exists\,\varepsilon>0$ such that

$$\forall\,k\in\mathbb{N}, \exists\,x_k,y_k\in K \text{ and } f_k\in B\ni d(x_k,y_k)<\frac{1}{k} \text{ but } \|f_k(x_k)-f_k(y_k)\|\geqslant\varepsilon\,.$$

Since B is pre-compact in $\big(\mathscr{C}(K;\mathcal{V}),\|\cdot\|_\infty\big)$, there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ converges to some function f in $\big(\mathscr{C}(K;\mathcal{V}),\|\cdot\|_\infty\big)$. By Proposition 5.45, $\{f_{k_j}\}_{j=1}^{\infty}$ converges uniformly to f on K; thus there exists $N_1>0$ such that

$$\big\|f_{k_j}(x)-f(x)\big\|<\frac{\varepsilon}{4} \quad \forall\,j\geqslant N_1 \text{ and } x\in K\,.$$

Since $f\in\mathscr{C}(K;\mathcal{V})$, by Theorem 4.52, f is uniformly continuous on K; thus

$$\exists\,\delta>0\ni\|f(x)-f(y)\|<\frac{\varepsilon}{4} \quad \text{if } d(x,y)<\delta \text{ and } x,y\in K\,.$$

Let $N=\max\big\{N_1,\big[\frac{1}{\delta}\big]+1\big\}$, and $j\geqslant N$. Then $d(x_{k_j},y_{k_j})<\delta$ and this further implies that

$$\varepsilon\leqslant\|f_{k_j}(x_{k_j})-f_{k_j}(y_{k_j})\|\leqslant\|f_{k_j}(x_{k_j})-f(x_{k_j})\|+\|f(x_{k_j})-f(y_{k_j})\|+\|f(y_{k_j})-f_{k_j}(y_{k_j})\|<\frac{3\varepsilon}{4}\,,$$

a contradiction. □

Corollary 5.55. *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a normed vector space, and $K \subseteq M$ be a compact subset. If $\{f_k\}_{k=1}^\infty$ converges uniformly on K, then $\{f_k\}_{k=1}^\infty$ is equi-continuous.*

Example 5.56. Corollary 5.55 fails to hold if the compactness of K is removed. For example, let $\{f_k\}_{k=1}^\infty$ be a sequence of identical functions $f_k(x) = \dfrac{1}{x}$ on $(0,1)$. Then $\{f_k\}_{k=1}^\infty$ converges uniformly on $(0,1)$ but $\{f_k\}_{k=1}^\infty$ is not equi-continuous since none of f_k is uniformly continuous on $(0,1)$ which violates Remark 5.50.

We have just shown that if $\{f_k\}_{k=1}^\infty$ converges uniformly on a compact set K, then $\{f_k\}_{k=1}^\infty$ must be equi-continuous. The inverse statement, on the other hand, cannot be true. For example, taking $\{f_k\}_{k=1}^\infty$ to be a sequence of constant functions $f_k(x) = k$. Then $\{f_k\}_{k=1}^\infty$ obviously does not converge, not even any subsequence. Therefore, we would like to study under what additional conditions, equi-continuity of a sequence of functions (defined on a compact set K) indeed converges uniformly. The following lemma is an answer to the question.

Lemma 5.57. *Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a Banach space, $K \subseteq M$ be a compact set, and $\{f_k\}_{k=1}^\infty \subseteq \mathscr{C}(K;\mathcal{V})$ be a equi-continuous sequence of functions. If $\{f_k\}_{k=1}^\infty$ converges pointwise on a dense subset E of K (that is, $E \subseteq K \subseteq \mathrm{cl}(E)$), then $\{f_k\}_{k=1}^\infty$ converges uniformly on K.*

Proof. Let $\varepsilon > 0$ be given. By the equi-continuity of $\{f_k\}_{k=1}^\infty$,

$$\exists\, \delta > 0 \ni \|f_k(x) - f_k(y)\| < \frac{\varepsilon}{3} \quad \text{if } d(x,y) < \delta,\ x,y \in K \text{ and } k \in \mathbb{N}\,.$$

Since K is compact, K is totally bounded; thus

$$\exists \{y_1,\cdots,y_m\} \subseteq K \ni K \subseteq \bigcup_{j=1}^m D\big(y_j, \frac{\delta}{2}\big)\,.$$

By the denseness of E in K, for each $j = 1,\cdots,m$, there exists $z_j \in E$ such that $d(z_j,y_j) < \dfrac{\delta}{2}$. Moreover, $D\big(y_j,\dfrac{\delta}{2}\big) \subseteq D(z_j,\delta)$; thus $K \subseteq \bigcup\limits_{j=1}^m D(z_j,\delta)$. Since $\{f_k\}_{k=1}^\infty$ converges pointwise on E, $\{f_k(z_j)\}_{k=1}^\infty$ converges as $k \to \infty$ for all $j = 1,\cdots,m$. Therefore,

$$\exists\, N_j > 0 \ni \|f_k(z_j) - f_\ell(z_j)\| < \frac{\varepsilon}{3} \qquad \forall\, k,\ell \geqslant N_j\,.$$

Let $N = \max\{N_1,\cdots,N_m\}$, then

$$\|f_k(z_j) - f_\ell(z_j)\| < \frac{\varepsilon}{3} \qquad \forall\, k,\ell \geqslant N \text{ and } j = 1,\cdots,m\,.$$

Now we are in the position of concluding the lemma. If $x \in K$, there exists $z_j \in E$ such that $d(x, z_j) < \delta$; thus if we further assume that $k, \ell \geqslant N$,

$$\|f_k(x) - f_\ell(x)\| \leqslant \|f_k(x) - f_k(z_j)\| + \|f_k(z_j) - f_\ell(z_j)\| + \|f_\ell(z_j) - f_\ell(x)\| < \varepsilon\,.$$

By Proposition 5.6, $\{f_k\}_{k=1}^\infty$ converges uniformly on K. □

Remark 5.58. Corollary 5.55 and Lemma 5.57 imply that "a sequence $\{f_k\}_{k=1}^\infty \subseteq \mathscr{C}(K;\mathcal{V})$ converges uniformly on K if and only if $\{f_k\}_{k=1}^\infty$ is equi-continuous and pointwise convergent (on a dense subset of K)".

5.5.2 Compact sets in $\mathscr{C}(K;\mathcal{V})$

The next subject in this section is to obtain a (useful) criterion of determining the compactness (or pre-compactness) of a subset $B \subseteq \mathscr{C}(K;\mathcal{V})$ which guarantees the existence of a convergent subsequence $\{f_{k_j}\}_{j=1}^\infty$ of a given sequence $\{f_k\}_{k=1}^\infty \subseteq B$ in $\big(\mathscr{C}(K;\mathcal{V}), \|\cdot\|_\infty\big)$.

Lemma 5.59 (Cantor's Diagonal Process)**.** *Let E be a countable set, $(\mathcal{V}, \|\cdot\|)$ be a Banach space, and $f_k : E \to \mathcal{V}$ be a sequence of functions. Suppose that for each $x \in E$, $\{f_k(x)\}_{k=1}^\infty$ is pre-compact in $\mathcal{V}$. Then there exists a subsequence of $\{f_k\}_{k=1}^\infty$ that converges pointwise on E.*

Proof. Since E is countable, $E = \{x_\ell\}_{\ell=1}^\infty$.

1. Since $\{f_k(x_1)\}_{k=1}^\infty$ is pre-compact in $(\mathcal{V}, \|\cdot\|)$, there exists a subsequence $\{f_{k_j}\}_{j=1}^\infty$ such that $\{f_{k_j}(x_1)\}_{j=1}^\infty$ converges in $(\mathcal{V}, \|\cdot\|)$.

2. Since $\{f_k(x_2)\}_{k=1}^\infty$ is pre-compact in $(\mathcal{V}, \|\cdot\|)$, the sequence $\{f_{k_j}(x_2)\}_{j=1}^\infty \subseteq \{f_k(x_2)\}_{k=1}^\infty$ has a convergent subsequence $\{f_{k_{j_\ell}}(x_2)\}_{\ell=1}^\infty$.

Continuing this process, we obtain a sequence of sequences S_1, S_2, $\cdots$ such that

1. S_k consists of a subsequence of $\{f_k\}_{k=1}^\infty$ which converges at x_k, and

2. $S_k \supseteq S_{k+1}$ for all $k \in \mathbb{N}$.

Let g_k be the k-th element of S_k. Then the sequence $\{g_k\}_{k=1}^\infty$ is a subsequence of $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ converges at each point of E. □

The condition that "$\big\{f_k(x)\big\}_{k=1}^{\infty}$ is pre-compact in $\mathcal{V}$ for each $x \in E$" in Lemma 5.59 motivates the following

Definition 5.60. Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, and $A \subseteq M$ be a subset. A subset $B \subseteq \mathscr{C}_b(A;\mathcal{V})$ is said to be ***pointwise*** $\begin{array}{c}\textbf{\textit{compact}}\\ \textbf{\textit{pre-compact}}\\ \textbf{\textit{bounded}}\end{array}$ if the set $B_x \equiv \big\{f(x) \,\big|\, f \in B\big\}$ is $\begin{array}{c}\text{compact}\\ \text{pre-compact}\\ \text{bounded}\end{array}$ in $(\mathcal{V}, \|\cdot\|)$ for all $x \in A$.

Example 5.61. Let $f_k : [0,1] \to \mathbb{R}$ be given in Example 5.53, and $B = \{f_k\}_{k=1}^{\infty}$. Then B is pointwise compact: for each $x \in [0,1]$, B_x is a finite set since if $f_k(0) = 0$ for all $k \in \mathbb{N}$, while if $x > 0$, $f_k(x) = 0$ for all k large enough which implies that $\#B_x < \infty$.

是時候可以來看 $\mathscr{C}(K;\mathcal{V})$ 裡面的 compact sets 有什麼等價條件了。首先我們先看何時 $B \subseteq \mathscr{C}(K;\mathcal{V})$ 是 compact set。給定一個函數列 $\{f_k\}_{k=1}^{\infty} \subseteq B$，我們想知道能不能找到一個在 sup-norm 下收斂的 subsequence $\big\{f_{k_j}\big\}_{j=1}^{\infty}$（即 sequentially compact）。由 Diagonal Process (Lemma 5.59) 知，我們得在 K 中找一個稠密的子集合 E 使得 $\{f_k\}_{k=1}^{\infty}$ 在 E 上是 pointwise pre-compact（這個部份只保證了可以找到 subsequence 逐點收斂），然後加上 Lemma 5.57 的幫助，馬上知道加上 equi-continuity 的條件之後，逐點收斂會變均勻收斂。因此，很自然地我們會要求 B 滿足 pointwise pre-compact 還有 equi-continuous 這兩個條件來證出 B 是 $\mathscr{C}(K;\mathcal{V})$ 中的 compact set。而在一個 compact set K 中能不能找到一個稠密子集合則是由下面這個 Lemma 所提供。

Lemma 5.62. *A compact set K in a metric space (M,d) is separable; that is, there exists a countable subset E of K such that* $\text{cl}(E) = K$.

