Chapter 4

Continuous Maps

4.1 Continuity

Definition 4.1. Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ and $f : A \to N$ be a map. For a given $x_0 \in A'$, we say that $b \in N$ is the limit of f at x_0, written

$$\lim_{x \to x_0} f(x) = b \quad \text{or} \quad f(x) \to b \text{ as } x \to x_0 \,,$$

if for every sequence $\{x_k\}_{k=1}^{\infty} \subseteq A \backslash \{x_0\}$ converging to x_0, the sequence $\big\{f(x_k)\big\}_{k=1}^{\infty}$ converges to b.

Proposition 4.2. *Let (M,d) and (N,ρ) be two metric spaces, $A \subseteq M$ and $f : A \to N$ be a map. Then $\lim\limits_{x \to x_0} f(x) = b$ if and only if*

$$\forall\, \varepsilon > 0, \exists\, \delta = \delta(x_0, \varepsilon) > 0 \ni \rho(f(x), b) < \varepsilon \text{ whenever } 0 < d(x, x_0) < \delta \text{ and } x \in A\,.$$

Proof. "$\Rightarrow$" Assume the contrary that $\exists\, \varepsilon > 0$ such that for all $\delta > 0$, there exists $x_\delta \in A$ with

$$0 < d(x_\delta, x_0) < \delta \quad \text{and} \quad \rho(f(x_\delta), b) \geqslant \varepsilon\,.$$

In particular, letting $\delta = \dfrac{1}{k}$, we can find $\{x_k\}_{k=1}^{\infty} \subseteq A \backslash \{x_0\}$ such that

$$0 < d(x_k, x_0) < \frac{1}{k} \quad \text{and} \quad \rho(f(x_k), b) \geqslant \varepsilon\,.$$

Then $x_k \to x_0$ as $k \to \infty$ but $f(x_k) \nrightarrow b$ as $k \to \infty$, a contradiction.

"$\Leftarrow$" Let $\{x_k\}_{k=1}^{\infty} \subseteq A\backslash\{x_0\}$ be such that $x_k \to x_0$ as $k \to \infty$, and $\varepsilon > 0$ be given. By assumption,

$$\exists\, \delta = \delta(x_0, \varepsilon) > 0 \ni \rho(f(x), b) < \varepsilon \text{ whenever } 0 < d(x, x_0) < \delta \text{ and } x \in A\,.$$

Since $x_k \to x_0$ as $k \to \infty$, $\exists\, N > 0 \ni d(x_k, x_0) < \delta$ if $k \geqslant N$. Therefore,

$$\rho(f(x_k), b) < \varepsilon \quad \forall\, k \geqslant N$$

which suggests that $\lim\limits_{k\to\infty} f(x_k) = b$. □

Remark 4.3. Let $(M, d) = (N, \rho) = (\mathbb{R}, |\cdot|)$, $A = (a, b)$, and $f : A \to N$. We write $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} f(x)$ for the limit $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to b} f(x)$, respectively, if the later exist. Following this notation, we have

$$\lim_{x\to a^+} f(x) = L \Leftrightarrow \forall\, \varepsilon > 0, \exists\, \delta > 0 \ni |f(x) - L| < \varepsilon \text{ if } 0 < x - a < \delta \text{ and } x \in (a, b)\,,$$
$$\lim_{x\to b^-} f(x) = L \Leftrightarrow \forall\, \varepsilon > 0, \exists\, \delta > 0 \ni |f(x) - L| < \varepsilon \text{ if } 0 < b - x < \delta \text{ and } x \in (a, b)\,.$$

Definition 4.4. Let (M, d) and (N, ρ) be two metric spaces, $A \subseteq M$, and $f : A \to N$ be a map. For a given $x_0 \in A$, f is said to be continuous at x_0 if either $x_0 \in A\backslash A'$ or $\lim\limits_{x\to x_0} f(x) = f(x_0)$.

Example 4.5. The identity map $f : \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^n \\ x & \mapsto & x \end{array}$ is continuous at each point of $\mathbb{R}^n$.

Example 4.6. The function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$ is continuous at each point of $(0, \infty)$.

Proposition 4.7. *Let (M, d) and (N, ρ) be two metric spaces, $A \subseteq M$, and $f : A \to N$ be a map. Then f is continuous at $x_0 \in A$ if and only if*

$$\forall\, \varepsilon > 0, \exists\, \delta = \delta(x_0, \varepsilon) > 0 \ni \rho(f(x), f(x_0)) < \varepsilon \textit{ whenever } x \in D(x_0, \delta) \cap A\,.$$

Proof. **Case 1**: If $x_0 \in A'$, then f is continuous at x_0 if and only if

$$\forall\, \varepsilon > 0, \exists\, \delta = \delta(x_0, \varepsilon) > 0 \ni \rho(f(x), f(x_0)) < \varepsilon \text{ whenever } x \in D(x_0, \delta) \cap A\backslash\{x_0\}\,.$$

Since $\rho(f(x_0), f(x_0)) = 0 < \varepsilon$, we find that the statement above is equivalent to that

$$\forall\, \varepsilon > 0, \exists\, \delta = \delta(x_0, \varepsilon) > 0 \ni \rho(f(x), f(x_0)) < \varepsilon \text{ whenever } x \in D(x_0, \delta) \cap A\,.$$

Case 2: Let $x_0 \in A \backslash A'$.

"$\Rightarrow$" then $\exists\, \delta > 0$ such that $D(x_0, \delta) \cap A = \{x_0\}$. Therefore, for this particular δ, we must have

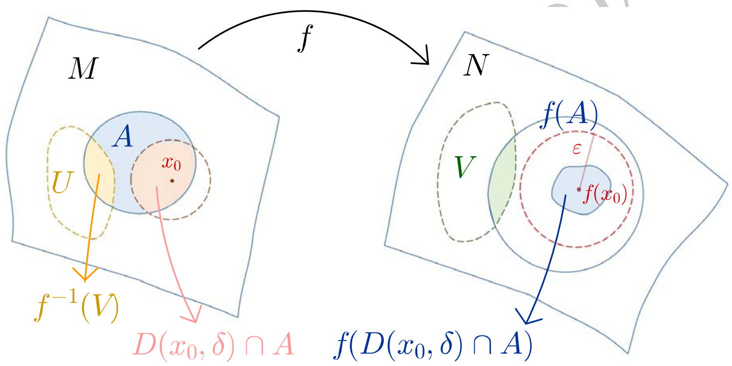
$$\rho(f(x), f(x_0)) = 0 < \varepsilon \quad \text{whenever } x \in D(x_0, \delta) \cap A\,.$$

"$\Leftarrow$" We note that if $x_0 \in A \backslash A'$, f is defined to be continuous at x_0. In other words,

f is continuous at each isolated point. □

Remark 4.8. We remark here that Proposition 4.7 suggests that f is continuous at $x_0 \in A$ if and only if

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni f(D(x_0, \delta) \cap A) \subseteq D(f(x_0), \varepsilon)\,.$$



Remark 4.9. In general the number δ in Proposition 4.7 also depends on the function f. For a function $f : A \to \mathbb{R}$ which is continuous at $x_0 \in A$, let $\delta(f, x_0, \varepsilon)$ denote the largest $\delta > 0$ such that if $x \in D(x_0, \delta) \cap A$, then $\rho\big(f(x), f(x_0)\big) < \varepsilon$. In other words,

$$\delta(f, x_0, \varepsilon) = \sup\big\{\delta > 0 \,\big|\, \rho\big(f(x), f(x_0)\big) < \varepsilon \text{ if } x \in D(x_0, \delta) \cap A\big\}\,.$$

This number provides another way for the understanding of the uniform continuity (in Section 4.5) and the equi-continuity (in Section 5.5). See Remark 4.51 and Remark 5.51 for further details.

Definition 4.10. Let (M, d) and (N, ρ) be metric spaces, and $A \subseteq M$. A map $f : A \to N$ is said to be continuous on the set $B \subseteq A$ if f is continuous at each point of B.

Theorem 4.11. *Let (M, d) and (N, ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be a map. Then the following assertions are equivalent:*

1. *f is continuous on A.*

2. *For each open set $\mathcal{V} \subseteq N$, $f^{-1}(\mathcal{V}) \subseteq A$ is open relative to A; that is, $f^{-1}(\mathcal{V}) = \mathcal{U} \cap A$ for some $\mathcal{U}$ open in M.*

3. *For each closed set $E \subseteq N$, $f^{-1}(E) \subseteq A$ is closed relative to A; that is, $f^{-1}(E) = F \cap A$ for some F closed in M.*

Proof. It should be clear that $2 \Leftrightarrow 3$ (left as an exercise); thus we show that $1 \Leftrightarrow 2$. Before proceeding, we recall that $B \subseteq f^{-1}(f(B))$ for all $B \subseteq A$ and $f(f^{-1}(B)) \subseteq B$ for all $B \subseteq N$.

"$1 \Rightarrow 2$" Let $a \in f^{-1}(\mathcal{V})$. Then $f(a) \in \mathcal{V}$. Since $\mathcal{V}$ is open in (N, ρ), $\exists\, \varepsilon_{f(a)} > 0$ such that $D(f(a), \varepsilon_{f(a)}) \subseteq \mathcal{V}$. By continuity of f (and Remark 4.8), there exists $\delta_a > 0$ such that

$$f(D(a, \delta_a) \cap A) \subseteq D\big(f(a), \varepsilon_{f(a)}\big)\,.$$

Therefore, by Proposition 0.16, for each $a \in f^{-1}(\mathcal{V})$, $\exists\, \delta_a > 0$ such that

$$D(a, \delta_a) \cap A \subseteq f^{-1}\big(f(D(a, \delta_a) \cap A)\big) \subseteq f^{-1}\big(D\big(f(a), \varepsilon_{f(a)}\big)\big) \subseteq f^{-1}(\mathcal{V})\,. \qquad (4.1.1)$$

Let $\mathcal{U} = \bigcup\limits_{a \in f^{-1}(\mathcal{V})} D(a, \delta_a)$. Then $\mathcal{U}$ is open (since it is the union of arbitrarily many open balls), and

(a) $\mathcal{U} \supseteq f^{-1}(\mathcal{V})$ since $\mathcal{U}$ contains every center of balls whose union forms $\mathcal{U}$;

(b) $\mathcal{U} \cap A \subseteq f^{-1}(\mathcal{V})$ by (4.1.1).

Therefore, $\mathcal{U} \cap A = f^{-1}(\mathcal{V})$.

"$2 \Rightarrow 1$" Let $a \in A$ and $\varepsilon > 0$ be given. Define $\mathcal{V} = D(f(a), \varepsilon)$. By assumption there exists $\mathcal{U}$ open in (M, d) such that $f^{-1}(\mathcal{V}) = \mathcal{U} \cap A$. Since $a \in f^{-1}(\mathcal{V})$, $a \in \mathcal{U}$; thus by the openness of $\mathcal{U}$, $\exists\, \delta > 0$ such that $D(a, \delta) \subseteq \mathcal{U}$. Therefore, by Proposition 0.16 we have

$$f(D(a, \delta) \cap A) \subseteq f(\mathcal{U} \cap A) = f(f^{-1}(\mathcal{V})) \subseteq \mathcal{V} = D(f(a), \varepsilon)$$

which suggests that f is continuous at a for all $a \in A$; thus f is continuous on A. ▫

Example 4.12. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Then $\big\{x \in \mathbb{R}^n \,\big|\, \|f(x)\|_2 < 1\big\}$ is open since

$$\big\{x \in \mathbb{R}^n \,\big|\, \|f(x)\|_2 < 1\big\} = f^{-1}\big(D(0, 1)\big)\,.$$

Remark 4.13. For a function f of two variable or more, it is important to distinguish the continuity of f and the continuity in each variable (by holding all other variables fixed). For example, let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 1 & \text{if either } x = 0 \text{ or } y = 0, \\ 0 & \text{if } x \neq 0 \text{ and } y \neq 0. \end{cases}$$

Observe that $f(0,0) = 1$, but f is not continuous at $(0,0)$. In fact, for any $\delta > 0$, $f(x,y) = 0$ for infinitely many values of $(x,y) \in D((0,0),\delta)$; that is, $|f(x,y) - f(0,0)| = 1$ for such values. However if we consider the function $g(x) = f(x,0) = 1$ or the function $h(y) = f(0,y) = 1$, then g,h are continuous. Note also that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exists but $\lim\limits_{x\to 0}(\lim\limits_{y\to 0} f(x,y)) = \lim\limits_{x\to 0} 0 = 0$.