Proof. Since K is compact, K is totally bounded; thus $\forall\, n \in \mathbb{N}$, $\exists\, E_n \subseteq K$ such that

$$\#E_n < \infty \quad \text{and} \quad K \subseteq \bigcup_{y \in E_n} D\big(y, \frac{1}{n}\big)\,.$$

Let $E = \bigcup_{n=1}^{\infty} E_n$. Then E is countable by Theorem 1.40. We claim that $\text{cl}(E) = K$.

To see this, first by the definition of the closure of a set, $\text{cl}(E) \subseteq K$ (since K is closed). Let $x \in K$. Since $K \subseteq \bigcup_{y \in E_n} D\big(y, \frac{1}{n}\big)$, $x \in D\big(y, \frac{1}{n}\big)$ for some $y \in E_n$. Therefore, $D\big(x, \frac{1}{n}\big) \cap E \neq \varnothing$ for all $n \in \mathbb{N}$. This implies that $x \in \bar{E} = \text{cl}(E)$. □

Theorem 5.63. *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a Banach space, $K \subseteq M$ be a compact set, and $B \subseteq \mathscr{C}(K; \mathcal{V})$ be equi-continuous and pointwise pre-compact. Then B is pre-compact in $\big(\mathscr{C}(K; \mathcal{V}), \|\cdot\|_\infty\big)$.*

Proof. We show that every sequence $\{f_k\}_{k=1}^\infty$ in B has a convergent subsequence. Since K is compact, there is a countable dense subset E of K (Lemma 5.62), and the diagonal process (Lemma 5.59) implies that there exists $\{f_{k_j}\}_{j=1}^\infty$ that converges pointwise on E. Since E is dense in K, by Lemma 5.57 $\{f_{k_j}\}_{j=1}^\infty$ converges uniformly on K; thus $\{f_{k_j}\}_{j=1}^\infty$ converges in $\big(\mathscr{C}(K; \mathcal{V}), \|\cdot\|_\infty\big)$ by Proposition 5.45. ▫

Remark 5.64. Lemma 5.54 and Theorem 5.63 imply that "a set $B \subseteq \mathscr{C}(K; \mathcal{V})$ is pre-compact if and only if B is equi-continuous and pointwise pre-compact". (That B is pre-compact implies that B is pointwise pre-compact is left as an exercise).

Corollary 5.65. *Let (M, d) be a metric space, and $K \subseteq M$ be a compact set. Assume that $B \subseteq \mathscr{C}(K; \mathbb{R})$ is equi-continuous and pointwise bounded on K. Then every sequence in B has a uniformly convergent subsequence.*

Proof. By the Bolzano-Weierstrass theorem the boundedness of $\big\{f_k(x)\big\}_{k=1}^\infty$ implies that $\big\{f_k(x)\big\}_{k=1}^\infty$ is pre-compact for all $x \in E$. Therefore, we can apply Theorem 5.63 under the setting $(\mathcal{V}, \|\cdot\|) = (\mathbb{R}, |\cdot|)$ to conclude the corollary. ▫

The following theorem provides how compact sets look like in $\mathscr{C}(K; \mathcal{V})$.

Theorem 5.66 (The Arzelà-Ascoli Theorem)**.** *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a Banach space, $K \subseteq M$ be a compact set, and $B \subseteq \mathscr{C}(K; \mathcal{V})$. Then B is compact in $\big(\mathscr{C}(K; \mathcal{V}), \|\cdot\|_\infty\big)$ if and only if B is closed, equi-continuous, and pointwise compact.*

Proof. "$\Leftarrow$" This direction is conclude by Theorem 5.63 and the fact that B is closed.

"$\Rightarrow$" By Lemma 3.10 and Lemma 5.54, it suffices to shows that B is pointwise compact. Let $x \in K$ and $\big\{f_k(x)\big\}_{k=1}^\infty$ be a sequence in B_x. Since B is compact, there exists a subsequence $\{f_{k_j}\}_{j=1}^\infty$ that converges uniformly to some function $f \in B$. In particular, $\big\{f_{k_j}(x)\big\}_{j=1}^\infty$ converges to $f(x) \in B_x$. In other words, we find a subsequence $\big\{f_{k_j}(x)\big\}_{j=1}^\infty$ of $\big\{f_k(x)\big\}_{k=1}^\infty$ that converges to a point in B_x. This implies that B_x is sequentially compact; thus B_x is compact. ▫

Example 5.67. Let $f_k : [0, 1] \to \mathbb{R}$ be a sequence of functions such that

(1) $|f_k(x)| \leqslant M_1$ for all $k \in \mathbb{N}$ and $x \in [0,1]$; (2) $|f_k'(x)| \leqslant M_2$ for all $k \in \mathbb{N}$ and $x \in [0,1]$.

Then $\{f_k\}_{k=1}^\infty$ is clearly pointwise bounded. Moreover, by the mean value theorem

$$\big|f_k(x) - f_k(y)\big| \leqslant M_2|x-y| \qquad \forall\, x, y \in [0,1], k \in \mathbb{N}$$

which implies that $\{f_k\}_{k=1}^\infty$ is equi-continuous. Therefore, by Corollary 5.65 there exists a subsequence $\big\{f_{k_j}\big\}_{j=1}^\infty$ that converges uniformly on $[0,1]$.

Question: If assumption (1) of Example 5.67 is omitted, can $\{f_k\}_{k=1}^\infty$ still have a convergent subsequence?

Answer: No! Take $f_k(x) = k$, then $\{f_k\}_{k=1}^\infty$ does not have a convergent subsequence (note that f_k is continuous and $f_k'(x) = 0$; that is, Assumption (2) is fulfilled).

Example 5.68. We show that Assumption (1) of Example 5.67 can be replaced by $f_k(0) = 0$ for all $k \in \mathbb{N}$.

Proof. (a) If $f_n(0) = 0$, then by the mean value theorem we have for all $x \in (0,1]$ and $k \in \mathbb{N}$, $f_k(x) - f_k(0) = f_k'(c_k)(x-0)$. Then Assumption (2) of Example 5.67 implies that

$$\big|f_k(x) - f_k(0)\big| = \big|f_k'(c_k)\big|\big|x\big| \leqslant M_2|x| \leqslant M_2$$

which shows that $\{f_k\}_{k=1}^\infty$ is uniformly bounded by M_2.

(b) $\{f_k\}_{k=1}^\infty$ are equi-continuous (same proof as in Example 5.67). □

5.6 The Stone-Weierstrass Theorem

Theorem 5.69 (Weierstrass). *Let $f : [0,1] \to \mathbb{R}$ be continuous and let $\varepsilon > 0$ be given. Then there is a polynomial $p : [0,1] \to \mathbb{R}$ such that $\|f - p\|_\infty < \varepsilon$. In other words, the collection of all polynomials is dense in the space $\big(\mathscr{C}([0,1];\mathbb{R}), \|\cdot\|_\infty\big)$.*

Proof. Let $r_k(x) = C_k^n x^k (1-x)^{n-k}$. By looking at the partial derivatives with respect to x of the identity $(x+y)^n = \sum\limits_{k=0}^n C_k^n x^k y^{n-k}$, we find that

1. $\sum\limits_{k=0}^n r_k(x) = 1$; 2. $\sum\limits_{k=0}^n k r_k(x) = nx$; 3. $\sum\limits_{k=0}^n k(k-1) r_k(x) = n(n-1)x^2$.

As a consequence,

$$\sum_{k=0}^n (k-nx)^2 r_k(x) = \sum_{k=0}^n \big[k(k-1) + (1-2nx)k + n^2x^2\big] r_k(x) = nx(1-x)\,.$$

Since $f : [0,1] \to \mathbb{R}$ is continuous on a compact $[0,1]$, f is uniformly continuous on $[0,1]$ (by Theorem 4.52); thus

$$\exists\, \delta > 0 \ni \big|f(x) - f(y)\big| < \frac{\varepsilon}{2} \qquad \text{if } |x-y| < \delta,\ x, y \in [0,1]\,.$$

Consider the ***Bernstein polynomial*** $p_n(x) = \sum\limits_{k=0}^{n} f\big(\frac{k}{n}\big) r_k(x)$. Note that

$$\begin{aligned}
\big|f(x) - p_n(x)\big| &= \Big|\sum_{k=0}^{n} \Big(f(x) - f\big(\frac{k}{n}\big)\Big) r_k(x)\Big| \leqslant \sum_{k=0}^{n} \Big|f(x) - f\big(\frac{k}{n}\big)\Big| r_k(x) \\
&\leqslant \Big(\sum_{|k-nx|<\delta n} + \sum_{|k-nx|\geqslant \delta n}\Big)\Big|f(x) - f\big(\frac{k}{n}\big)\Big| r_k(x) \\
&< \frac{\varepsilon}{2} + 2\|f\|_\infty \sum_{|k-nx|\geqslant\delta n} \frac{(k-nx)^2}{(k-nx)^2} r_k(x) \\
&\leqslant \frac{\varepsilon}{2} + \frac{2\|f\|_\infty}{n^2\delta^2} \sum_{k=0}^{n} (k-nx)^2 r_k(x) \leqslant \frac{\varepsilon}{2} + \frac{2\|f\|_\infty}{n\delta^2} x(1-x) \leqslant \frac{\varepsilon}{2} + \frac{\|f\|_\infty}{2n\delta^2}\,.
\end{aligned}$$

Choose N large enough such that $\dfrac{\|f\|_\infty}{2N\delta^2} < \dfrac{\varepsilon}{2}$. Then for all $n \geqslant N$,

$$\|f - p_n\|_\infty = \sup_{x\in[0,1]} \big|f(x) - p_n(x)\big| < \varepsilon\,. \qquad \square$$

Remark 5.70. A polynomial of the form $p_n(x) = \sum\limits_{k=0}^{n} \beta_k r_k(x)$ is called a ***Bernstein polynomial of degree*** n, and the coefficients β_k are called Bernstein coefficients.

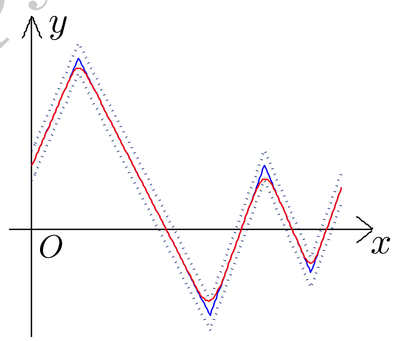

Figure 5.3: Using a Bernstein polynomial of degree 350 (the red curve) to approximate a "saw-tooth" function (the blue curve)

Corollary 5.71. *The collection of polynomials on* $[a,b]$ *is dense in* $\big(\mathscr{C}([a,b];\mathbb{R}), \|\cdot\|_\infty\big)$.

Proof. We note that $g \in \mathscr{C}([a,b];\mathbb{R})$ if and only if $f(x) = g\big(x(b-a)+a\big) \in \mathscr{C}([0,1];\mathbb{R})$; thus

$$\big|f(x) - p(x)\big| < \varepsilon \ \forall\, x \in [0,1] \Leftrightarrow \Big|g(x) - p(\frac{x-a}{b-a})\Big| < \varepsilon \ \forall\, x \in [a,b]\,. \qquad \square$$

Example 5.72.
Question: Let $f \in \mathscr{C}([0,1],\mathbb{R})$ be such that $\|p_n - f\|_\infty \to 0$ as $n \to \infty$; that is, $\{p_n\}_{n=1}^\infty$ converges uniformly to f on $[0,1]$, where $p_n \in \mathscr{P}([0,1])$. Is f differentiable?
Answer: No! Take any continuous but not differentiable function f (for example, let $f(x) = \big|x - \dfrac{1}{2}\big|$). By Theorem 5.69, $\exists\, p_n$: polynomial $\ni \|p_n - f\|_\infty \to 0$ as $n \to \infty$.

Definition 5.73. Let (M,d) be a metric space, and $E \subseteq M$ be a subset. A family $\mathcal{A}$ of functions defined on E is called an ***algebra*** if

1. $f + g \in \mathcal{A}$ for all $f, g \in \mathcal{A}$;

2. $f \cdot g \in \mathcal{A}$ for all $f, g \in \mathcal{A}$;

3. $\alpha f \in \mathcal{A}$ for all $f \in \mathcal{A}$ and $\alpha \in \mathbb{R}$.

In other words, $\mathcal{A}$ is an algebra if $\mathcal{A}$ is closed under addition, multiplication, and scalar multiplication.

Example 5.74. A function $g : [a,b] \to \mathbb{R}$ is called ***simple*** if we can divide up $[a,b]$ into sub-intervals on which g is constant except perhaps at the end-points. In other words, g is called simple if there is a partition $\mathcal{P} = \{x_0, x_1, \cdots, x_N\}$ of $[a,b]$ such that

$$g(x) = g\big(\frac{x_{i-1}+x_i}{2}\big) \qquad \text{if } \ x \in (x_{i-1}, x_i)\,.$$

Then the collection of all simple functions is an algebra.

Proposition 5.75. *Let (M,d) be a metric space, and $A \subseteq M$ be a subset. If $\mathcal{A} \subseteq \mathscr{C}_b(A;\mathbb{R})$ is an algebra, then* $\mathrm{cl}(\mathcal{A})$ *is also an algebra.*

Proof. Let $f, g \in \mathrm{cl}(\mathcal{A})$. Then $\exists\,\{f_k\}_{k=1}^\infty, \{g_k\}_{k=1}^\infty \subseteq \mathcal{A}$ such that $\{f_k\}_{k=1}^\infty$ converges uniformly to f on A, and $\{g_k\}_{k=1}^\infty$ converges uniformly to g on A. Since $\mathcal{A}$ is an algebra, $f_k + g_k, f_k \cdot g_k$ and αf_k belong to $\mathcal{A}$ for all $k \in \mathbb{N}$. As a consequence, the uniform limit of $f_k + g_k, f_k \cdot g_k$ and αf_k belong to $\mathrm{cl}(\mathcal{A})$ which implies that $f + g, f \cdot g$ and αf belong to $\mathrm{cl}(\mathcal{A})$. Therefore, $\mathrm{cl}(\mathcal{A})$ is an algebra. $\square$

Definition 5.76. Let (M, d) be a metric space, and $A \subseteq M$ be a subset. A family $\mathscr{F}$ of functions defined on A is said to

1. ***separate points*** on A if for all $x, y \in A$ and $x \neq y$, there exists $f \in \mathscr{F}$ such that $f(x) \neq f(y)$.

2. ***vanish at no point*** of A if for each $x \in A$ there is $f \in \mathscr{F}$ such that $f(x) \neq 0$.

Example 5.77. Let $\mathscr{P}([a, b])$ denote the collection of polynomials defined on $[a, b]$ is an algebra. Moreover, $\mathscr{P}([a, b])$ separates points on $[a, b]$ since $p(x) = x$ does the separation, and $\mathscr{P}([a, b])$ vanishes at no point of $[a, b]$.

Example 5.78. Let $\mathscr{P}_{\text{even}}([a, b])$ denote the collection of all polynomials $p(x)$ of the form

$$p(x) = \sum_{k=0}^{n} a_k x^{2k} = a_n x^{2n} + a_{n-1} x^{2n-2} + \cdots + a_0 .$$

Then $\mathscr{P}_{\text{even}}([a, b])$ is an algebra. Moreover, $\mathscr{P}_{\text{even}}([a, b])$ vanishes at no point of $[a, b]$ since the constant functions are polynomials (since constant functions belongs to $\mathscr{P}([a, b])$. However, if $ab < 0$, $\mathscr{P}_{\text{even}}([a, b])$ does not separate points on $[a, b]$. On the other hand, if $ab \geqslant 0$, then $\mathscr{P}_{\text{even}}([a, b])$ separates points on $[a, b]$ since $p(x) = x^2$ does the job.

Lemma 5.79. *Let (M, d) be a metric space, and $A \subseteq M$ be a subset. Suppose that $\mathcal{A}$ is an algebra of functions defined on A, $\mathcal{A}$ separates points on A, and $\mathcal{A}$ vanishes at no point of A. Then for all $x_1, x_2 \in A$, $x_1 \neq x_2$, and $c_1, c_2 \in \mathbb{R}$ (c_1, c_2 could be the same), there exists $f \in \mathcal{A}$ such that $f(x_1) = c_1$ and $f(x_2) = c_2$.*

Proof. Since $\mathcal{A}$ separates points on A, $\exists\, g \in \mathcal{A}$ such that $g(x_1) \neq g(x_2)$, and since $\mathcal{A}$ vanishes at no point of A, $\exists\, h, k \in \mathcal{A}$ such that $h(x_1) \neq 0$ and $k(x_2) \neq 0$. Then

$$f(x) = c_1 \frac{\big[g(x) - g(x_2)\big] h(x)}{\big[g(x_1) - g(x_2)\big] h(x_1)} + c_2 \frac{\big[g(x) - g(x_1)\big] k(x)}{\big[g(x_2) - g(x_1)\big] k(x_2)}$$

has the desired property. □

Theorem 5.80 (Stone). *Let (M, d) be a metric space, $K \subseteq M$ be a compact set, and $\mathcal{A} \subseteq \mathscr{C}(K; \mathbb{R})$ satisfying*

1. *$\mathcal{A}$ is an algebra.* 2. *$\mathcal{A}$ vanishes at no point of K.* 3. *$\mathcal{A}$ separates points on K.*

Then $\mathcal{A}$ is dense in $\mathscr{C}(K;\mathbb{R})$.

Example 5.81. Let $K = [-1,1] \times [-1,1] \subseteq \mathbb{R}^2$. Consider the set $\mathscr{P}(K)$ of all polynomials $p(x,y)$ in two variables $(x,y) \in K$. Then $\mathscr{P}(K)$ is dense in $\mathscr{C}(K;\mathbb{R})$.

Reason: Since K is compact, and $\mathscr{P}(K)$ is definitely an algebra and the constant function $p(x,y) = 1 \in \mathscr{P}(K)$ vanishes at no point of K, it suffices to show that $\mathscr{P}(K)$ separates points. Let (a_1,b_1) and (a_2,b_2) be two different points in K. Then the polynomial

$$p(x,y) = (x-a_1)^2 + (y-b_1)^2$$

has the property that $p(a_1,b_1) \neq p(a_2,b_2)$. Therefore, $\mathscr{P}(K)$ separates points in K,

Proof of Theorem 5.80. We divide the proof into the following four steps:

Step 1: We claim that if $f \in \bar{\mathcal{A}}$, then $|f| \in \bar{\mathcal{A}}$.

Proof of claim: Let $\mathrm{M} = \sup\limits_{x\in K} |f(x)|$, and $\varepsilon > 0$ be given. By Corollary 5.71, for every $\varepsilon > 0$ there is a polynomial $p(y)$ such that $\big|p(y) - |y|\big| < \varepsilon$ for all $y \in [-\mathrm{M},\mathrm{M}]$. Since $\mathcal{A}$ is an algebra, by Proposition 5.75 $\mathrm{cl}(\mathcal{A})$ is also an algebra; thus $g \equiv p(f) \in \mathrm{cl}(\mathcal{A})$ if $f \in \mathrm{cl}(\mathcal{A})$. Nevertheless,

$$\big|g(x) - |f(x)|\big| < \varepsilon \qquad \forall\, x \in K$$

which shows that $|f| \in \bar{\mathcal{A}}$.