4.2 Operations on Continuous Maps

Definition 4.14. Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a (real) normed space, $A \subseteq M$, and $f,g : A \to \mathcal{V}$ be maps, $h : A \to \mathbb{R}$ be a function. The maps $f+g$, $f-g$ and hf, mapping from A to $\mathcal{V}$, are defined by

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \qquad \forall\, x \in A\,, \\ (f-g)(x) &= f(x) - g(x) \qquad \forall\, x \in A\,, \\ (hf)(x) &= h(x)f(x) \qquad \forall\, x \in A\,. \end{aligned}$$

The map $\dfrac{f}{h} : A \backslash \{x \in A \,|\, h(x) = 0\} \to \mathcal{V}$ is defined by

$$\Big(\frac{f}{h}\Big)(x) = \frac{f(x)}{h(x)} \qquad \forall\, x \in A \backslash \{x \in A \,|\, h(x) = 0\}\,.$$

Proposition 4.15. *Let (M,d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a (real) normed space, $A \subseteq M$, and $f,g : A \to \mathcal{V}$ be maps, $h : A \to \mathbb{R}$ be a function. Suppose that $x_0 \in A'$, and $\lim\limits_{x\to x_0} f(x) = a$, $\lim\limits_{x\to x_0} g(x) = b$, $\lim\limits_{x\to x_0} h(x) = c$. Then*

$$\begin{aligned} \lim_{x\to x_0} (f+g)(x) &= a + b\,, \\ \lim_{x\to x_0} (f-g)(x) &= a - b\,, \\ \lim_{x\to x_0} (hf)(x) &= ca\,, \\ \lim_{x\to x_0} \Big(\frac{f}{h}\Big) &= \frac{a}{c} \quad \text{if } c \neq 0\,. \end{aligned}$$

Corollary 4.16. *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a (real) normed space, $A \subseteq M$, and $f, g : A \to \mathcal{V}$ be maps, $h : A \to \mathbb{R}$ be a function. Suppose that f, g, h are continuous at $x_0 \in A$. Then the maps $f + g$, $f - g$ and hf are continuous at x_0, and $\dfrac{f}{h}$ is continuous at x_0 if $h(x_0) \neq 0$.*

Corollary 4.17. *Let (M, d) be a metric space, $(\mathcal{V}, \|\cdot\|)$ be a (real) normed space, $A \subseteq M$, and $f, g : A \to \mathcal{V}$ be continuous maps, $h : A \to \mathbb{R}$ be a continuous function. Then the maps $f + g$, $f - g$ and hf are continuous on A, and $\dfrac{f}{h}$ is continuous on $A\backslash\{x \in A \,|\, h(x) = 0\}$.*

Definition 4.18. Let (M, d), (N, ρ) and (P, r) be metric space, $A \subseteq M$, $B \subseteq N$, and $f : A \to N$, $g : B \to P$ be maps such that $f(A) \subseteq B$. The composite function $g \circ f : A \to P$ is the map defined by

$$(g \circ f)(x) = g\big(f(x)\big) \qquad \forall\, x \in A\,.$$

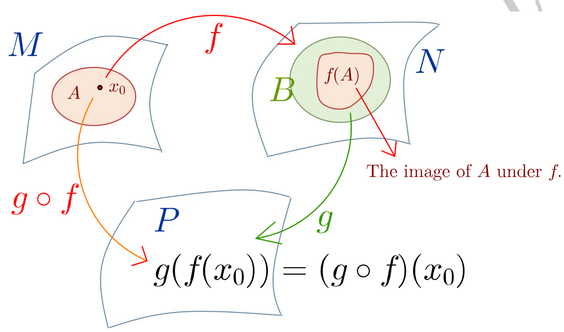


Figure 4.1: The composition of functions

Theorem 4.19. *Let (M, d), (N, ρ) and (P, r) be metric space, $A \subseteq M$, $B \subseteq N$, and $f : A \to N$, $g : B \to P$ be maps such that $f(A) \subseteq B$. Suppose that f is continuous at x_0, and g is continuous at $f(x_0)$. Then the composite function $g \circ f : A \to P$ is continuous at x_0.*

Proof. Let $\varepsilon > 0$ be given. Since g is continuous at $f(x_0)$, $\exists\, r > 0$ such that

$$g(D(f(x_0), r) \cap B) \subseteq D\big((g \circ f)(x_0), \varepsilon\big)\,.$$

Since f is continuous at x_0, $\exists\, \delta > 0$ such that

$$f(D(x_0, \delta) \cap A) \subseteq D\big(f(x_0), r\big)\,.$$

Since $f(A) \subseteq B$, $f(D(x_0, \delta) \cap A) \subseteq D\big(f(x_0), r\big) \cap B$; thus

$$(g \circ f)(D(x_0, \delta) \cap A) \subseteq g(D(f(x_0), r) \cap B) \subseteq D\big((g \circ f)(x_0), \varepsilon\big)\,. \qquad \square$$

Corollary 4.20. *Let* (M, d), (N, ρ) *and* (P, r) *be metric space,* $A \subseteq M$, $B \subseteq N$, *and* $f : A \to N$, $g : B \to P$ *be continuous maps such that* $f(A) \subseteq B$. *Then the composite function* $g \circ f : A \to P$ *is continuous on* A.

Alternative Proof of Corollary 4.20. Let $\mathcal{W}$ be an open set in (P, r). By Theorem 4.11, there exists $\mathcal{V}$ open in (N, ρ) such that $g^{-1}(\mathcal{W}) = \mathcal{V} \cap B$. Since $\mathcal{V}$ is open in (N, ρ), by Theorem 4.11 again there exists $\mathcal{U}$ open in (M, d) such that $f^{-1}(\mathcal{V}) = \mathcal{U} \cap A$. Then

$$(g \circ f)^{-1}(\mathcal{W}) = f^{-1}\big(g^{-1}(\mathcal{W})\big) = f^{-1}(\mathcal{V} \cap B) = f^{-1}(\mathcal{V}) \cap f^{-1}(B) = \mathcal{U} \cap A \cap f^{-1}(B)\,,$$

while the fact that $f(A) \subseteq B$ further suggests that

$$(g \circ f)^{-1}(\mathcal{W}) = \mathcal{U} \cap A\,.$$

Therefore, by Theorem 4.11 we find that $(g \circ f)$ is continuous on A. □

4.3 Images of Compact Sets under Continuous Maps

Theorem 4.21. *Let* (M, d) *and* (N, ρ) *be metric spaces,* $A \subseteq M$, *and* $f : A \to N$ *be a continuous map.*

1. *If* $K \subseteq A$ *is compact, then* $f(K)$ *is compact in* (N, ρ).
2. *Moreover, if* $(N, \rho) = (\mathbb{R}, |\cdot|)$, *then there exist* $x_0, x_1 \in K$ *such that*

$$f(x_0) = \inf f(K) = \inf \big\{ f(x) \,\big|\, x \in K \big\} \quad \textit{and} \quad f(x_1) = \sup f(K) = \sup \big\{ f(x) \,\big|\, x \in K \big\}\,.$$

Proof. 1. Let $\{\mathcal{V}_\alpha\}_{\alpha \in I}$ be an open cover of $f(K)$. Since $\mathcal{V}_\alpha$ is open, by Theorem 4.11 there exists $\mathcal{U}_\alpha$ open in (M, d) such that $f^{-1}(\mathcal{V}_\alpha) = \mathcal{U}_\alpha \cap A$. Since $f(K) \subseteq \bigcup\limits_{\alpha \in I} \mathcal{V}_\alpha$,

$$K \subseteq f^{-1}(f(K)) \subseteq \bigcup_{\alpha \in I} f^{-1}(\mathcal{V}_\alpha) = A \cap \bigcup_{\alpha \in I} \mathcal{U}_\alpha$$

which implies that $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ is an open cover of K. Therefore,

$$\exists\, J \subseteq I, \#J < \infty \ni K \subseteq A \cap \bigcup_{\alpha \in J} \mathcal{U}_\alpha = \bigcup_{\alpha \in J} f^{-1}(\mathcal{V}_\alpha)\,;$$

thus $f(K) \subseteq \bigcup\limits_{\alpha \in J} f(f^{-1}(\mathcal{V}_\alpha)) \subseteq \bigcup\limits_{\alpha \in J} \mathcal{V}_\alpha$.

2. By 1, $f(K)$ is compact; thus sequentially compact. Corollary 3.5 then implies that $\inf f(K) \in f(K)$ and $\sup f(K) \in f(K)$. ▫

Alternative Proof of Part 1. Let $\{y_n\}_{n=1}^\infty$ be a sequence in $f(K)$. Then there exists $\{x_n\}_{n=1}^\infty \subseteq K$ such that $y_n = f(x_n)$. Since K is sequentially compact, there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^\infty$ with limit $x \in K$. Let $y = f(x) \in f(K)$. By the continuity of f,

$$\lim_{k\to\infty} \rho\big(y_{n_k}, y\big) = \lim_{k\to\infty} \rho\big(f(x_{n_k}), f(x)\big) = 0$$

which implies that the sequence $\big\{y_{n_k}\big\}_{k=1}^\infty$ converges to $y \in f(K)$. Therefore, $f(K)$ is sequentially compact. ▫

Corollary 4.22 (The Extreme Value Theorem (極值定理)). *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f attains its maximum and minimum in $[a, b]$; that is, there are $x_0 \in [a, b]$ and $x_1 \in [a, b]$ such that*

$$f(x_0) = \inf\big\{f(x) \,\big|\, x \in [a, b]\big\} \quad \textit{and} \quad f(x_1) = \sup\big\{f(x) \,\big|\, x \in [a, b]\big\}. \tag{4.3.1}$$

Proof. The Heine-Borel Theorem suggests that $[a, b]$ is a compact set in $\mathbb{R}$; thus Theorem 4.21 implies that $f([a, b])$ must be compact in $\mathbb{R}$. By the Heine-Borel Theorem again $f([a, b])$ is closed and bounded, so

$$\inf f([a, b]) \in f([a, b]) \quad \text{and} \quad \sup f([a, b]) \in f([a, b])$$

which further imply (4.3.1). ▫

Remark 4.23. If f attains its maximum (or minimum) on a set B, we use $\max\big\{f(x) \,\big|\, x \in B\big\}$ $\big($or $\min\big\{f(x) \,\big|\, x \in B\big\}\big)$ to denote $\sup\big\{f(x) \,\big|\, x \in B\big\}$ $\big($or $\inf\big\{f(x) \,\big|\, x \in B\big\}\big)$. Therefore, (4.3.1) can be rewritten as

$$f(x_0) = \min\big\{f(x) \,\big|\, x \in [a, b]\big\} \quad \text{and} \quad f(x_1) = \max\big\{f(x) \,\big|\, x \in [a, b]\big\}.$$