Step 2: Let the functions $\max\{f,g\}$ and $\min\{f,g\}$ be defined by

$$\max\{f,g\}(x) = \max\big\{f(x), g(x)\big\}, \quad \min\{f,g\}(x) = \min\big\{f(x), g(x)\big\}.$$

Since $\max\{f,g\} = \dfrac{f+g}{2} + \dfrac{|f-g|}{2}$ and $\min\{f,g\} = \dfrac{f+g}{2} - \dfrac{|f-g|}{2}$, we find that if $f, g \in \bar{\mathcal{A}}$, then $\max\{f,g\} \in \bar{\mathcal{A}}$ and $\min\{f,g\} \in \bar{\mathcal{A}}$. As a consequence, if $f_1, \cdots, f_n \in \bar{\mathcal{A}}$,

$$\max\{f_1,\cdots,f_n\} \in \bar{\mathcal{A}} \quad \text{and} \quad \min\{f_1,\cdots,f_n\} \in \bar{\mathcal{A}}.$$

Step 3: We claim that for any given $f \in \mathscr{C}(K;\mathbb{R})$, $a \in K$ and $\varepsilon > 0$, there exists a function $g_a \in \bar{\mathcal{A}}$ such that

$$g_a(a) = f(a) \quad \text{and} \quad g_a(x) > f(x) - \varepsilon \qquad \forall\, x \in K. \tag{5.6.1}$$

Proof of claim: Since $\mathcal{A}$ separates points on K and $\mathcal{A}$ vanishes at no point of K, so does $\bar{\mathcal{A}}$. Therefore, Lemma 5.79 implies that for every $b \in K$ with $b \neq a$, there exists $h_b \in \bar{\mathcal{A}}$ such that $h_b(a) = f(a)$ and $h_b(b) = f(b)$. Note that every function in $\bar{\mathcal{A}}$ is continuous (by Theorem 5.7); thus the continuity of h_b provides $\delta = \delta_b > 0$ such that

$$h_b(x) > f(x) - \varepsilon \qquad \forall\, x \in \big[D\big(b, \delta_b\big) \cup D\big(a, \delta_b\big)\big] \cap K\,.$$

Let $\mathcal{U}_b = D\big(b, \delta_b\big) \cup D\big(a, \delta_b\big)$. Then $\mathcal{U}_b$ is open. Since $K \subseteq \bigcup\limits_{\substack{b \in K \\ b \neq a}} \mathcal{U}_b$ and K is compact, there exists a finite set $\{b_1, \cdots, b_n\} \subseteq K$ such that $K \subseteq \bigcup\limits_{j=1}^{n} \mathcal{U}_{b_j}$. Define $g_a = \max\big\{h_{b_1}, \cdots h_{b_n}\big\}$. Then $g_a \in \bar{\mathcal{A}}$, and $g_a(a) = f(a)$. Moreover, if $x \in K$, $x \in \mathcal{U}_{b_j}$ for some j; thus

$$g_a(x) \geqslant h_{b_j}(x) > f(x) - \varepsilon$$

which implies that g satisfies (5.6.1).

Step 4: Let $f \in \mathscr{C}(K; \mathbb{R})$ and $\varepsilon > 0$ be given. For any $a \in K$, let $g_a \in \bar{\mathcal{A}}$ be a function provided in Step 3 satisfying

$$g_a(a) = f(a) \quad \text{and} \quad g_a(x) > f(x) - \frac{\varepsilon}{2} \qquad \forall\, x \in K\,. \tag{5.6.2}$$

By the continuity of g_a, there exists $\delta = \delta_a > 0$ such that

$$g_a(x) < f(x) + \frac{\varepsilon}{2} \qquad \forall\, x \in D(a, \delta_a) \cap K\,. \tag{5.6.3}$$

Similar to Step 3, $\exists\, \{a_1, \cdots, a_m\} \subseteq K$ such that

$$K \subseteq \bigcup_{j=1}^{m} D\big(a_j, \delta_{a_j}\big)\,. \tag{5.6.4}$$

Define $h = \min\big\{g_{a_1}, \cdots, g_{a_m}\big\}$. Then $h \in \bar{\mathcal{A}}$, and (5.6.2) shows that

$$h(x) > f(x) - \frac{\varepsilon}{2} \qquad \forall\, x \in K.$$

Moreover, similar to Step 3 (5.6.3) and (5.6.4) imply that

$$h(x) < f(x) + \frac{\varepsilon}{2} \qquad \forall\, x \in K.$$

On the other hand, since $h \in \bar{\mathcal{A}}$, there exists $p \in \mathcal{A}$ such that

$$\big|p(x) - h(x)\big| < \frac{\varepsilon}{2} \qquad \forall\, x \in K\,;$$

thus

$$\big|p(x) - f(x)\big| \leqslant \big|p(x) - h(x)\big| + \big|h(x) - f(x)\big| < \varepsilon \qquad \forall\, x \in K$$

which concludes the theorem. □

Example 5.82. Consider $\mathscr{P}_{\text{even}}([0,1]) = \Big\{p(x) = \sum\limits_{k=0}^{n} a_k x^{2k} \,\Big|\, a_k \in \mathbb{R}\Big\}$ (see Example 5.78). Then $\mathcal{A} = \mathscr{P}_{\text{even}}([0,1])$ satisfies all the conditions in the Stone theorem, so $\mathscr{P}_{\text{even}}([0,1])$ is dense in $\mathscr{C}([0,1];\mathbb{R})$.

On the other hand, if $K = [-1,1]$, then $\mathscr{P}_{\text{even}}([-1,1])$ does not separate points on K since if $p \in \mathscr{P}_{\text{even}}([-1,1])$, $p(x) = p(-x)$; thus the Stone theorem cannot be applied to conclude the denseness of $\mathscr{P}_{\text{even}}([-1,1])$ in $\mathscr{C}([-,1];\mathbb{R})$. In fact, $\mathscr{P}_{\text{even}}([-1,1])$ is not dense in $\mathscr{C}([-1,1];\mathbb{R})$ since polynomials in $\mathscr{P}_{\text{even}}([-1,1])$ are all even functions and $f(x) = x$ cannot be approximated by even functions.

Corollary 5.83. *Let $\mathscr{C}(\mathbb{T})$ be the collection of all 2π-periodic continuous functions, and $\mathscr{P}_n(\mathbb{T})$ be the collection of all trigonometric polynomials of degree n; that is,*

$$\mathscr{P}_n(\mathbb{T}) = \Big\{\frac{c_0}{2} + \sum_{k=1}^{n} c_k \cos kx + s_k \sin kx \,\Big|\, \{c_k\}_{k=0}^{n}, \{s_k\}_{k=1}^{n} \subseteq \mathbb{R}\Big\}.$$

Let $\mathscr{P}(\mathbb{T}) = \bigcup\limits_{n=0}^{\infty} \mathscr{P}_n(\mathbb{T})$. Then $\mathscr{P}(\mathbb{T})$ is dense in $\mathscr{C}(\mathbb{T})$. In other words, if $f \in \mathscr{C}(\mathbb{T})$ and $\varepsilon > 0$ is given, there exists $p \in \mathscr{P}(\mathbb{T})$ such that

$$\big|f(x) - p(x)\big| < \varepsilon \qquad \forall\, x \in \mathbb{R}.$$

Proof. We note that $\mathscr{C}(\mathbb{T})$ can be viewed as the collection of all continuous functions defined on the unit circle $\mathbb{S}^1$ in the sense that every $f \in \mathscr{C}(\mathbb{T})$ corresponds to a unique $F \in \mathscr{C}(\mathbb{S}^1;\mathbb{R})$ such that $f(x) = F(\cos x, \sin x)$, and vice versa. Since $\mathbb{S}^1 \subseteq [-1,1] \times [-1,1]$ is compact, Example 5.81 provides that $\mathscr{P}(\mathbb{S}^1)$, the collection of all polynomials defined on $\mathbb{S}^1$, is an algebra that separates points of $\mathbb{S}^1$ and vanishes at no point on $\mathbb{S}^1$. The Stone-Weierstrass Theorem then implies that there exists $P \in \mathscr{P}(\mathbb{S}^1)$ such that

$$\big|F(x,y) - P(x,y)\big| < \varepsilon \qquad \forall\,(x,y) \in \mathbb{S}^1 \text{ (that is, } x^2 + y^2 = 1).$$

Let $p(x) = P(\cos x, \sin x)$. Note that

$$\begin{aligned}\cos^n x = \Big(\frac{e^{ix} + e^{-ix}}{2}\Big)^n &= \sum_{k=0}^{n} \frac{1}{2^n} C_k^n e^{ikx} e^{-i(n-k)x} = \sum_{k=0}^{n} \frac{1}{2^n} C_k^n e^{i(2k-n)x} \\ &= \sum_{k=0}^{n} \frac{1}{2^n} C_k^n \big(\cos(2k-n)x + i\sin(2k-n)x\big) = \sum_{k=0}^{n} \frac{1}{2^n} C_k^n \cos(2k-n)x \in \mathscr{P}_n(\mathbb{T}),\end{aligned}$$

and similarly, $\sin^m x \in \mathscr{P}_m(\mathbb{T})$. Therefore, if $P(x,y) = \sum_{k,\ell=0}^{n} a_{k,\ell} x^k y^\ell$, then $P(\cos x, \sin x) \in \mathscr{P}_{2n}(\mathbb{T})$ because of the identities

$$\begin{aligned}\cos\theta\cos\varphi &= \frac{1}{2}\Big[\cos(\theta-\varphi)+\cos(\theta+\varphi)\Big],\\ \sin\theta\cos\varphi &= \frac{1}{2}\Big[\sin(\theta+\varphi)+\sin(\theta-\varphi)\Big],\\ \sin\theta\sin\varphi &= \frac{1}{2}\Big[\cos(\theta-\varphi)-\cos(\theta+\varphi)\Big].\end{aligned}$$

As a consequence, we conclude that

$$\big|f(x)-p(x)\big| = \big|F(\cos x,\sin x) - P(\cos x, \sin x)\big| < \varepsilon \qquad \forall\, x \in \mathbb{R}\,. \qquad \square$$

5.7 The Contraction Mapping Principle（收縮映射原理）and its Applications

Definition 5.84. Let (M,d) be a metric space, and $\Phi : M \to M$ be a mapping. Φ is said to be a ***contraction mapping*** if there exists a constant $k \in [0,1)$ such that

$$d\big(\Phi(x),\Phi(y)\big) \leqslant k d(x,y) \qquad \forall\, x,y \in M\,.$$

Remark 5.85. A contraction mapping must be (uniformly) continuous.
Reason: Given $\varepsilon > 0$, take $\delta = \dfrac{\varepsilon}{k}$, where k is set as in the definition of contraction. Now if $d(x,y) < \delta$, then

$$d\big(\Phi(x),\Phi(y)\big) \leqslant k d(x,y) < k\cdot\frac{\varepsilon}{k} = \varepsilon\,.$$

Example 5.86. For what $r < 1$ do we have $f : [0,r] \to [0,r]$ where $f(x) = x^2$ a contraction?
Answer: By the mean value theorem, $f(x) - f(y) = f'(c)(x-y)$, c between x,y; thus

$$\big|f(x)-f(y)\big| = \big|f'(c)\big||x-y| = 2c|x-y| \leqslant 2r|x-y|\,.$$

Hence for all $r < \dfrac{1}{2}$, the map $f : [0,r] \to [0,r]$ is a contraction where $f(x) = x^2$.