Example 4.24. Two norms $\|\cdot\|$ and $|||\cdot|||$ on a real vector space $\mathcal{V}$ are called equivalent if there are positive constants C_1 and C_2 such that

$$C_1\|x\| \leqslant |||x||| \leqslant C_2\|x\| \qquad \forall\, x \in \mathcal{V}.$$

We note that equivalent norms on a vector space $\mathcal{V}$ induce the same topology; that is, if $\|\cdot\|$ and $|||\cdot|||$ are equivalent norms on $\mathcal{V}$, then $\mathcal{U}$ is open in the normed space $(\mathcal{V}, \|\cdot\|)$ if and only if $\mathcal{U}$ is open in the normed space $(\mathcal{V}, |||\cdot|||)$. In fact, let $\mathcal{U}$ be an open set in $(\mathcal{V}, \|\cdot\|)$. Then for any $x \in \mathcal{U}$, there exists $r > 0$ such that

$$D_{\|\cdot\|}(x,r) \equiv \{y \in \mathcal{V} \,|\, \|x-y\| < r\} \subseteq \mathcal{U}\,.$$

Let $\delta = C_1 r$. Then if $z \in D_{|||\cdot|||}(x,\delta) \equiv \{y \in \mathcal{V} \,|\, |||x-y||| < \delta\}$,

$$\|x-z\| \leqslant \frac{1}{C_1}|||x-z||| < \frac{1}{C_1}\cdot C_1 r = r$$

which implies that $D_{|||\cdot|||}(x,\delta) \subseteq D_{\|\cdot\|}(x,r) \subseteq \mathcal{U}$. Therefore, $\mathcal{U}$ is open in $(\mathcal{V}, |||\cdot|||)$. Similarly, if $\mathcal{U}$ is open in $(\mathcal{V}, |||\cdot|||)$, then the inequality $|||x||| \leqslant C_2\|x\|$ suggests that $\mathcal{U}$ is open in $(\mathcal{V}, \|\cdot\|)$.
Claim: Any two norms on $\mathbb{R}^n$ are equivalent.
Proof of claim: It suffices to show that any norm $\|\cdot\|$ on $\mathbb{R}^n$ is equivalent to the two-norm $\|\cdot\|_2$ (check). Let $\{e_k\}_{k=1}^n$ be the standard basis of $\mathbb{R}^n$; that is,

$$e_k = (\underbrace{0,\cdots,0}_{(k-1)\text{ zeros}},1,0,\cdots,0)\,.$$

Every $x \in \mathbb{R}^n$ can be written as $x = \sum\limits_{i=1}^n x_i e_i$, and $\|x\|_2 = \sqrt{\sum\limits_{i=1}^n |x_i|^2}$. By the definition of norms and the Cauchy-Schwarz inequality,

$$\|x\| \leqslant \sum_{i=1}^n |x_i|\|e_i\| \leqslant \|x\|_2\sqrt{\sum_{i=1}^n \|e_i\|^2}\,; \tag{4.3.2}$$

thus letting $C_2 = \sqrt{\sum\limits_{i=1}^n \|e_i\|^2}$ we have $\|x\| \leqslant C_2\|x\|_2$.

On the other hand, define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \|x\| = \Big\|\sum_{i=1}^n x_i e_i\Big\|\,.$$

Because of (4.3.2), f is continuous on $\mathbb{R}^n$. In fact, for $x, y \in \mathbb{R}^n$,

$$\big|f(x)-f(y)\big| = \big|\|x\|-\|y\|\big| \leqslant \|x-y\| \leqslant C_2\|x-y\|_2$$

which guarantees the continuity of f on $\mathbb{R}^n$. Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \,|\, \|x\|_2 = 1\}$. Then $\mathbb{S}^{n-1}$ is a compact set in $(\mathbb{R}^n, \|\cdot\|_2)$ (since it is closed and bounded); thus by Theorem 4.21 f attains its minimum on $\mathbb{S}^{n-1}$ at some point $a = (a_1, \cdots, a_n)$. Moreover, $f(a) > 0$ (since if $f(a) = 0$, $a = 0 \notin \mathbb{S}^{n-1}$). Then for all $x \in \mathbb{R}^n$, $\dfrac{x}{\|x\|_2} \in \mathbb{S}^{n-1}$; thus

$$f\Big(\frac{x}{\|x\|_2}\Big) \geqslant f(a)\,.$$

The inequality above further implies that $f(a)\|x\|_2 \leqslant f(x) = \|x\|$; thus letting $C_1 = f(a)$ we have $C_1\|x\|_2 \leqslant \|x\|$.

Remark 4.25.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$. Then f is continuous. Note that $\{0\} \subseteq \mathbb{R}$ is compact ($\because$ closed and bounded), but $f^{-1}(\{0\}) = \mathbb{R}$ is not compact.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then f is continuous. Note that $C = \{1\}$ is connected, but $f^{-1}(C) = \{1, -1\}$ is not connected.

Remark 4.26.

1. If K is not compact, then Theorem 4.21 is not true. Consider the following counter example: $K = (0, 1), f : K \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$. Then $f(K)$ is unbounded.

2. If f is not continuous, then Theorem 4.21 is not true either.

   (a) Counter example 1: $f : K = [0, 1] \to \mathbb{R}$ defined by

   $$f(x, y) = \begin{cases} \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   Then $f(K)$ is unbounded $\Rightarrow \nexists\, x_1 \in K \ni f(x_1) = \sup f(K)$.

   (b) Counter example 2: $f : [0, 1] \to \mathbb{R}$ by

   $$f(x, y) = \begin{cases} x & \text{if } x \neq 1, \\ 0 & \text{if } x = 1. \end{cases}$$

   Then there is no $x_1 \in [0, 1]$ such that $f(x_1) = \sup\limits_{x\in[0,1]} f(x) = 1$.

Example 4.27 (An example show that x_0, x_1 in Theorem 4.21 are not unique)**.** Let $f : [-2, 2] \to \mathbb{R}$ be defined by $f(x) = (x^2 - 1)^2$.

1. Critical point: $f'(x) = 2(x^2 - 1) \cdot 2x = 0 \Leftrightarrow x = 0, \pm 1$.

2. Comparison: $f(0) = 1$, $f(1) = f(-1) = 0$, $f(2) = f(-2) = 9$. Then

$$f(2) = f(-2) = \sup_{x \in [-2,2]} f(x) \quad \text{and} \quad f(1) = f(-1) = \inf_{x \in [-2,2]} f(x)\,.$$

Corollary 4.28. *Let (M, d) be a metric space, $K \subseteq M$ be a compact set, and $f : K \to \mathbb{R}$ be continuous. Then the set*

$$\big\{x \in K \,\big|\, f(x) \text{ is the maximum of } f \text{ on } K\big\}$$

is a non-empty compact set.

Proof. Let $\mathrm{M} = \sup f(K)$. Then the set defined above is $f^{-1}(\{\mathrm{M}\})$, and

1. $f^{-1}(\{\mathrm{M}\})$ is non-empty by Theorem 4.21;

2. $f^{-1}(\{\mathrm{M}\})$ is closed since $\{\mathrm{M}\}$ is a closed set in $(\mathbb{R}, |\cdot|)$ and f is continuous on K.

Lemma 3.11 suggests that $f^{-1}(\{\mathrm{M}\})$ is compact. □

4.4 Images of Connected and Path Connected Sets under Continuous Maps

Definition 4.29. Let (M, d) be a metric space. A subset $A \subseteq M$ is said to be **path connected** if for every $x, y \in A$, there exists a continuous map $\varphi : [0, 1] \to A$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

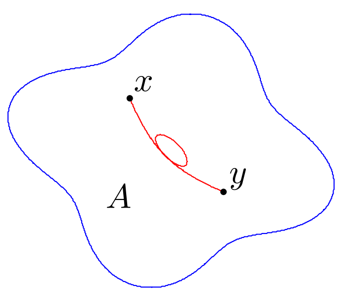


Figure 4.2: Path connected sets

Example 4.30. A set A in a vector space $\mathcal{V}$ is called **convex** if for all $x, y \in A$, the line segment joining x and y, denoted by $\overline{xy}$, lies in A. Then a convex set in a normed space is path connected. In fact, for $x, y \in A$, define $\varphi(t) = ty + (1 - t)x$. Then

1. $\varphi : [0,1] \to \overline{xy} \subseteq A$, $\varphi(0) = x$, $\varphi(1) = y$;

2. $\varphi : [0,1] \to A$ is continuous.

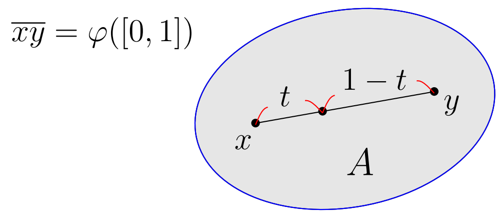


Figure 4.3: Convex sets

Example 4.31. A set S in a vector space $\mathcal{V}$ is called **star-shaped** if there exists $p \in S$ such that for any $q \in S$, the line segment joining p and q lies in S. A star-shaped set in a normed space is path connected. In fact, for $x, y \in S$, define

$$\varphi(t) = \begin{cases} 2tp + (1-2t)x & \text{if } t \in \left[0, \frac{1}{2}\right], \\ (2t-1)y + (2-2t)p & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then

1. $\varphi : [0,1] \to \overline{xp} \cup \overline{py} \subseteq S$, $\varphi(0) = x$, $\varphi(1) = y$;

2. $\varphi : [0,1] \to A$ is continuous.

Theorem 4.32. *Let (M,d) be a metric space, and $A \subseteq M$. If A is path connected, then A is connected.*

Proof. Assume the contrary that there are two open sets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that

1. $A \cap \mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$; 2. $A \cap \mathcal{V}_1 \neq \varnothing$; 3. $A \cap \mathcal{V}_2 \neq \varnothing$; 4. $A \subseteq \mathcal{V}_1 \cup \mathcal{V}_2$.

Since A is path connected, for $x \in A \cap \mathcal{V}_1$ and $y \in A \cap \mathcal{V}_2$, there exists $\varphi : [0,1] \to A$ such that $\varphi(0) = x$ and $\varphi(1) = y$. By Theorem 4.11, there exist $\mathcal{U}_1$ and $\mathcal{U}_2$ open in $(\mathbb{R}, |\cdot|)$ such that $\varphi^{-1}(\mathcal{V}_1) = \mathcal{U}_1 \cap [0,1]$ and $\varphi^{-1}(\mathcal{V}_2) = \mathcal{U}_2 \cap [0,1]$. Therefore,

$$[0,1] = \varphi^{-1}(A) \subseteq \varphi^{-1}(\mathcal{V}_1) \cup \varphi^{-1}(\mathcal{V}_2) \subseteq \mathcal{U}_1 \cup \mathcal{U}_2 .$$

Since $0 \in \mathcal{U}_1$, $1 \in \mathcal{U}_2$, and $[0,1] \cap \mathcal{U}_1 \cap \mathcal{U}_2 = \varphi^{-1}(A \cap \mathcal{V}_1 \cap \mathcal{V}_2) = \varnothing$, we conclude that $[0,1]$ is disconnected, a contradiction. □

Example 4.33. Let $A = \Big\{\Big(x, \sin\dfrac{1}{x}\Big) \,\Big|\, x \in (0,1]\Big\} \cup \big(\{0\} \times [-1,1]\big)$. Then A is connected in $(\mathbb{R}^2, \|\cdot\|_2)$, but A is not path connected.

To see this, we assume the contrary that A is path connected such that there is a continuous function $\varphi : [0,1] \to A$ such that $\varphi(0) = (x_0, y_0) \in \Big\{\Big(x, \sin\dfrac{1}{x}\Big) \,\Big|\, x \in (0,1)\Big\}$ and $\varphi(1) = (0,0) \in \{0\} \times [-1,1]$. Let $t_0 = \inf\big\{t \in [0,1] \,\big|\, \varphi(t) \in \{0\} \times [-1,1]\big\}$. In other words, at $t = t_0$ the path touches $0 \times [-1,1]$ for the "first time". By the continuity of φ, $\varphi(t_0) \in \{0\} \times [-1,1]$. Since $\varphi(0) \notin \{0\} \times [-1,1]$, $\varphi([0,t_0)) \subseteq \Big\{\Big(x, \sin\dfrac{1}{x}\Big) \,\Big|\, x \in (0,1)\Big\}$.

Suppose that $\varphi(t_0) = (0, \bar{y})$ for some $\bar{y} \in [-1,1]$, and $\varphi(t) = \Big(x(t), \sin\dfrac{1}{x(t)}\Big)$ for $0 \leqslant t < t_0$. By the continuity of φ, there exists $\delta > 0$ such that if $|t - t_0| < \delta$, $|\varphi(t) - \varphi(t_0)| < 1$. In particular,

$$x(t)^2 + \Big(\sin\frac{1}{x(t)} - \bar{y}\Big)^2 < 1 \qquad \forall\, t \in (t_0 - \delta, t)\,.$$

On the other hand, since φ is continuous, $x(t)$ is continuous on $[0,t_0)$; thus by the fact that $[0,t_0)$ is connected, $x([0,t_0))$ is connected. Therefore, $x([0,t_0)) = (0,\bar{x}]$ for some $\bar{x} > 0$. Since $\lim\limits_{t\to t_0} x(t) = 0$, there exists $\{t_n\}_{n=1}^{\infty} \in [0,t_0)$ such that $t_n \to t_0$ as $n \to \infty$ and $\Big|\sin\dfrac{1}{x(t_n)} - \bar{y}\Big| \geqslant 1$. For $n \gg 1$, $t_n \in (t_0 - \delta, t_0)$ but

$$x(t_n)^2 + \Big(\sin\frac{1}{x(t_n)} - \bar{y}\Big)^2 \geqslant 1\,,$$

a contradiction.

On the other hand, A is the closure of the connected set $B = \Big\{\Big(x, \sin\dfrac{1}{x}\Big) \,\Big|\, x \in (0,1)\Big\}$ (the connectedness of B follows from the fact that the function $\psi(x) = \Big(x, \sin\dfrac{1}{x}\Big)$ is continuous on the connected set $(0,1)$). Therefore, by Problem 9 of Exercise 8, $A = \bar{B}$ is connected.

Theorem 4.34. *Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be a continuous map.*

1. *If $C \subseteq A$ is connected, then $f(C)$ is connected in (N,ρ).*

2. *If $C \subseteq A$ is path connected, then $f(C)$ is path connected in (N,ρ).*

Proof. 1. Suppose that there are two open sets $\mathcal{V}_1$ and $\mathcal{V}_2$ in (N,ρ) such that

(a) $f(C) \cap \mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$; (b) $f(C) \cap \mathcal{V}_1 \neq \varnothing$; (c) $f(C) \cap \mathcal{V}_2 \neq \varnothing$; (d) $f(C) \subseteq \mathcal{V}_1 \cup \mathcal{V}_2$.

By Theorem 4.11, there are $\mathcal{U}_1$ and $\mathcal{U}_2$ open in (M,d) such that $f^{-1}(\mathcal{V}_1) = \mathcal{U}_1 \cap A$ and $f^{-1}(\mathcal{V}_2) = \mathcal{U}_2 \cap A$. By (d),

$$C \subseteq f^{-1}(f(C)) \subseteq f^{-1}(\mathcal{V}_1) \cup f^{-1}(\mathcal{V}_2) = (\mathcal{U}_1 \cup \mathcal{U}_2) \cap A \subseteq \mathcal{U}_1 \cup \mathcal{U}_2 .$$

Moreover, by (a) we find that

$$\begin{aligned} C \cap \mathcal{U}_1 \cap \mathcal{U}_2 &= C \cap (\mathcal{U}_1 \cap A) \cap (\mathcal{U}_2 \cap A) = C \cap f^{-1}(\mathcal{V}_1) \cap f^{-1}(\mathcal{V}_2) \\ &\subseteq f^{-1}(f(C) \cap \mathcal{V}_1 \cap \mathcal{V}_2) = \varnothing \end{aligned}$$

which implies $C \cap \mathcal{U}_1 \cap \mathcal{U}_2 = \varnothing$. Finally, (b) implies that for some $x \in C$, $f(x) \in \mathcal{V}_1$. Therefore, $x \in f^{-1}(\mathcal{V}_1) = \mathcal{U}_1 \cap A$ which suggests that $x \in \mathcal{U}_1$; thus $C \cap \mathcal{U}_1 \neq \varnothing$. Similarly, $C \cap \mathcal{U}_2 \neq \varnothing$. Therefore, C is disconnected which is a contradiction.

2. Let $y_1, y_2 \in f(C)$. Then $\exists\, x_1, x_2 \in C$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since C is path connected, $\exists\, r : [0,1] \to C$ such that r is continuous on $[0,1]$ and $r(0) = x_1$ and $r(1) = x_2$. Let $\varphi : [0,1] \to f(C)$ be defined by $\varphi = f \circ r$. By Corollary 4.20 φ is continuous on $[0,1]$, and $\varphi(0) = y_1$ and $\varphi(1) = y_2$. □

Corollary 4.35 (The Intermediate Value Theorem（中間值定理）). *Let $f : [a,b] \to \mathbb{R}$ be continuous. If $f(a) \neq f(b)$, then for all d in between $f(a)$ and $f(b)$, there exists $c \in (a,b)$ such that $f(c) = d$.*

Proof. The closed interval $[a,b]$ is connected by Theorem 3.38, so Theorem 4.34 implies that $f([a,b])$ must be connected in $\mathbb{R}$. By Theorem 3.38 again, if d is in between $f(a)$ and $f(b)$, then d belongs to $f([a,b])$. Therefore, for some $c \in (a,b)$ we have $f(c) = d$. □

Example 4.36. Let $f : [0,1] \to [0,1]$ be continuous. Then $\exists\, x_0 \in [0,1] \ni f(x_0) = x_0$.

Proof. Let $g(x) = x - f(x)$. Then

1. $g(0) = 0$ or $g(1) = 0 \Rightarrow x_0 = 0$ or 1.

2. $g(0) \neq 0$ or $g(1) \neq 0 \Rightarrow g(0) < 0$ and $g(1) > 0$. Since $g : [0,1] \to \mathbb{R}$ is continuous,

   $\exists\, x_0 \in [0,1] \ni g(x_0) = 0 \Rightarrow \exists\, x_0 \in (0,1) \ni f(x_0) = x_0$. □

Remark 4.37. Such an x_0 in Example 4.36 is called a **fixed-point** of f.

Example 4.38. Let $f : [1,2] \to [0,3]$ be continuous, and $f(1) = 0$ and $f(2) = 3$. Then $\exists\, x_0 \in [1,2] \ni f(x_0) = x_0$.

Proof. Let $g(x) = x - f(x)$. Then $g : [1,2] \to \mathbb{R}$ is continuous. Moreover,

$$g(1) = 1 - f(1) = 1, \quad g(2) = 2 - f(2) = -1\,;$$

thus $\exists\, x_0 \in (1,2) \ni g(x_0) = 0$. □

Example 4.39. Let p be a cubic polynomial; that is, $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ for some $a_0, a_1, a_2 \in \mathbb{R}$ and $a_3 \neq 0$. Then p has a real root x_0 (that is, $\exists\, x_0 \in \mathbb{R}$ such that $p(x_0) = 0$).

Proof. Note that p is obviously continuous and $\mathbb{R}$ is connected. Write

$$p(x) = a_3x^3\big(1 + \frac{a_2}{a_3x} + \frac{a_1}{a_3x^2} + \frac{a_0}{a_3x^3}\big)\,.$$

Now $\lim\limits_{x\to\pm\infty} \dfrac{\alpha}{\beta x^n} = 0$ if $n > 0$ and $\beta \neq 0$, so

$$\lim_{x\to\pm\infty} \big(1 + \frac{a_2}{a_3x} + \frac{a_1}{a_3x^2} + \frac{a_0}{a_3x^3}\big) = 1\,.$$

Moreover,

$$\lim_{x\to\infty} ax^3 = \begin{cases} \infty & \text{if } a > 0, \\ -\infty & \text{if } a < 0. \end{cases}$$

Suppose that $a > 0$. Then $\lim\limits_{x\to\infty} ax^3 = \infty$ and $\lim\limits_{x\to-\infty} ax^3 = -\infty \Rightarrow \exists\, x, y \in \mathbb{R} \ni p(x) < 0 < p(y)$. By Corollary 4.35 $\exists\, r \in \mathbb{R} \ni p(r) = 0$. The case that $a < 0$ is similar. □

4.5 Uniform Continuity（均勻連續）

Definition 4.40. Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be a map. For a set $B \subseteq A$, f is said to be **uniformly continuous on B** if for any two sequences $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \subseteq B$ with the property that $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$, one has $\lim\limits_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) = 0$.

Proposition 4.41. *Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be a map. If f is uniformly continuous on A, then f is continuous on A.*

Proof. Let $x_0 \in A \cap A'$, and $\{x_k\}_{k=1}^\infty \subseteq A$ be a sequence such that $x_k \to x_0$ as $k \to \infty$. Let $\{y_k\}_{k=1}^\infty$ be a constant sequence with value x_0; that is, $y_k = x_0$ for all $k \in \mathbb{N}$. Then $\{y_k\}_{k=1}^\infty \subseteq A$ and $d(x_k, y_k) \to 0$ as $k \to \infty$. By the uniform continuity of f on A,

$$\lim_{k\to\infty} \rho\big(f(x_k), f(x_0)\big) = \lim_{k\to\infty} \rho\big(f(x_k), f(y_k)\big) = 0$$

which implies that f is continuous on x_0. □

Example 4.42. Let $f : [0,1] \to \mathbb{R}$ be the Dirichlet function; that is,

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}\,, \\ 1 & \text{if } x \in \mathbb{Q}^\complement. \end{cases}$$

and $B = \mathbb{Q} \cap [0,1]$. Then f is continuous **nowhere** in $[0,1]$, but f is uniformly continuous on B. However, the proposition above guarantees that if f is uniformly continuous on A, then f must be continuous on A (Check why the proof of Proposition 4.41 does not go through if B is a proper subset of A).

Example 4.43. The function $f(x) = |x|$ is uniformly continuous on $\mathbb{R}$.

Proof. By the triangle inequality,

$$\big|f(x) - f(y)\big| = \big||x| - |y|\big| \leqslant |x - y|\,;$$

thus if $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are sequences in $\mathbb{R}$ and $\lim\limits_{n\to\infty} |x_n - y_n| = 0$, by the Sandwich lemma we must have $\lim\limits_{n\to\infty} \big|f(x_n) - f(y_n)\big| = 0$. □

Example 4.44. The function $f : (0,\infty) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$ is uniformly continuous on $[a,\infty)$ for all $a > 0$. However, it is not uniformly continuous on $(0,\infty)$.