On the other hand, we show that f cannot be a contraction if $r = \dfrac{1}{2}$. Suppose the contrary that there exists $k \in [0,1)$ such that for all $x,y \in \big[0,\frac{1}{2}\big]$, $\big|x^2-y^2\big| \leqslant k\big|x-y\big|$. Then

$$\sup_{x\neq y, x,y\in[0,\frac{1}{2}]} \frac{\big|x^2-y^2\big|}{|x-y|} \leqslant k < 1\,.$$

But we can take $x = \dfrac{1}{2}$, $y_n = \dfrac{1}{2} - \dfrac{1}{n}$, $n = 1, 2, \cdots, x, y_n \in \big[0, \frac{1}{2}\big]$. So

$$\lim_{n\to\infty} \frac{|x^2 - {y_n}^2|}{|x - y_n|} = \lim_{n\to\infty} |x + y_n| = \lim_{n\to\infty} \Big(\frac{1}{2} + \frac{1}{2} - \frac{1}{n}\Big) = 1\,.$$

This means $\displaystyle\sup_{x\neq y, x,y\in[0,\frac{1}{2}]} \frac{|x^2 - y^2|}{|x - y|} < 1$ is not possible.

Definition 5.87. Let (M, d) be a metric space, and $\Phi : M \to M$ be a mapping. A point $x_0 \in M$ is called a ***fixed-point*** for Φ if $\Phi(x_0) = x_0$.

Example 5.88. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be given by $\Phi(x) = \dfrac{x^2 + 2}{3}$. Then 1 is a fixed-point, and 2 is also a fixed-point.

Theorem 5.89 (Contraction Mapping Principle)**.** *Let (M, d) be a complete metric space, and $\Phi : M \to M$ be a contraction mapping. Then Φ has a unique fixed-point.*

Proof. Let $x_0 \in M$, and define $x_{n+1} = \Phi(x_n)$ for all $n \in \mathbb{N} \cup \{0\}$. Then

$$d(x_{n+1}, x_n) = d\big(\Phi(x_n), \Phi(x_{n-1})\big) \leqslant kd(x_n, x_{n-1}) \leqslant k^n d(x_1, x_0)\,;$$

thus if $n > m$,

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leqslant (k^m + k^{m+1} + \cdots + k^{n-1}) d(x_1, x_0) \\ &\leqslant k^m(1 + k + k^2 + \cdots) d(x_1, x_0) = \frac{k^m}{1 - k} d(x_1, x_0)\,. \end{aligned} \tag{5.7.1}$$

Since $k \in [0, 1)$, $\displaystyle\lim_{m\to\infty} \frac{k^m}{1 - k} d(x_1, x_0) = 0$; thus

$$\forall\, \varepsilon > 0, \exists\, N > 0 \ni d(x_n, x_m) < \varepsilon \quad \forall\, n, m \geqslant N\,.$$

In other words, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since (M, d) is complete, $x_n \to x$ as $n \to \infty$ for some $x \in M$. Finally, since $\Phi(x_n) = x_{n+1}$ for all $n \in \mathbb{N}$, by the continuity of Φ we obtain that

$$\Phi(x) = \lim_{n\to\infty} \Phi(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

which guarantees the existence of a fixed-point.

Suppose that for some $x, y \in M$, $\Phi(x) = x$ and $\Phi(y) = y$. Then

$$d(x, y) = d\big(\Phi(x), \Phi(y)\big) \leqslant kd(x, y)$$

which implies that $d(x, y) = 0$ or $x = y$. Therefore, the fixed-point of Φ is unique. □

Remark 5.90. The proof of the contraction mapping principle also provides an iterative way, $x_{k+1} = \Phi(x_k)$, of finding the fixed-point of a contraction mapping Φ. Using (5.7.1), the convergence rate of $\{x_m\}_{m=1}^{\infty}$ to the fixed-point x is measured by

$$d(x_m, x) = \lim_{n\to\infty} d(x_m, x_n) \leqslant \frac{k^m}{1-k} d(x_1, x_0)\,.$$

Therefore, the smaller the contraction constant k, the faster the convergence.

Remark 5.91. Theorem 5.89 sometimes is also called the ***Banach fixed-point theorem***.

Example 5.92. The condition $k < 1$ in Theorem 5.89 is necessary. For example, let $M = \mathbb{R}$, $d(x,y) = |x-y|$, and $\Phi : \mathbb{R} \to \mathbb{R}$ be given by $\Phi(x) = x+1$. Then $\big|\Phi(x) - \Phi(y)\big| = \big|x-y\big|$. Suppose $x_\star$ is a fixed-point of Φ. Then $x_\star = \Phi(x_\star) = x_\star + 1$ which leads to a contradiction that $0 = 1$.

Example 5.93. Let $\Phi : [1,\infty) \to [1,\infty)$ be given by $\Phi(x) = x + \dfrac{1}{x}$. Then if $x \neq y$,

$$\big|\Phi(x) - \Phi(y)\big| = \Big|x - y + \frac{1}{x} - \frac{1}{y}\Big| = \Big|(x-y)\Big(1 - \frac{1}{xy}\Big)\Big| < |x-y|\,.$$

However, there is no fixed-point of Φ.

Example 5.94 (The secant method)**.** Suppose that f is continuously differentiable, $f'(x) > 0$ for all $x \in [a,b]$ and $f(a)f(b) < 0$. By the intermediate value theorem there must be a (unique) zero of f. How do we find this zero?

Assume that $\sup\limits_{x\in[a,b]} f'(x) < \infty$. Let

$$M = \max\Big\{\sup_{x\in[a,b]} f'(x), -\frac{f(a)}{b-a}, \frac{f(b)}{b-a}\Big\} + 1$$

be a positive constant, and consider $\Phi(x) = x - \dfrac{f(x)}{M}$. Then by the mean value theorem,

$$\big|\Phi(x) - \Phi(y)\big| = \Big|(x-y)(1 - \frac{f'(\xi)}{M})\Big| \leqslant \Big(1 - \frac{\min_{\xi\in[a,b]} f'(\xi)}{M}\Big)|x-y| \leqslant k|x-y|\,,$$

where $k \in [0,1)$ is a fixed constant. Moreover, $\Phi'(x) = 1 - \dfrac{f'(x)}{M} > 0$; thus Φ is strictly increasing. Since the choice of M implies that $a < \Phi(a) < \Phi(b) < b$; thus $\Phi : [a,b] \to [a,b]$. Therefore, the contraction mapping principle implies that one can find the fixed-point of Φ (which is the zero of f) using the iterative scheme $x_{k+1} = \Phi(x_k)$ (by picking any arbitrary initial guess $x_0 \in [a,b]$).

5.7.1 The existence and uniqueness of the solution to ODEs

In this sub-section we are concerned with if there is a solution to the initial value problem of ordinary differential equation:

$$x'(t) = f\big(x(t), t\big) \qquad \forall\, t \in [t_0, t_0 + \Delta t], \tag{5.7.2a}$$

$$x(t_0) = x_0\,, \tag{5.7.2b}$$

where $x : [t_0, t_0 + \Delta t] \to \mathbb{R}^n$ and $f : \mathbb{R}^n \times [t_0, t_0 + \Delta t] \to \mathbb{R}^n$ are vector-valued functions, and $x_0 \in \mathbb{R}^n$ is a vector. Another question we would like to answer is "if (5.7.2) indeed has a solution, is the solution unique?"

Theorem 5.95 (Fundamental Theorem of ODE)**.** *Suppose that for some $r > 0$, $f : D(x_0, r) \times [t_0, T] \to \mathbb{R}^n$ is continuous and is Lipschitz in the spatial variable; that is,*

$$\exists K > 0 \ni \big\|f(x,t) - f(y,t)\big\|_2 \leqslant K\|x - y\|_2 \qquad \forall\, x, y \in D(x_0, r) \text{ and } t \in [t_0, T]\,.$$

Then there exists $0 < \Delta t \leqslant T - t_0$ such that there exists a unique solution to (5.7.2)*.*

Proof. For any $x \in \mathscr{C}([t_0, T]; \mathbb{R}^n)$, define

$$\Phi(x)(t) = x_0 + \int_{t_0}^{t} f\big(x(s), s\big) ds\,.$$

We note that if $x(t)$ is a solution to (5.7.2), then x is a fixed point of Φ (for $t \in [t_0, t_0 + \Delta t]$). Therefore, the problem of finding a solution to (5.7.2) transforms to a problem of finding a fixed-point of Φ.

To guarantee the existence of a unique fixed-point, we appeal to the contraction mapping principle. To be able to apply the contraction mapping principle, we need to specify the metric space (M, d). Let

$$\Delta t = \min\Big\{T - t_0, \frac{r}{Kr + 2\|f(x_0, \cdot)\|_\infty}, \frac{1}{2K}\Big\}\,, \tag{5.7.3}$$

and define

$$M = \Big\{x \in \mathscr{C}\big([t_0, t_0 + \Delta t]; \mathbb{R}^n\big) \,\Big|\, \|x - x_0\|_\infty \leqslant \frac{r}{2}\Big\}$$

with the metric induced by the sup-norm $\|\cdot\|_\infty$ of $\mathscr{C}\big([t_0, t_0 + \Delta t]; \mathbb{R}^n\big)$. Then

1. We first show that $\Phi : M \to M$. To see this, we observe that

$$\begin{aligned}\big\|\Phi(x) - x_0\big\|_\infty &= \Big\|\int_{t_0}^t f\big(x(s), s\big)ds\Big\|_\infty = \Big\|\int_{t_0}^t \big[f\big(x(s), s\big) - f(x_0, s)\big]ds + \int_{t_0}^t f(x_0, s)ds\Big\|_\infty \\ &\leqslant \int_{t_0}^{t_0+\Delta t} \big\|f\big(x(s), s\big) - f(x_0, s)\big\|_2 ds + \int_{t_0}^{t_0+\Delta t} \big\|f(x_0, s)\big\|_2 ds \\ &\leqslant K \int_{t_0}^{t_0+\Delta t} \|x(s) - x_0\|_2 ds + \Delta t\big\|f(x_0, \cdot)\big\|_\infty \\ &\leqslant \Delta t\Big[K\|x - x_0\|_\infty + \big\|f(x_0, \cdot)\big\|_\infty\Big]\,;\end{aligned}$$

thus if $x \in M$, (5.7.3) implies that $\|\Phi(x) - x_0\|_\infty \leqslant \dfrac{r}{2}$.

2. Next we show that Φ is a contraction mapping. To see this, we compute $\big\|\Phi(x)-\Phi(y)\big\|_\infty$ for $x, y \in M$ and find that

$$\begin{aligned}\big\|\Phi(x) - \Phi(y)\big\|_\infty &\leqslant \Big\|\int_{t_0}^t \big[f\big(x(s), s\big) - f\big(y(s), s\big)\big]ds\Big\|_\infty \\ &\leqslant \int_{t_0}^{t_0+\Delta t} K\|x(s) - y(s)\|_2 ds \leqslant K\Delta t\|x - y\|_\infty \leqslant \frac{1}{2}\|x - y\|_\infty\,;\end{aligned}$$

thus $\Phi : M \to M$ is a contraction mapping.

3. Finally we show that (M, d) is complete. It suffices to show that M is a closed subset of $\mathscr{C}([t_0, t_0 + \Delta t]; \mathbb{R}^n)$. Let $\{x_k\}_{k=1}^\infty$ be a uniformly convergent sequence with limit x. Since $\|x_k(t) - x_0\|_2 \leqslant \dfrac{r}{2}$ for all $t \in [t_0, t_0 + \Delta t]$, passing k to the limit we find that $\|x(t) - x_0\|_2 \leqslant \dfrac{r}{2}$ for all $t \in [t_0, t_0 + \Delta t]$ which implies that $\|x - x_0\|_\infty \leqslant \dfrac{r}{2}$; thus $x \in M$.

Therefore, by the contraction mapping principle, there exists a unique fixed point $x \in M$ which implies that there exists a unique solution to (5.7.2). □

Example 5.96. Let

$$x_c(t) = \begin{cases} 0 & \text{if } 0 \leqslant t < c\,, \\ \dfrac{1}{4}(t-c)^2 & \text{if } t \geqslant c\,. \end{cases}$$

Then for all $c > 0$, $x_c(t)$ is a solution to $x'(t) = x(t)^{\frac{1}{2}}$ for all $t > 0$ with initial value $x(0) = 0$. The reason for not having unique solution is that if $f(x, t) = \sqrt{x}$, $f : D(0, r) \times \mathbb{R} \to \mathbb{R}$ is not Lipschitz in the spatial variabvle for all $r > 0$. In other words, for all $r, K > 0$, there exists $x, y \in D(0, r)$ satisfying $\big|f(x) - f(y)\big| \geqslant K|x - y|$.

Example 5.97. Find a function $x(t)$ satisfying $x'(t) = x(t)$ with initial value $x(0) = 1$. Define $\Phi(x)(t) = 1 + \int_0^t x(s)ds$, $x_0(t) = 1$ and $x_{k+1}(t) = \Phi(x_k)(t)$. Then

$$\begin{aligned}
x_1(t) = 1 + \int_0^t x_0(s)ds = 1 + t \Rightarrow x_2(t) = 1 + \int_0^t x_1(s)ds = 1 + t + \frac{t^2}{2} \\
\Rightarrow x_3(t) = 1 + \int_0^t x_2(s)ds = 1 + t + \frac{t^2}{2} + \frac{t^3}{3\cdot 2} \\
\Rightarrow \cdots\cdots \\
\Rightarrow \text{By induction, we have } x_k(t) = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^k}{k!}
\end{aligned}$$

which converges to $x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t$.

Example 5.98. Find a function $x(t)$ satisfying $x'(t) = tx(t)$ with initial value $x(0) = 3$. Define $\Phi(x)(t) = 3 + \int_0^t sx(s)ds$, $x_0(t) = 3$ and $x_{k+1}(t) = \Phi(x_k)(t)$. Then

$$\begin{aligned}
x_1(t) = 3 + \int_0^t 3s ds = 3 + \frac{3t^2}{2} \Rightarrow x_2(t) = 3 + \int_0^t sx_1(s)ds = 3 + \frac{3t^2}{2} + \frac{3t^4}{2\cdot 4} \\
\Rightarrow x_3(t) = 3 + \int_0^t sx_2(s)ds = 3 + \frac{3t^2}{2} + \frac{3t^4}{2\cdot 4} + \frac{3t^6}{2\cdot 4\cdot 6}.
\end{aligned}$$

We can conjecture and prove that

$$x_k(t) = 3 + \frac{3t^2}{2} + \frac{3t^4}{2\cdot 4} + \cdots + \frac{3t^{2k}}{2\cdot 4\cdots(2k)};$$

thus $x_k(t) \to x(t) = 3 + 3\sum_{k=1}^{\infty} \frac{t^{2k}}{2\cdot 4\cdots(2k)}$. To see what $x(t)$ is, we observe that

$$1 + \sum_{k=1}^{\infty} \frac{t^{2k}}{2\cdot 4\cdots(2k)} = \sum_{k=0}^{\infty} \frac{t^{2k}}{2^k k!} = \sum_{k=0}^{\infty} \frac{(t^2/2)^k}{k!} = \exp\left(\frac{t^2}{2}\right);$$

thus the solution is $x(t) = 3\exp\left(\frac{t^2}{2}\right)$.

Remark 5.99. In the iterative process above of solving ODE, the iterative relation

$$x_{k+1}(t) = x_0 + \int_{t_0}^t f\big(x_k(s), s\big)ds$$

is called the ***Picard iteration***.

Example 5.100. Is there a solution to the Fredholm equation

$$x(t) = \lambda \int_a^b K(t,s)x(s)ds + \varphi(t)\,? \tag{5.7.4}$$

Define $\Phi : \mathscr{C}([a,b];\mathbb{R}) \to \mathscr{C}([a,b];\mathbb{R})$ by

$$\Phi(x)(t) = \lambda \int_a^b K(t,s)x(s)ds + \varphi(t)\,.$$

Then if $K : [a,b] \times [a,b] \to \mathbb{R}$ is continuous, and $\varphi : [a,b] \to \mathbb{R}$ is continuous, $\Phi(x) \in \mathscr{C}([a,b];\mathbb{R})$ as long as $x \in \mathscr{C}([a,b];\mathbb{R})$. Moreover,

$$\big|\Phi(x)(t) - \Phi(y)(t)\big| \leqslant \Big|\lambda \int_a^b K(t,s)\big(x(s) - y(s)\big)ds\Big| \leqslant |\lambda|\|K\|_\infty|b-a|\|x-y\|_\infty\,;$$

thus if $|\lambda|\|K\|_\infty|b-a| < 1$, Φ is a contraction mapping. As a consequence, if

1. $K : [a,b] \times [a,b] \to \mathbb{R}$ is continuous;

2. $\varphi : [a,b] \to \mathbb{R}$ is continuous;

3. $|\lambda|\|K\|_\infty|b-a| < 1$,

there exists a unique function $x(t)$ satisfying (5.7.4).

5.8 Exercises

§5.1 Pointwise and Uniform Convergence

Problem 5.1. Let (M,d) be a metric space, $A \subseteq M$, and $f_k : A \to \mathbb{R}$ be a sequence of functions (not necessary continuous) which converges uniformly on A. Suppose that $a \in \mathrm{cl}(A)$ and

$$\lim_{x\to a} f_k(x) = A_k$$

exists for all $k \in \mathbb{N}$. Show that $\{A_k\}_{k=1}^\infty$ converges, and

$$\lim_{x\to a}\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty}\lim_{x\to a} f_k(x)\,.$$

Problem 5.2. Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f_k : A \to N$ be uniformly continuous functions, and $\{f_k\}_{k=1}^\infty$ converges uniformly to $f : A \to N$ on A. Show that f is uniformly continuous on A.

Problem 5.3. Complete the following.

(a) Suppose that $f_k, f, g : [0, \infty) \to \mathbb{R}$ are functions such that

1. $\forall\, R > 0$, f_k and g are Riemann integrable on $[0, R]$;
2. $|f_k(x)| \leqslant g(x)$ for all $k \in \mathbb{N}$ and $x \in [0, \infty)$;
3. $\forall\, R > 0$, $\{f_k\}_{k=1}^{\infty}$ converges to f uniformly on $[0, R]$;
4. $\displaystyle\int_0^\infty g(x)dx \equiv \lim_{R\to\infty} \int_0^R g(x)dx < \infty$.

Show that $\displaystyle\lim_{k\to\infty} \int_0^\infty f_k(x)dx = \int_0^\infty f(x)dx$; that is,

$$\lim_{k\to\infty} \lim_{R\to\infty} \int_0^R f_k(x)dx = \lim_{R\to\infty} \lim_{k\to\infty} \int_0^R f_k(x)dx\,.$$

(b) Let $f_k(x)$ be given by $f_k(x) = \begin{cases} 1 & \text{if } k-1 \leqslant x < k\,, \\ 0 & \text{otherwise.} \end{cases}$ Find the (pointwise) limit f of the sequence $\{f_k\}_{k=1}^{\infty}$, and check whether $\displaystyle\lim_{k\to\infty} \int_0^\infty f_k(x)dx = \int_0^\infty f(x)dx$ or not. Briefly explain why one can or cannot apply (a).

(c) Let $f_k : [0, \infty) \to \mathbb{R}$ be given by $f_k(x) = \dfrac{x}{1 + kx^4}$. Find $\displaystyle\lim_{k\to\infty} \int_0^\infty f_k(x)dx$.

§5.2 The Weierstrass M-Test

Problem 5.4. Show that the series

$$\sum_{k=1}^{\infty} (-1)^k \frac{x^2 + k}{k^2}$$

converges uniformly on every bounded interval.

Problem 5.5. Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{1 + k^2 x}\,.$$

On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? If f bounded?

Problem 5.6. Determine which of the following real series $\sum\limits_{k=1}^{\infty} g_k$ converge (pointwise or uniformly). Check the continuity of the limit in each case.

1. $g_k(x) = \begin{cases} 0 & \text{if } x \leqslant k\,, \\ (-1)^k & \text{if } x > k\,. \end{cases}$

2. $g_k(x) = \begin{cases} \dfrac{1}{k^2} & \text{if } |x| \leqslant k\,, \\ \dfrac{1}{x^2} & \text{if } |x| > k\,. \end{cases}$

3. $g_k(x) = \Big(\dfrac{(-1)^k}{\sqrt{k}}\Big)\cos(kx)$ on $\mathbb{R}$.

4. $g_k(x) = x^k$ on $(0, 1)$.

§5.3 Integration and Differentiation of Series

Problem 5.7. In the following series of functions defined on $\mathbb{R}$, find its domain of convergence (classify it into domain of absolute and conditional convergence). If the series is a power series, find its radius of convergence. Also discuss whether the series is uniformly convergent in every compact subsets of its domain of convergence. Determine which series can be differentiated or integrated term by term in its domain of convergence.