Proof. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be sequences in $[a,\infty)$ such that $\lim\limits_{n\to\infty} |x_n - y_n| = 0$. Then

$$|f(x_n) - f(y_n)| = \Big|\frac{1}{x_n} - \frac{1}{y_n}\Big| = \frac{|x_n - y_n|}{|x_n y_n|} \leqslant \frac{|x_n - y_n|}{a^2} \to 0 \quad \text{as} \quad n \to \infty$$

which implies that f is uniformly continuous on $[a,\infty)$ if $a > 0$. However, by choosing $x_n = \dfrac{1}{n}$ and $y_n = \dfrac{1}{2n}$, we find that

$$|x_n - y_n| = \frac{1}{2n} \quad \text{but} \quad |f(x_n) - f(y_n)| = n \geqslant 1\,;$$

thus f cannot be uniformly continuous on $(0,\infty)$. □

Remark 4.45. Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f : B \subseteq A \to N$ be a map. Then the following four statements are equivalent:

(1) f is **not** uniformly continuous on B.

(2) $\exists \{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \subseteq B \ni \lim\limits_{n\to\infty} d(x_n, y_n) = 0$ and $\limsup\limits_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) > 0$.

(3) $\exists \{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \subseteq B \ni \lim\limits_{n\to\infty} d(x_n, y_n) = 0$ and $\lim\limits_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) > 0$.

(4) $\exists \varepsilon > 0 \ni \forall n > 0, \exists x_n, y_n \in B$ and $d(x_n, y_n) < \dfrac{1}{n} \ni \rho\big(f(x_n), f(y_n)\big) \geqslant \varepsilon$.

Example 4.46. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Then f is continuous in $\mathbb{R}$ but not uniformly continuous on $\mathbb{R}$. Let $\varepsilon = 1$, $x_n = n$, and $y_n = n + \dfrac{1}{2n}$,

$$\big|f(x_n) - f(y_n)\big| = \big|n^2 - (n + \frac{1}{2n})^2\big| = \big|n^2 - n^2 - 1 - \frac{1}{4n^2}\big| > 1 \quad \forall\, n > 0\,.$$

Example 4.47. The function $f(x) = \sin(x^2)$ is not uniform continuous on $\mathbb{R}$.

Proof. Let $\varepsilon = 1$, $x_n = 2n\sqrt{\pi} + \dfrac{\sqrt{\pi}}{8n}$ and $y_n = 2n\sqrt{\pi} - \dfrac{\sqrt{\pi}}{8n}$. Then

$$\big|\sin(x_n^2) - \sin(y_n^2)\big| = \Big|\sin\big(4n^2\pi + \frac{\pi}{2} + \frac{\pi}{64n^2}\big) - \sin\big(4n^2\pi - \frac{\pi}{2} + \frac{\pi}{64n^2}\big)\Big| = 2\cos\frac{\pi}{64n^2}\,;$$

thus if n is large enough, $\big|\sin(x_n^2) - \sin(y_n^2)\big| \geqslant 1$. ▫

Example 4.48. The function $f : (0,1) \to \mathbb{R}$ defined by $f(x) = \sin\dfrac{1}{x}$ is not uniformly continuous.

Proof. Let $\varepsilon = 1$, $x_n = \big(2n\pi + \frac{\pi}{2}\big)^{-1}$ and $y_n = \big(2n\pi - \frac{\pi}{2}\big)^{-1}$. Then

$$\big|\sin\frac{1}{x_n} - \sin\frac{1}{y_n}\big| = 2\,,$$

while $|x_n - y_n| = \dfrac{\pi}{4n^2\pi^2 - \frac{\pi^2}{4}} = \dfrac{1}{(4n^2 - \frac{1}{4})\pi} \leqslant \dfrac{1}{n}$ for all $n \in \mathbb{N}$. ▫

Theorem 4.49. *Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be a map. For a set $B \subseteq A$, f is uniformly continuous on B if and only if*

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni \rho\big(f(x), f(y)\big) < \varepsilon \text{ whenever } d(x,y) < \delta \text{ and } x, y \in B\,.$$

Proof. "$\Leftarrow$" Suppose the contrary that f is not uniformly continuous on B. Then there are two sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ in B such that

$$\lim_{k\to\infty} d(x_n, y_n) = 0 \quad \text{but} \quad \limsup_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) > 0\,.$$

Let $\varepsilon = \dfrac{1}{2}\limsup\limits_{n\to\infty} \rho\big(f(x_n), f(y_n)\big)$. Then by the definition of the limit and the limit superior (or Proposition 1.121) we conclude that there exist subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ and $\{y_{n_k}\}_{k=1}^{\infty}$ such that

$$\rho\big(f(x_{n_k}), f(y_{n_k})\big) \geqslant \limsup_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) - \varepsilon = \varepsilon > 0$$

while $\lim\limits_{k\to\infty} d(x_{n_k}, y_{n_k}) = 0$, a contradiction.

"$\Rightarrow$" Suppose the contrary that there exists $\varepsilon > 0$ such that for all $\delta = \dfrac{1}{n} > 0$, there exist two points x_n and $y_n \in B$ such that

$$d(x_n, y_n) < \frac{1}{n} \quad \text{but} \quad \rho\big(f(x_n), f(y_n)\big) \geqslant \varepsilon\,.$$

These points form two sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ in B such that $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$, while the limit of $\rho\big(f(x_n), f(y_n)\big)$, if exists, does not converges to zero as $n \to \infty$. As a consequence, f is not uniformly continuous on B, a contradiction. ▫

Remark 4.50. The theorem above provides another way (the blue color part) of defining the uniform continuity of a function over a subset of its domain. Moreover, according to this alternative definition, if $f : A \to N$ is uniformly continuous on $B \subseteq A$, then

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni \forall\, b \in M, f\Big(D\big(b, \frac{\delta}{2}\big) \cap B\Big) \subseteq D\big(c, \frac{\varepsilon}{2}\big) \text{ for some } c \in N\,;$$

that is, the diameter of the image, under f, of subsets of B whose diameter is not greater than δ is not greater than ε（在 B 中直徑不超過 δ 的子集合被函數 f 映過去之後，在對應域中的直徑不會超過 ϵ）.

Remark 4.51. In terms of the number $\delta(f, x, \varepsilon)$ defined in Remark 4.9, the uniform continuity of a function $f : A \to N$ is equivalent to that

$$\delta_f(\varepsilon) \equiv \inf_{x\in A} \delta(f, x, \varepsilon) > 0 \qquad \forall\, \varepsilon > 0\,.$$

The function $\delta_f(\cdot)$ is <u>**the inverse of the modulus of continuity of (a uniform continuous) function f**</u>.

Theorem 4.52. *Let (M, d) and (N, ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be a map. If $K \subseteq A$ is compact and f is continuous on K, then f is uniformly continuous on K.*

Proof. Let $\varepsilon > 0$ be given. Since f is continuous on K,

$$\forall\, a \in K, \exists\, \delta = \delta(a) > 0 \ni \rho\big(f(x), f(a)\big) < \frac{\varepsilon}{2} \text{ whenever } x \in D(a, \delta) \cap A\,.$$

Then $\Big\{D\big(a, \frac{\delta(a)}{2}\big)\Big\}_{a \in K}$ is an open cover of K; thus

$$\exists\, \{a_1, \cdots, a_N\} \subseteq K \ni K \subseteq \bigcup_{i=1}^{N} D\big(a_i, \frac{\delta_i}{2}\big)\,,$$

where $\delta_i = \delta(a_i)$. Let $\delta = \frac{1}{2}\min\{\delta_1, \cdots, \delta_N\}$. Then $\delta > 0$, and if $x_1, x_2 \in K$ and $d(x_1, x_2) < \delta$, there must be $j = 1, \cdots, N$ such that $x_1, x_2 \in B(a_j, \delta_j)$. In fact, since $x_1 \in D\big(a_j, \frac{\delta_j}{2}\big)$ for some $j = 1, \cdots, N$, then

$$d(x_2, a_j) \leqslant d(x_1, x_2) + d(x_1, a_j) < \delta + \frac{\delta_j}{2} < \delta_j\,.$$

Therefore, $x_1, x_2 \in D(a_j, \delta_j) \cap A$ for some $j = 1, \cdots, N$; thus

$$\rho\big(f(x_1), f(x_2)\big) \leqslant \rho\big(f(x_1), f(a_j)\big) + \rho\big(f(x_2), f(a_j)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,. \qquad \square$$

Alternative proof. Assume the contrary that f is not uniformly continuous on K. Then ((3) of Remark 4.45 implies that) there are sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in K such that

$$\lim_{n\to\infty} d(x_n, y_n) = 0 \quad \text{but} \quad \lim_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) > 0\,.$$

Since K is (sequentially) compact, there exist convergent subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ and $\{y_{n_k}\}_{k=1}^{\infty}$ with limits $x, y \in K$. On the other hand, $\lim_{n\to\infty} d(x_n, y_n) = 0$, we must have $x = y$; thus by the continuity of f (on K),

$$0 = \rho\big(f(x), f(x)\big) = \lim_{k\to\infty} \rho\big(f(x_{n_k}), f(y_{n_k})\big) = \lim_{n\to\infty} \rho\big(f(x_n), f(y_n)\big) > 0\,,$$

a contradiction. $\square$

Lemma 4.53. *Let (M, d) and (N, ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be uniformly continuous. If $\{x_k\}_{k=1}^{\infty} \subseteq A$ is a Cauchy sequence, so is $\big\{f(x_k)\big\}_{k=1}^{\infty}$.*

Proof. Let $\{x_k\}_{k=1}^{\infty}$ be a Cauchy sequence in (M,d), and $\varepsilon > 0$ be given. Since $f : A \to N$ is uniformly continuous,

$$\exists\, \delta > 0 \ni \rho\big(f(x), f(y)\big) < \varepsilon \text{ whenever } d(x,y) < \delta \text{ and } x, y \in A\,.$$

For this particular δ, $\exists\, N > 0 \ni d(x_k, x_\ell) < \delta$ if $k, \ell \geqslant N$. Therefore,

$$\rho\big(f(x_k), f(x_\ell)\big) < \varepsilon \text{ if } k, \ell \geqslant N\,.$$

□

Corollary 4.54. *Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f : A \to N$ be uniformly continuous. If N is complete, then f has a unique extension to a continuous function on $\bar{A}$; that is, $\exists\, g : \bar{A} \to N$ such that*

(1) *g is uniformly continuous on $\bar{A}$;*

(2) *$g(x) = f(x)$ for all $x \in A$;*

(3) *if $h : \bar{A} \to N$ is a continuous map satisfying (1) and (2), then $h = g$.*

Proof. Let $x \in \bar{A}\backslash A$. Then $\exists\, \{x_k\}_{k=1}^{\infty} \subseteq A$ such that $x_k \to x$ as $k \to \infty$. Since $\{x_k\}_{k=1}^{\infty}$ is Cauchy, by Lemma 4.53 $\big\{f(x_k)\big\}_{k=1}^{\infty}$ is a Cauchy sequence in (N,ρ); thus is convergent. Moreover, if $\{z_k\}_{k=1}^{\infty} \subseteq A$ is another sequence converging to x, we must have $d(x_k, z_k) \to 0$ as $k \to \infty$; thus $\rho(f(x_k), f(z_k)) \to 0$ as $k \to \infty$, so the limit of $\big\{f(x_k)\big\}_{k=1}^{\infty}$ and $\big\{f(z_k)\big\}_{k=1}^{\infty}$ must be the same.

Define $g : \bar{A} \to N$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in A\,, \\ \lim\limits_{k\to\infty} f(x_k) & \text{if } x \in \bar{A}\backslash A, \text{ and } \{x_k\}_{k=1}^{\infty} \subseteq A \text{ converging to } x \text{ as } k \to \infty\,. \end{cases}$$

Then the argument above shows that g is well-defined, and (2), (3) hold.