(1) $\sum\limits_{k=1}^{\infty} \dfrac{x}{k^\alpha + k^\beta x^2}$, $\alpha \geqslant 0$, $\beta > 0$;

(2) $\sum\limits_{k=1}^{\infty} \dfrac{1}{2^k}\sqrt{1 - x^{2k}}$;

(3) $\sum\limits_{k=1}^{\infty} \dfrac{1 \cdot 3 \cdots (2k-1)}{2 \cdot 4 \cdots (2k)}\big(1 + \dfrac{1}{2} + \cdots + \dfrac{1}{k}\big)x^{2k}$;

(4) $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k-1}}{k\log(k+1)}x^{k!}$;

(5) $\sum\limits_{k=1}^{\infty} a_k x^k$, where $\{a_k\}_{k=1}^{\infty}$ is defined by the recursive relation $a_k = 3a_{k-1} - 2a_{k-2}$ for $k \geqslant 2$, and $a_0 = 1$, $a_1 = 2$.

Also find the sum of the series in (5).

Problem 5.8. In this problem we investigate the differentiability of a complex power series. This requires a new proof of $\frac{d}{dx}\sum\limits_{k=0}^{\infty} a_k x^k = \sum\limits_{k=1}^{\infty} k a_k x^{k-1}$ instead of making use of Theorem 5.10.

Let $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$ be a real sequence, and $f(x) = \sum\limits_{k=0}^{\infty} a_k x^k$ be a (real) power series with radius of convergence $R > 0$. Let $s_n(x) = \sum\limits_{k=0}^{n} a_k x^k$ be the n-th partial sum, $R_n(x) = f(x) - s_n(x)$, and $g(x) = \sum\limits_{k=1}^{\infty} k a_k x^{k-1}$. For $x, x_0 \in (-\rho, \rho) \subsetneq (-R, R)$, where $x \neq x_0$, write

$$\frac{f(x) - f(x_0)}{x - x_0} - g(x) = \frac{s_n(x) - s_n(x_0)}{x - x_0} - s_n'(x_0) + \big(s_n'(x_0) - g(x_0)\big) + \frac{R_n(x) - R_n(x_0)}{x - x_0}.$$

1. Show that
$$\Big|\frac{R_n(x) - R_n(x_0)}{x - x_0}\Big| \leqslant \sum_{k=n+1}^{\infty} k|a_k|\rho^{k-1},$$
and use the inequality above to show that $\lim\limits_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g(x_0)$.

2. Generalize the conclusion to complex power series; that is, show that if $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{C}$ and the power series $\sum\limits_{k=0}^{\infty} a_k z^k$ has radius of convergence $R > 0$, then
$$\frac{d}{dz}\sum_{k=0}^{\infty} a_k z^k = \sum_{k=1}^{\infty} k a_k z^{k-1} \qquad \forall\, |z| < R.$$
Assume that you have known $\frac{d}{dz}\sum\limits_{k=0}^{n} a_k z^k = \sum\limits_{k=1}^{n} k a_k z^{k-1}$ for all $n \in \mathbb{N} \cup \{0\}$ (this can be proved using the definition of differentiability of functions with values in normed vector spaces provided in Chapter 6).

Problem 5.9. Suppose that the series $\sum\limits_{n=0}^{\infty} a_n = 0$, and $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ for $-1 < x \leqslant 1$. Show that f is continuous at $x = 1$ by complete the following.

1. Write $s_n = a_0 + a_1 + \cdots + a_n$ and $s_n(x) = a_0 + a_1 x + \cdots + a_n x^n$. Show that
$$s_n(x) = (1 - x)(s_0 + s_1 x + \cdots + s_{n-1} x^{n-1}) + s_n x^n$$
and $f(x) = (1 - x)\sum\limits_{n=0}^{\infty} s_n x^n$.

2. Using the representation of f from above to conclude that $\lim\limits_{x \to 1^-} f(x) = 0$.

3. What if $\sum\limits_{n=0}^{\infty} a_n$ is convergent but not zero?

Problem 5.10. Construct the function $g(x)$ by letting $g(x) = |x|$ if $x \in \big[-\frac{1}{2}, \frac{1}{2}\big]$ and extending g so that it becomes periodic (with period 1). Define

$$f(x) = \sum_{k=1}^{\infty} \frac{g(4^{k-1}x)}{4^{k-1}}\,.$$

1. Use the Weierstrass M-test to show that f is continuous on $\mathbb{R}$.

2. Prove that f is differentiable at no point.

(So there exists a continuous which is nowhere differentiable!)
Hint: Google Blancmange function!

§5.4 The Space of Continuous Functions

Problem 5.11. Let $\delta : (\mathscr{C}([0,1];\mathbb{R}), \|\cdot\|_\infty) \to \mathbb{R}$ be defined by $\delta(f) = f(0)$. Show that δ is linear and continuous.

Problem 5.12. Let (M, d) be a metric space, and $K \subseteq M$ be a compact subset.

1. Show that the set $U = \big\{f \in \mathscr{C}(K;\mathbb{R}) \,\big|\, a < f(x) < b \text{ for all } x \in K\big\}$ is open in $\big(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty\big)$ for all $a, b \in \mathbb{R}$.

2. Show that the set $F = \big\{f \in \mathscr{C}(K;\mathbb{R}) \,\big|\, a \leqslant f(x) \leqslant b \text{ for all } x \in K\big\}$ is closed in $\big(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty\big)$ for all $a, b \in \mathbb{R}$.

3. Let $A \subseteq M$ be a subset, not necessarily compact. Prove or disprove that the set $B = \big\{f \in \mathscr{C}_b(A;\mathbb{R}) \,\big|\, f(x) > 0 \text{ for all } x \in A\big\}$ is open in $\big(\mathscr{C}_b(A;\mathbb{R}), \|\cdot\|_\infty\big)$.

§5.5 The Arzelà-Ascoli Theorem

Problem 5.13. Which of the following set B of continuous functions are equi-continuous in the metric space M? Are the closure $\bar{B}$ compact in M?

1. $B = \big\{\sin kx \,\big|\, k = 0, 1, 2, \cdots\big\}$, $M = \mathscr{C}([0,\pi];\mathbb{R})$.

2. $B = \big\{\sin\sqrt{x + 4k^2\pi^2} \,\big|\, k = 0, 1, 2, \cdots\big\}$, $M = \mathscr{C}_b([0,\infty);\mathbb{R})$.

3. $B = \left\{ \dfrac{x^2}{x^2 + (1 - kx)^2} \,\Big|\, k = 0, 1, 2, \cdots \right\}$, $M = \mathscr{C}([0,1]; \mathbb{R})$.

4. $B = \left\{ (k+1)x^k(1-x) \,\big|\, k \in \mathbb{N} \right\}$, $M = \mathscr{C}([0,1]; \mathbb{R})$.

Problem 5.14. Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a normed space, and $A \subseteq M$ be a subset. Suppose that $B \subseteq \mathscr{C}_b(A; \mathcal{V})$ be equi-continuous. Prove or disprove that $\text{cl}(B)$ is equi-continuous.

Problem 5.15. Let $f_k : [a, b] \to \mathbb{R}$ be a sequence of differentiable functions such that $f_k(a)$ is bounded and $|f_k'(x)| \leqslant M$ for all $x \in [a, b]$ and $k \in \mathbb{N}$. Show that $\{f_k\}_{k=1}^{\infty}$ contains an uniformly convergent subsequence. Must the limit function differentiable?

Problem 5.16. Let $\mathscr{C}^{0,\alpha}([0, 1]; \mathbb{R})$ denote the "space"

$$\mathscr{C}^{0,\alpha}([0,1]; \mathbb{R}) \equiv \left\{ f \in \mathscr{C}([0,1]; \mathbb{R}) \,\Big|\, \sup_{x,y \in [0,1]} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty \right\},$$

where $\alpha \in (0, 1]$. For each $f \in \mathscr{C}^{0,\alpha}([0, 1]; \mathbb{R})$, define

$$\|f\|_{\mathscr{C}^{0,\alpha}} = \sup_{x \in [0,1]} |f(x)| + \sup_{\substack{x,y \in [0,1] \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

1. Show that $\left(\mathscr{C}^{0,\alpha}([0, 1]; \mathbb{R}), \|\cdot\|_{\mathscr{C}^{0,\alpha}}\right)$ is a complete normed space.

2. Show that the set $B = \left\{ f \in \mathscr{C}([0, 1]; \mathbb{R}) \,\big|\, \|f\|_{\mathscr{C}^{0,\alpha}} < 1 \right\}$ is equi-continuous.

3. Show that $\text{cl}(B)$ is compact in $\left(\mathscr{C}([0, 1]; \mathbb{R}), \|\cdot\|_\infty\right)$.

Problem 5.17. Given $f : \mathbb{R} \to \mathbb{R}$ a continuous periodic function of period 1; that is, $f(x + 1) = f(x)$ for all $x \in \mathbb{R}$, and $x_1, \cdots, x_m \in [0, 1]$ arbitrary m points, define a new function

$$I(f; x_1, \cdots, x_m)(x) = \frac{1}{m}\big(f(x + x_1) + \cdots f(x + x_m)\big) \qquad \forall\, x \in \mathbb{R}.$$

Prove that the set

$$B = \left\{ I(f; x_1, \cdots, x_m) \,\big|\, x_1, \cdots, x_m \in [0, 1], m \in \mathbb{N} \right\}$$

is uniformly bounded and equi-continuous in the space $\mathscr{C}([0, 1]; \mathbb{R})$. Moreover, show that the derived set $B' = \left\{ \int_0^1 f(x)dx \right\}$; that is, the derived set of B consists of one single function which is a constant function $y = \int_0^1 f(x)dx$.

Problem 5.18. Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a Banach space, $K \subseteq M$ be compact, and $\{f_k\}_{k=1}^{\infty} \subseteq \mathscr{C}(K; \mathcal{V})$ be a sequence of continuous functions. Suppose that for all $x \in K$, if $\{x_k\}_{k=1}^{\infty}, \{y_k\}_{k=1}^{\infty} \subseteq K$ and $\lim\limits_{k\to\infty} x_k = \lim\limits_{k\to\infty} y_k = x$, the limits $\lim\limits_{k\to\infty} f_k(x_k)$ and $\lim\limits_{k\to\infty} f_k(y_k)$ exist and are identical. Show that $\{f_k\}_{k=1}^{\infty}$ converges uniformly on K. How about if K is not compact?

Problem 5.19. Assume that $\{f_k\}_{k=1}^{\infty}$ is a sequence of monotone increasing functions on $\mathbb{R}$ with $0 \leqslant f_k(x) \leqslant 1$ for all x and $k \in \mathbb{N}$.