Let $\varepsilon > 0$ be given. Since $f : A \to N$ is uniformly continuous,

$$\exists\, \delta > 0 \ni \rho\big(f(x), f(y)\big) < \frac{\varepsilon}{3} \text{ whenever } d(x,y) < 2\delta \text{ and } x, y \in A\,.$$

Suppose that $x, y \in \bar{A}$ such that $d(x,y) < \delta$. Let $\{x_k\}_{k=1}^{\infty}, \{y_k\}_{k=1}^{\infty} \subseteq A$ be sequences converging to x and y, respectively. Then $\exists\, N > 0$ such that

$$d(x_k, x) < \frac{\delta}{2}\,, d(y_k, y) < \frac{\delta}{2} \text{ and } \rho\big(f(x_k), g(x)\big) < \frac{\varepsilon}{3}\,, \rho\big(f(y_k), g(y)\big) < \frac{\varepsilon}{3} \quad \forall\, k \geqslant N\,.$$

In particular, due to the triangle inequality,

$$d(x_N, y_N) \leqslant d(x_N, x) + d(x, y) + d(y, y_N) < \frac{\delta}{2} + \delta + \frac{\delta}{2} = 2\delta\,;$$

thus $\rho\big(f(x_N), f(y_N)\big) < \frac{\varepsilon}{3}$. As a consequence,

$$\rho\big(g(x), g(y)\big) \leqslant \rho\big(g(x), f(x_N)\big) + \rho\big(f(x_N), f(y_N)\big) + \rho\big(f(y_N), f(y)\big) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,. \qquad \square$$

4.6 Differentiation of Functions of One Variable

Definition 4.55. A function $f : (a, b) \to \mathbb{R}$ is said to be **differentiable at x_0** if there exists a number m such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - m(x - x_0)}{x - x_0} = 0\,.$$

The (unique) number m is usually denoted by $f'(x_0)$, and is called the **derivative** of f at x_0.

Remark 4.56. The derivative of f at x_0 can be computed by

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}\,.$$

Remark 4.57. By the definition of the limit of functions, $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$ if and only if there exists $m \in \mathbb{R}$, denoted by $f'(x_0)$, such that

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni \big|f(x) - f(x_0) - f'(x_0)(x - x_0)\big| \leqslant \varepsilon|x - x_0| \text{ if } |x - x_0| < \delta\,.$$

Definition 4.58. A function $f : (a, b) \to \mathbb{R}$ is said to be **differentiable** (on (a, b)) if f is differentiable at each $x_0 \in (a, b)$.

Proposition 4.59. *Suppose that a function $f : (a, b) \to \mathbb{R}$ is differentiable at x_0. Then f is continuous at x_0.*

Proof. For $x \neq x_0$, $f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)$; thus Proposition 4.15 implies that

$$\lim_{x \to x_0} \big(f(x) - f(x_0)\big) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) = f'(x_0) \cdot 0 = 0\,. \qquad \square$$

Theorem 4.60. *Suppose that functions $f, g : (a, b) \to \mathbb{R}$ are differentiable at x_0, and $k \in \mathbb{R}$ is a constant. Then*

1. $(kf)'(x_0) = kf'(x_0)$.
2. $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$.
3. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.
4. $\left(\dfrac{f}{g}\right)'(x_0) = \dfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$ *if* $g(x_0) \neq 0$.

Theorem 4.61 (Chain Rule)**.** *Suppose that a function* $f : (a,b) \to \mathbb{R}$ *is differentiable at* x_0*, and* $g : (c,d) \to \mathbb{R}$ *is differentiable at* $y_0 = f(x_0) \in (c,d)$*. Then* $g \circ f$ *is differentiable at* x_0*, and*

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)\,.$$

Proof. Let $\varepsilon > 0$ be given. Since $f : (a,b) \to \mathbb{R}$ is differentiable at x_0 and $g : (c,d) \to \mathbb{R}$ is differentiable at $y_0 = f(x_0)$,

$$\exists\, \delta_1 > 0 \ni \big|f(x) - f(x_0) - f'(x_0)(x - x_0)\big| \leqslant \min\Big\{1, \frac{\varepsilon}{2(1 + |g'(y_0)|)}\Big\}|x - x_0| \text{ if } |x - x_0| < \delta_1$$

and

$$\exists\, \delta_2 > 0 \ni \big|g(y) - g(y_0) - g'(y_0)(y - y_0)\big| \leqslant \frac{\varepsilon|y - y_0|}{2(1 + |f'(x_0)|)} \text{ if } |y - y_0| < \delta_2\,.$$

Moreover, by Proposition 4.59 f is continuous at x_0; thus

$$\exists\, \delta_3 > 0 \ni \big|f(x) - f(x_0)\big| < \delta_2 \text{ if } |x - x_0| < \delta_3 \text{ and } x \in (a,b)\,.$$

Let $\delta = \min\{\delta_1, \delta_3\}$, and denote $f(x)$ by y. Then if $|x - x_0| < \delta$, we have $|y - y_0| < \delta_2$ and

$$\begin{aligned}
&\big|(g \circ f)(x) - (g \circ f)(x_0) - g'(y_0)f'(x_0)(x - x_0)\big| = \big|g(y) - g(y_0) - g'(y_0)f'(x_0)(x - x_0)\big| \\
&\quad = \big|g(y) - g(y_0) - g'(y_0)(y - y_0) + g'(y_0)(f(x) - f(x_0) - f'(x_0)(x - x_0)\big| \\
&\quad \leqslant \frac{\varepsilon\big|f(x) - f(x_0)\big|}{2(1 + |f'(x_0)|)} + \big|g'(y_0)\big|\frac{\varepsilon|x - x_0|}{2(1 + |g'(y_0)|)} \\
&\quad \leqslant \frac{\varepsilon}{2(1 + |f'(x_0)|)}\big(|x - x_0| + |f'(x_0)||x - x_0|\big) + \frac{\varepsilon}{2}|x - x_0| = \varepsilon|x - x_0|\,.
\end{aligned}$$

By Remark 4.57, $g \circ f$ is differentiable at x_0 with derivative $g'(f(x_0))f'(x_0)$. □

Proposition 4.62. *If* $f : (a,b) \to \mathbb{R}$ *is differentiable at* $x_0 \in (a,b)$ *and* f *attains a local minimum or maximum at* x_0*, then* $f'(x_0) = 0$.

Proof. W.L.O.G. we assume that f attains its local minimum at x_0. Then $f(x) - f(x_0) \geqslant 0$ for all $x \in I$, where I is an open interval containing x_0. Therefore,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \leqslant 0$$

and

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \geqslant 0 \,.$$

As a consequence, $f'(x_0) = 0$. □

Theorem 4.63 (Rolle)**.** *Suppose that a function $f : [a, b] \to \mathbb{R}$ is continuous, and is differentiable on (a, b). If $f(a) = f(b)$, then $\exists\, c \in (a, b)$ such that $f'(c) = 0$.*

Proof. By the Extreme Value Theorem, there exists x_0 and x_1 in $[a, b]$ such that

$$f(x_0) = \min f([a, b]) \quad \text{and} \quad f(x_1) = \max f([a, b]) \,.$$

Case 1. $f(x_0) = f(x_1)$, then f is constant on $[a, b]$; thus $f'(x) = 0$ for all $x \in (a, b)$.

Case 2. One of $f(x_0)$ and $f(x_1)$ is different from $f(a)$. W.L.O.G. we may assume that $f(x_0) \neq f(a)$. Then $x_0 \in (a, b)$, and f attains its global minimum at x_0. By Proposition

4.62, $f'(x_0) = 0$. □

Theorem 4.64 (Cauchy's Mean Value Theorem)**.** *Suppose that functions $f, g : [a, b] \to \mathbb{R}$ are continuous, and $f, g : (a, b) \to \mathbb{R}$ are differentiable. If $g(a) \neq g(b)$ and $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists $c \in (a, b)$ such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \,.$$

Proof. Consider the function

$$h(x) \equiv \big(f(x) - f(a)\big)\big(g(b) - g(a)\big) - \big(f(b) - f(a)\big)\big(g(x) - g(a)\big) \,.$$

Then $h : [a, b] \to \mathbb{R}$ is continuous, and is differentiable on (a, b). Moreover, $h(b) = h(a) = 0$. By Rolle's theorem, there exists $c \in (a, b)$ such that

$$h'(c) = f'(c)\big(g(b) - g(a)\big) - \big(f(b) - f(a)\big)g'(c) = 0 \,.$$ □

Corollary 4.65 (Mean Value Theorem)**.** *Suppose that a function $f:[a,b]\to\mathbb{R}$ is continuous, and $f:(a,b)\to\mathbb{R}$ is differentiable. Then there exists $c\in(a,b)$ such that*

$$f'(c)=\frac{f(b)-f(a)}{b-a}\,.$$

Proof. Apply the Cauchy Mean Value Theorem for the case that $g(x)=x$. ▫

Corollary 4.66. *Suppose that a function $f:[a,b]\to\mathbb{R}$ is continuous and $f'(x)=0$ for all $x\in(a,b)$. Then f is constant.*

Proof. Let $x\in(a,b)$ be given. By Mean Value Theorem, there exists $c\in(a,x)$ such that

$$f(x)-f(a)=f'(c)(x-a)=0\,.$$

Therefore, $f(x)=f(a)$; thus for all $x\in(a,b)$, $f(x)=f(a)$. Now by continuity, $f(b)=\lim\limits_{x\to b^-}f(x)=f(a)$. ▫

Corollary 4.67 (L'Hôspital's rule)**.** *Let $f,g:(a,b)\to\mathbb{R}$ be differentiable functions. Suppose that for some $x_0\in(a,b)$, $f(x_0)=g(x_0)=0$, $g'(x)\neq 0$ for all $x\neq x_0$, and the limit $\lim\limits_{x\to x_0}\frac{f'(x)}{g'(x)}$ exists. Then the limit $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}$ also exists, and*

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}\,.$$

Proof. We first note that $g(x)\neq g(x_0)$ for all $x\neq x_0$ since if not, the Mean Value Theorem implies that the existence of c in between x and x_0 such that $g'(c)=0$ which contradicts to the condition that $g'(x)\neq 0$ for all $x\neq x_0$. By Cauchy's Mean Value Theorem, for all $x\in(a,b)$ and $x\neq x_0$, there exists $\xi=\xi(x)$ in between x and x_0 such that

$$\frac{f(x)}{g(x)}=\frac{f(x)-f(x_0)}{g(x)-g(x_0)}=\frac{f'(\xi)}{g'(\xi)}$$

Since $\xi\to x_0$ as $x\to x_0$, we have

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{\xi\to x_0}\frac{f'(\xi)}{g'(\xi)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}\,.$$ ▫

Theorem 4.68 (Taylor)**.** *Suppose that for some $k\in\mathbb{N}$, $f:(a,b)\to\mathbb{R}$ be $(k+1)$-times differentiable and $c\in(a,b)$. Then for all $x\in(a,b)$, there exists d in between c and x such that*

$$f(x)=\sum_{j=0}^{k}\frac{f^{(j)}(c)}{j!}(x-c)^j+\frac{f^{(k+1)}(d)}{(k+1)!}(x-c)^{(k+1)},$$

where $f^{(j)}$ denotes the j-th derivative of f.

Proof. Let $g(x) = f(x) - \sum\limits_{j=0}^{k} \frac{f^{(j)}(c)}{j!}(x-c)^j$, and $h(x) = (x-c)^{k+1}$. Then for $1 \leqslant j \leqslant k$,

$$g^{(j)}(c) = h^{(j)}(c) = 0\,;$$

thus by the Cauchy mean value theorem (Theorem 4.64), there exists ξ_1 in between x and c, ξ_2 in between ξ_1 and c, $\cdots$, ξ_{k+1} in between ξ_k and c such that

$$\begin{aligned}\frac{g(x)}{h(x)} &= \frac{g(x)-g(c)}{h(x)-h(c)} = \frac{g'(\xi_1)}{h'(\xi_1)} = \frac{g'(\xi_1)-g'(c)}{h'(\xi_1)-h'(c)} = \frac{g''(\xi_2)}{h''(\xi_2)} = \cdots \\ &= \frac{g^{(k)}(\xi_k)}{h^{(k)}(\xi_k)} = \frac{g^{(k)}(\xi_k)-g^{(k)}(c)}{h^{(k)}(\xi_k)-h^{(k)}(c)} = \frac{g^{(k+1)}(\xi_{k+1})}{h^{(k+1)}(\xi_k)} = \frac{f^{(k+1)}(\xi_{k+1})}{(k+1)!}\,.\end{aligned}$$

Letting $d = \xi_{k+1}$ we conclude the theorem. ▫

Example 4.69. A function $f : [a,b] \to \mathbb{R}$ is said to be **Lipschitz continuous** if $\exists\, M > 0$ such that

$$|f(x_1) - f(x_2)| \leqslant M|x_1 - x_2| \qquad \forall\, x_1, x_2 \in [a,b]\,.$$

If the derivative of a differentiable function $f : (a,b) \to \mathbb{R}$ is bounded; that is, $\exists\, M > 0$ $\ni |f'(x)| \leqslant M$ for all $x \in (a,b)$, then the Mean Value Theorem implies that f is Lipschitz continuous. A Lipschitz continuous function must be uniformly continuous.

Definition 4.70. A function $f : (a,b) \to \mathbb{R}$ is said to be **increasing** / **decreasing** / **strictly increasing** / **strictly decreasing** (on (a,b)) if $f(x_1) \;\leqslant\; / \;\geqslant\; / \;<\; / \;>\; f(x_2)$ if $a < x_1 < x_2 < b$. f is said to be **monotone** if f is either increasing or decreasing on (a,b), and **strictly monotone** if f is either strictly increasing or strictly decreasing.

Theorem 4.71. *Suppose that $f : (a,b) \to \mathbb{R}$ is differentiable.*

1. *f is increasing on (a,b) if and only if $f'(x) \geqslant 0$ for all $x \in (a,b)$.*

2. *f is decreasing on (a,b) if and only if $f'(x) \leqslant 0$ for all $x \in (a,b)$.*

3. *If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing.*

4. *If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing.*

Theorem 4.72 (Inverse Function Theorem)**.** *Let $f : (a, b) \to \mathbb{R}$ be differentiable, and f' be sign-definite; that is, $f'(x) > 0$ for all $x \in (a, b)$ or $f'(x) < 0$ for all $x \in (a, b)$. Then $f : (a, b) \to f((a, b))$ is a bijection, and f^{-1}, the inverse function of f, is differentiable on $f((a, b))$, and*

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)} \qquad \forall\, x \in (a, b)\,. \tag{4.6.1}$$

Proof. W.L.O.G. we assume that $f'(x) > 0$ for all $x \in (a, b)$. By Theorem 4.71 f is strictly increasing; thus f^{-1} exists.
Claim: $f^{-1} : f((a, b)) \to (a, b)$ is continuous.
Proof of claim: Let $y_0 = f(x_0) \in f((a, b))$, and $\varepsilon > 0$ be given. Then $f((x_0 - \varepsilon, x_0 + \varepsilon)) = \big(f(x_0 - \varepsilon), f(x_0 + \varepsilon)\big)$ since f is continuous on (a, b) and $(x_0 - \varepsilon, x_0 + \varepsilon)$ is connected. Let $\delta = \min\{f(x_0) - f(x_0 - \varepsilon), f(x_0 + \varepsilon) - f(x_0)\big\}$. Then $\delta > 0$, and

$$(y_0 - \delta, y_0 + \delta) = \big(f(x_0) - \delta, f(x_0) + \delta\big) \subseteq f((x_0 - \varepsilon, x_0 + \varepsilon))\,;$$

thus by the injectivity of f,

$$f^{-1}((y_0 - \delta, y_0 + \delta)) \subseteq f^{-1}(f((x_0 - \varepsilon, x_0 + \varepsilon))) = (x_0 - \varepsilon, x_0 + \varepsilon) = (f^{-1}(y_0) - \varepsilon, f^{-1}(y_0) + \varepsilon)\,.$$

The inclusion above implies that f^{-1} is continuous at y_0.

Writing $y = f(x)$ and $x = f^{-1}(y)$. Then if $y_0 = f(x_0) \in f((a, b))$,

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)}\,.$$

Since f^{-1} is continuous on $f((a, b))$, $x \to x_0$ as $y \to y_0$; thus

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}$$

which implies that f^{-1} is differentiable at y_0. □

4.7 Integration of Functions of One Variable

Definition 4.73. Let $A \subseteq \mathbb{R}$ be a bounded subset. A collection $\mathcal{P}$ of finitely many points $\{x_0, x_1, \cdots, x_n\}$ is called a **partition** of A if $\inf A = x_0 < x_1 < \cdots < x_{n-1} < x_n = \sup A$. The **mesh size** of the partition $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, is defined by

$$\|\mathcal{P}\| = \max\big\{x_k - x_{k-1}\,\big|\, k = 1, \cdots, n\big\}\,.$$

Definition 4.74. Let $A \subseteq \mathbb{R}$ be a bounded subset, and $f : A \to \mathbb{R}$ be a bounded function. For any partition $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ of A, the **upper sum** and the **lower sum** of f with respect to the partition $\mathcal{P}$, denoted by $U(f, \mathcal{P})$ and $L(f, \mathcal{P})$ respectively, are numbers defined by

$$U(f,\mathcal{P}) = \sum_{k=1}^{n} \sup_{x\in[x_{k-1},x_k]} \bar{f}(x)(x_k - x_{k-1}) = \sum_{k=0}^{n-1} \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k)\,,$$
$$L(f,\mathcal{P}) = \sum_{k=1}^{n} \inf_{x\in[x_{k-1},x_k]} \bar{f}(x)(x_k - x_{k-1}) = \sum_{k=0}^{n-1} \inf_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k)\,,$$

where $\bar{f}$ is an extension of f given by

$$\bar{f}(x) = \begin{cases} f(x) & x \in A\,, \\ 0 & x \notin A\,. \end{cases} \tag{4.7.1}$$

The two numbers

$$\overline{\int_A} f(x)dx \equiv \inf\big\{U(f,\mathcal{P}) \,\big|\, \mathcal{P} \text{ is a partition of } A\big\}\,,$$

and

$$\underline{\int_A} f(x)dx \equiv \sup\big\{L(f,\mathcal{P}) \,\big|\, \mathcal{P} \text{ is a partition of } A\big\}$$

are called the **upper integral** and **lower integral** of f over A, respective. The function f is said to be **Riemann (Darboux) integrable** (over A) if $\overline{\int_A} f(x)dx = \underline{\int_A} f(x)dx$, and in this case, we express the upper and lower integral as $\int_A f(x)dx$, called the **integral** of f over A. The upper integral, the lower integral, and the integral of f over $[a, b]$ sometimes are also denoted by $\overline{\int_a^b} f(x)dx$, $\underline{\int_a^b} f(x)dx$, and $\int_a^b f(x)dx$.

Example 4.75. $\underline{\int_a^b} f(x)dx$ and $\overline{\int_a^b} f(x)dx$ are not always the same. For example, define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1]\backslash\mathbb{Q}, \\ 0 & \text{if } x \in [0,1] \cap \mathbb{Q}. \end{cases}$$

Let $\mathcal{P} = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ be any partition on $[0, 1]$. Then for any $k =$

$0, 1, \cdots, n-1$, $\sup\limits_{x\in[x_k,x_{k+1}]} f(x) = 1$ and $\inf\limits_{x\in[x_k,x_{k+1}]} f(x) = 0$; thus

$$\begin{aligned} U(f,\mathcal{P}) &= \sum_{k=0}^{n-1} \sup_{x\in[x_k,x_{k+1}]} f(x)(x_k - x_{k-1}) = \sum_{k=0}^{n}(x_k - x_{k-1}) \\ &= (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) = x_n - x_0 = 1 - 0 = 1 \end{aligned}$$

and

$$L(f,\mathcal{P}) = \sum_{i=1}^{n} 0(x_i - x_{i-1}) = 0\,.$$

As a consequence,

$$\begin{aligned} \overline{\int_0^1} f(x)dx &= \inf\big\{U(f,\mathcal{P})\,\big|\,\mathcal{P} \text{ is a partition on } [0,1]\big\} = 1\,, \\ \underline{\int_0^1} f(x)dx &= \sup\big\{L(f,\mathcal{P})\,\big|\,\mathcal{P} \text{ is a partition on } [0,1]\big\} = 0\,; \end{aligned}$$

hence f is not Riemann integrable over $[0,1]$.

Example 4.76. Suppose $f : [a,b] \to \mathbb{R}$ is integrable and $f \geqslant 0$ on $[a,b]$, then $\int_a^b f(x)dx \geqslant 0$. Reason: Since $f \geqslant 0$ on $[a,b] \Rightarrow \sup\limits_{x\in[x_k,x_{k+1}]} f(x) \geqslant 0$ for $k = 0, 1, \dots, n-1$. Therefore, $U(f,\mathcal{P}) \geqslant 0$ for all partition $\mathcal{P}$ on $[a,b]$, so

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \inf\{U(f,\mathcal{P})\,\big|\,\mathcal{P} \text{ is a partition on } [a,b]\big\} \geqslant 0\,.$$

Definition 4.77. A partition $\mathcal{P}'$ of a bounded set $A \subseteq \mathbb{R}$ is said to be a **refinement** of another partition $\mathcal{P}$ if $\mathcal{P} \subseteq \mathcal{P}'$.

Proposition 4.78. *Let $A \subseteq \mathbb{R}$ be a bounded subset, and $f : A \to \mathbb{R}$ be a bounded function. If $\mathcal{P}$ and $\mathcal{P}'$ are partitions of A and $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then*

$$L(f,\mathcal{P}) \leqslant L(f,\mathcal{P}') \leqslant U(f,\mathcal{P}') \leqslant U(f,\mathcal{P})\,.$$

Proof. Let $\bar{f}$ be the extension of f given by (4.7.1). Suppose that $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$, $\mathcal{P}' = \{y_0, y_1, \cdots, y_m\}$, and $\mathcal{P} \subseteq \mathcal{P}'$. For any fixed $k = 0, 1, \cdots, n-1$, either $\mathcal{P}' \cap (x_k, x_{k+1}) = \varnothing$ or $\mathcal{P}' \cap (x_k, x_{k+1}) \neq \varnothing$.