1. Show that there is a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ which converges pointwise to a function f (This is usually called the ***Helly selection theorem***).

2. If the limit f is continuous, show that $\{f_{k_j}\}_{j=1}^{\infty}$ converges uniformly to f on any compact set of $\mathbb{R}$.

§5.6 The Stone-Weierstrass Theorem

Problem 5.20. Define B to be the set of all even functions in the space $\mathscr{C}([-1, 1]; \mathbb{R})$; that is, $f \in B$ if and only if f is continuous on $[-1, 1]$ and $f(x) = f(-x)$ for all $x \in [-1, 1]$. Prove that B is closed but not dense in $\mathscr{C}([-1, 1]; \mathbb{R})$. Hence show that even polynomials are dense in B, but not in $\mathscr{C}([-1, 1]; \mathbb{R})$.

Problem 5.21. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function.

1. Suppose that
$$\int_0^1 f(x)x^n dx = 0 \qquad \forall\, n \in \mathbb{N} \cup \{0\}\,.$$
Show that $f = 0$ on $[0, 1]$.

2. Suppose that for some $m \in \mathbb{N}$,
$$\int_0^1 f(x)x^n dx = 0 \qquad \forall\, n \in \{0, 1, \cdots, m\}\,.$$
Show that $f(x) = 0$ has at least m distinct real roots around which $f(x)$ change signs.

Problem 5.22. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Show that
$$\lim_{n\to\infty} \int_0^1 f(x) \sin(nx)\, dx = 0\,.$$

Problem 5.23. Put $p_0 = 0$ and define

$$p_{k+1}(x) = p_k(x) + \frac{x^2 - p_k^2(x)}{2} \qquad \forall\, k \in \mathbb{N} \cup \{0\}\,.$$

Show that $\{p_k\}_{k=1}^{\infty}$ converges uniformly to $|x|$ on $[-1, 1]$.
Hint: Use the identity

$$|x| - p_{k+1}(x) = \Big[|x| - p_k(x)\Big]\Big[1 - \frac{|x| + p_k(x)}{2}\Big]$$

to prove that $0 \leqslant p_k(x) \leqslant p_{k+1}(x) \leqslant |x|$ if $|x| \leqslant 1$, and that

$$|x| - p_k(x) \leqslant |x|\Big(1 - \frac{|x|}{2}\Big)^k < \frac{2}{k+1}$$

if $|x| \leqslant 1$.

Problem 5.24. Let $f : [0, 1] \to \mathbb{R}$ be continuous and $\varepsilon > 0$. Prove that there is a simple function g (defined in Example 5.74) such that $\|f - g\|_\infty < \varepsilon$.

Problem 5.25. Suppose that p_n is a sequence of polynomials converging uniformly to f on $[0, 1]$ and f is not a polynomial. Prove that the degrees of p_n are not bounded.
Hint: An Nth-degree polynomial p is uniquely determined by its values at $N + 1$ points $x_0, \cdots, x_N$ via Lagrange's interpolation formula

$$p(x) = \sum_{k=0}^{N} \pi_k(x) \frac{p(x_k)}{\pi_k(x_k)}\,,$$

where $\pi_k(x) = (x - x_0)(x - x_1) \cdots (x - x_N)/(x - x_k) = \prod\limits_{\substack{1 \leqslant j \leqslant N \\ j \neq k}} (x - x_j)$.

Problem 5.26. Consider the set of all functions on $[0, 1]$ of the form

$$h(x) = \sum_{j=1}^{n} a_j e^{b_j x}\,,$$

where a_j, $b_j \in \mathbb{R}$. Is this set dense in $\mathscr{C}([0, 1]; \mathbb{R})$?

§5.7 The Contraction Mapping Principle and its Applications

Problem 5.27. Suppose that $f : [a, b] \to \mathbb{R}$ is twice continuous differentiable; that is, $f', f'' : [a, b] \to \mathbb{R}$ are continuous, and $f(a) < 0 = f(c) < f(b)$, and $f'(x) \neq 0$ for all $x \in [a, b]$. Consider the function

$$\Phi(x) = x - \frac{f(x)}{f'(x)}\,.$$

1. Show that $\Phi : [a, b] \to \mathbb{R}$ satisfies

$$|\Phi(x) - \Phi(y)| \leqslant k|x - y| \quad \forall\, x, y \in [a, b]$$

for some $k \in [0, 1)$ if $|b - a|$ are small enough.

2. Suppose that $f''(x) > 0$ for all $x \in [a, b]$. Show that there exists $a \leqslant \tilde{a} < c$ such that $\Phi : [\tilde{a}, b] \to [\tilde{a}, b]$.

3. Under the condition of 2, show that if $x_0 \in [\tilde{a}, b]$, then the iteration

$$x_{k+1} = \Phi(x_k) \quad \forall\, k \in \mathbb{N} \cup \{0\}$$

provides a convergent sequence $\{x_k\}_{k=1}^{\infty}$ with limit c.

(The iteration scheme above of finding the zero c of f is called the Newton method.)

Problem 5.28. Let (M, d) be a complete metric space, and $f : M \to M$. Define $f_k = f \circ f \circ \cdots \circ f$, here the composition was taken for $k - 1$ times. Assume that there exists a sequence $\{\alpha_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ such that

1. $\alpha_k \to 0$ as $k \to \infty$.

2. $d\big(f_k(x), f_k(y)\big) \leqslant \alpha_k d(x, y)$ for all $k \in \mathbb{N}$, $x, y \in M$.

Show that f has a unique fixed-point.

Problem 5.29. Let (M, d) be a metric space, and $K \subseteq M$ be a compact set.

(1) Given $f : M \to M$ a continuous map, define $f_k = f \circ f \circ \cdots \circ f$ (as in the previous problem) to the the k-th iterate of f. Prove that if f_k has a unique fixed-point x_0, then $f(x_0) = x_0$.

(2) Let $f : K \to K$ be continuous and $d\big(f(x), f(y)\big) < d(x, y)$ for all $x, y \in K$. Show that f has a unique fixed-point in K. Show that the conclusion is false if K is not compact.

(3) Let $K = [0, 1]$ be a closed interval in (2) and $\big|f(x) - f(y)\big| \leqslant |x - y|$ for all $x, y \in K$. Given any $x_1 \in [0, 1]$, define a sequence $\{x_k\}_{k=1}^{\infty}$ by

$$x_{k+1} = \frac{1}{2}\big(x_k + f(x_k)\big) \qquad \forall\, k > 1\,.$$

Show that $\{x_k\}_{k=1}^{\infty}$ converges to a fixed-point of f.

Problem 5.30. Let (M, d) be a metric space, and $f : M \to M$ be such that $d\big(f(x), f(y)\big) < d(x, y)$ for all $x, y \in M$, $x \neq y$.

1. Fix $x_0 \in M$. Let $x_{n+1} = f(x_n)$, and $c_n = d(x_n, x_{n+1})$. Show that $\{c_n\}_{n=1}^{\infty}$ is a decreasing sequence; thus $c = \lim\limits_{n\to\infty} c_n$ exists.

2. Assume that there is a subsequence $\big\{x_{n_j}\big\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_j} \to x$ as $j \to \infty$. Show that
$$c = d\big(x, f(x)\big) = d\big(f(x), f(f(x))\big)\,.$$
and deduce that x is a fixed-point of f.

3. Suppose further that M is compact. Show that the sequence $\{x_n\}_{n=1}^{\infty}$ itself converges to x.

Problem 5.31. Find an upper bound on $r > 0$ such that the mapping $T : \mathscr{C}([0, r]; \mathbb{R}) \to \mathscr{C}([0, r]; \mathbb{R})$ defined by
$$T(f)(x) = 1 + 3\int_0^x t f(t)\,dt$$
is a contraction mapping. what is its fixed-point?

Problem 5.32. Let $A = [a, b] \times [a, b]$ be a closed square in $\mathbb{R}^2$, $M = \mathscr{C}([a, b]; \mathbb{R})$, and $K : A \to \mathbb{R}$ be a continuous function. For $f \in \mathscr{C}([a, b]; \mathbb{R})$, define
$$T(f)(x) = \int_a^b K(x, y) f(y) dy \qquad \forall\, x \in [a, b]\,.$$

(1) Show that $T(f) \in M$ for all $f \in M$, and $T : M \to M$ is Lipschitz continuous. Find a Lipschitz constant for T.

(2) If $B \subseteq M$ is a bounded subset of M, show that the image $T(B) = \big\{T(f) \,\big|\, f \in B\big\}$ is uniformly bounded and equi-continuous.

(3) If the norm $\|K\|_\infty < \dfrac{1}{b-a}$, show that T is a contraction mapping. What is its fixed-point?

(4) If K satisfies the assumption in (3), show that the mapping $S : M \to M$ defined by $S(f) = f - T(f)$ is a homeomorphism.

(5) Let $a = 0$, $b = 1$, and $K(x, y) = \frac{1}{4}e^{x+y-1}$. Show that K satisfies the assumption in (3). Given $g \in M$, find $f \in M$ such that $S(f) = g$.

Problem 5.33. Let $A = [a, b] \times [a, b]$ be a closed square in $\mathbb{R}^2$, and $K : A \to \mathbb{R}$ be continuous on A. Define

$$T(f)(x) = \int_a^b K(x, y)g(y)dy\,,$$

where f is a real-valued function defined on $[a, b]$ such that the integral makes sense. For a family of functions $\mathcal{F}$ consisting of f such that $T(f)$ is well-defined and $|f(y)| \leqslant M$ for all $y \in [a, b]$, let $\mathcal{G} = T(\mathcal{F})$. Show that each sequence of $\mathcal{G}$ contains a uniformly convergent subsequence.

Problem 5.34 (True or False). Determine whether the following statements are true or false. If it is true, prove it. Otherwise, give a counter-example.

1. Let $f_n : [a, b] \to \mathbb{R}$ be an uniformly convergent sequence of continuous functions. Then the sequence of the indefinite integrals $g_n(x)$ defined by

$$g_n(x) = \int_a^x f_n(t)\,dt \qquad \forall\, x \in [a, b]$$

converges uniformly to a continuously differentiable function.

2. Let $f_n : [0, 1] \to \mathbb{R}$ be a equi-continuous sequence of functions such that the sequence $\{f_n(\frac{1}{2})\}_{n=1}^{\infty}$ is bounded in $\mathbb{R}$. Then $\{f_n\}_{n=1}^{\infty}$ contains a convergent subsequence.

3.

4.