1. If $\mathcal{P}' \cap (x_k, x_{k+1}) = \varnothing$, then $x_k = y_\ell$ and $x_{k+1} = y_{\ell+1}$ for some ℓ. Therefore,

$$\sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k) = \sup_{x\in[y_\ell,y_{\ell+1}]} \bar{f}(x)(y_{\ell+1} - y_\ell).$$

2. If $\mathcal{P}' \cap (x_k, x_{k+1}) = \{y_{\ell+1}, y_{\ell+2}, \cdots, y_{\ell+p}\}$, then $x_k = y_\ell$ and $x_{k+1} = y_{\ell+p+1}$. Therefore,

$$\begin{aligned}\sum_{i=1}^{p+1} \sup_{x\in[y_{\ell+i-1},y_{\ell+i}]} \bar{f}(x)(y_{\ell+i} - y_{\ell+i-1}) &= \sup_{x\in[y_\ell,y_{\ell+1}]} \bar{f}(x)(y_{\ell+1} - y_\ell) \\ &\quad + \sup_{x\in[y_{\ell+1},y_{\ell+2}]} \bar{f}(x)(y_{\ell+2} - y_{\ell+1}) + \cdots + \sup_{x\in[y_{\ell+p},y_{\ell+p+1}]} \bar{f}(x)(y_{\ell+p+1} - y_{\ell+p}) \\ &\leqslant \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(y_{\ell+1} - y_\ell) + \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(y_{\ell+2} - y_{\ell+1}) + \cdots \\ &\quad + \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(y_{\ell+p+1} - y_{\ell+p}) = \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k)\,.\end{aligned}$$

In either case,

$$\sum_{[y_{\ell-1},y_\ell]\subseteq[x_k,x_{k+1}]} \sup_{x\in[y_{\ell-1},y_\ell]} \bar{f}(x)(y_\ell - y_{\ell-1}) \leqslant \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k)\,.$$

As a consequence,

$$\begin{aligned}U(f, \mathcal{P}') &= \sum_{\ell=0}^{m-1} \sup_{x\in[y_\ell,y_{\ell+1}]} \bar{f}(x)(y_{\ell+1} - y_\ell) = \sum_{k=0}^{n-1} \sum_{[y_{\ell-1},y_\ell]\subseteq[x_k,x_{k+1}]} \bar{f}(x)(y_\ell - y_{\ell-1}) \\ &\leqslant \sum_{k=0}^{n-1} \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k) = U(f, \mathcal{P})\,.\end{aligned}$$

Similarly, $L(f, \mathcal{P}) \leqslant L(f, \mathcal{P}')$; thus the fact that $L(f, \mathcal{P}') \leqslant U(f, \mathcal{P}')$ concludes the proposition. □

Corollary 4.79. *Let $f : [a, b] \to \mathbb{R}$ be a function bounded by M; that is, $|f(x)| \leqslant M$ for all $a \leqslant x \leqslant b$. Then for all partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a, b]$,*

$$-M(b - a) \leqslant L(f, \mathcal{P}_1) \leqslant \underline{\int_a^b} f(x)dx \leqslant \overline{\int_a^b} f(x)dx \leqslant U(f, \mathcal{P}_2) \leqslant M(b - a)\,.$$

Proof. It suffices to show that $\underline{\int_a^b} f(x)dx \leqslant \overline{\int_a^b} f(x)dx$. By the definition of infimum and supremum, for any given $\varepsilon > 0$, $\exists$ partitions $\bar{\mathcal{P}}$ and $\widetilde{\mathcal{P}}$ such that

$$\underline{\int_a^b} f(x)dx - \frac{\varepsilon}{2} < L(f, \bar{\mathcal{P}}) \leqslant \underline{\int_a^b} f(x)dx \quad \text{and} \quad \overline{\int_a^b} f(x)dx \leqslant U(f, \widetilde{\mathcal{P}}) < \overline{\int_a^b} f(x)dx + \frac{\varepsilon}{2}\,.$$

Let $\mathcal{P} = \bar{\mathcal{P}} \cup \widetilde{\mathcal{P}}$. Then $\mathcal{P}$ is a refinement of both $\bar{\mathcal{P}}$ and $\widetilde{\mathcal{P}}$; thus

$$\underline{\int_a^b} f(x)dx - \frac{\varepsilon}{2} < L(f, \bar{\mathcal{P}}) \leqslant L(f, \mathcal{P}) \leqslant U(f, \mathcal{P}) \leqslant U(f, \widetilde{\mathcal{P}}) < \overline{\int_a^b} f(x)dx + \frac{\varepsilon}{2}\,.$$

Since $\varepsilon > 0$ is given arbitrarily, we must have $\underline{\int_a^b} f(x)dx \leqslant \overline{\int_a^b} f(x)dx$. □

Proposition 4.80 (Riemann's condition). *Let $A \subseteq \mathbb{R}$ be a bounded set, and $f : A \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable over A if and only if*

$$\forall\, \varepsilon > 0, \exists \text{ a partition } \mathcal{P} \text{ of } A \ni U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon\,.$$

Proof. "⇒" Let $\varepsilon > 0$ be given. Since f is integrable over A,

$$\inf_{\mathcal{P}:\text{ Partition of } A} U(f,\mathcal{P}) = \sup_{\mathcal{P}:\text{ Partition of } A} L(f,\mathcal{P}) = \int_A f(x)dx\,;$$

thus there exist $\mathcal{P}_1$ and $\mathcal{P}_2$, partitions of A, such that

$$\int_A f(x)dx - \frac{\varepsilon}{2} < L(f,\mathcal{P}_1) \leqslant \int_A f(x)dx \leqslant U(f,\mathcal{P}_2) < \int_A f(x)dx + \frac{\varepsilon}{2}\,.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $\mathcal{P}$ is a refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$; thus

$$\begin{aligned}\int_A f(x)dx - \frac{\varepsilon}{2} < L(f,\mathcal{P}_1) \leqslant L(f,\mathcal{P}) &\leqslant \int_A f(x)dx \\ &\leqslant U(f,\mathcal{P}) \leqslant U(f,\mathcal{P}_2) < \int_A f(x)dx + \frac{\varepsilon}{2}\end{aligned}$$

which implies that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$.

"⇐" We note that for any partition $\mathcal{P}$ of A,

$$L(f,\mathcal{P}) \leqslant \underline{\int_A} f(x)dx \leqslant \overline{\int_A} f(x)dx \leqslant U(f,\mathcal{P})\,;$$

so we have that for all partition $\mathcal{P}$ of A,

$$\overline{\int_A} f(x)dx - \underline{\int_A} f(x)dx < U(f,\mathcal{P}) - L(f,\mathcal{P})\,.$$

Let $\varepsilon > 0$ be given. By choosing $\mathcal{P}$ so that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$, we conclude that

$$\overline{\int_A} f(x)dx - \underline{\int_A} f(x)dx < \varepsilon\,.$$

Since $\varepsilon > 0$ is given arbitrarily, $\overline{\int_A} f(x)dx = \underline{\int_A} f(x)dx$; thus f is Riemann integrable over A. □

Proposition 4.81. *Suppose that $f, g : [a,b] \to \mathbb{R}$ are Riemann integrable, and $k \in \mathbb{R}$. Then*

1. *kf is Riemann integrable, and $\displaystyle\int_a^b (kf)(x)dx = k\int_a^b f(x)dx$.*
2. *$f \pm g$ are Riemann integrable, and $\displaystyle\int_a^b (f \pm g)(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.*
3. *If $f \leqslant g$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)dx \leqslant \int_a^b g(x)dx$.*
4. *If f is also Riemann integrable over $[b,c]$, then f is Riemann integrable over $[a,c]$, and*
$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx\,. \tag{4.7.2}$$
5. *The function $|f|$ is also Riemann integrable, and $\displaystyle\left|\int_a^b f(x)dx\right| \leqslant \int_a^b |f(x)|dx$.*

Proof. 1. Case 1. $k \geqslant 0$. We note that

$$\inf_{x\in[x_{i-1},x_i]}(kf)(x) = k\inf_{x\in[x_{i-1},x_i]} f(x) \quad \text{and} \quad \sup_{x\in[x_{i-1},x_i]}(kf)(x) = k\sup_{x\in[x_{i-1},x_i]} f(x)\,.$$

Then

$$\begin{aligned} L(kf,\mathcal{P}) &= \sum_{i=1}^n \inf_{x\in[x_{i-1},x_i]}(kf)(x)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n k\inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) = kL(f,\mathcal{P})\,. \end{aligned}$$

Similarly, $U(kf,\mathcal{P}) = kU(f,\mathcal{P})$ for every partition $\mathcal{P}$. So

$$\begin{aligned} \underline{\int_a^b}(kf)(x)dx &= \sup_{\mathcal{P}:\text{ Partition of }[a,b]} L(kf,\mathcal{P}) = k\sup_{\mathcal{P}:\text{ Partition of }[a,b]} L(f,\mathcal{P}) \\ &= k\underline{\int_a^b} f(x)dx = k\int_a^b f(x)dx\,. \end{aligned}$$

Similarly, $\displaystyle\overline{\int_a^b}(kf)(x)dx = k\int_a^b f(x)dx$. Hence kf is integrable and

$$\int_a^b (kf)(x)dx = \underline{\int_a^b}(kf)(x)dx = k\underline{\int_a^b} f(x)dx = k\int_a^b f(x)dx\,.$$

Case 2. $k < 0$. We have

$$\inf_{x\in[x_{i-1},x_i]}(kf)(x) = k \sup_{x\in[x_{i-1},x_i]} f(x) \quad \text{and} \quad \sup_{x\in[x_{i-1},x_i]}(kf)(x) = k \inf_{x\in[x_{i-1},x_i]} f(x)\,.$$

Then $L(kf,\mathcal{P}) = kU(f,\mathcal{P})$ and $U(kf,\mathcal{P}) = kL(f,\mathcal{P})$; thus

$$\begin{aligned}\underline{\int_a^b}(kf)(x)dx &= \sup_{\mathcal{P}\text{: Partition of }[a,b]} L(kf,\mathcal{P}) = \sup_{\mathcal{P}\text{: Partition of }[a,b]} kU(f,\mathcal{P}) \\ &= k \inf_{\mathcal{P}\text{: Partition of }[a,b]} U(f,\mathcal{P}) = k\overline{\int_a^b} f(x)dx = k\int_a^b f(x)dx\,.\end{aligned}$$

Similarly, $\overline{\int_a^b}(kf)(x)dx = k\int_a^b f(x)dx$. Hence kf is Riemann integrable over $[a,b]$ and

$$\int_a^b (kf)(x)dx = \underline{\int_a^b}(kf)(x)dx = k\underline{\int_a^b} f(x)dx = k\int_a^b f(x)dx\,.$$

2. We prove the case of summation. For ant partition $\mathcal{P}$, we have

$$\begin{aligned}L(f+g,\mathcal{P}) &= \sum_{i=1}^n \inf_{x\in[x_{i-1},x_i]}(f+g)(x)(x_i - x_{i-1}) \\ &\geqslant \sum_{i=1}^n \inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) + \sum_{i=1}^n \inf_{x\in[x_{i-1},x_i]} g(x)(x_i - x_{i-1}) \\ &= L(f,\mathcal{P}) + L(g,\mathcal{P})\,.\end{aligned}$$

Similarly, $U(f+g,\mathcal{P}) \leqslant U(f,\mathcal{P}) + U(g,\mathcal{P})$. Therefore,

$$L(f,\mathcal{P}) + L(g,\mathcal{P}) \leqslant L(f+g,\mathcal{P}) \leqslant U(f+g,\mathcal{P}) \leqslant U(f,\mathcal{P}) + U(g,\mathcal{P})\,. \tag{4.7.3}$$

Let $\varepsilon > 0$ be given. By Proposition 4.80, $\exists\, \mathcal{P}_1, \mathcal{P}_2$ partitions of $[a,b]$ such that

$$U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) < \frac{\varepsilon}{2} \quad \text{and} \quad U(g,\mathcal{P}_2) - L(g,\mathcal{P}_2) < \frac{\varepsilon}{2}\,.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. By (4.7.3),

$$\begin{aligned}U(f+g,\mathcal{P}) - L(f+g,\mathcal{P}) &\leqslant \big(U(f,\mathcal{P}) + U(g,\mathcal{P})\big) - \big(L(f,\mathcal{P}) + L(g,\mathcal{P})\big) \\ &= \big(U(f,\mathcal{P}) - L(f,\mathcal{P})\big) + \big(U(g,\mathcal{P}) - L(g,\mathcal{P})\big) \\ &\leqslant \big(U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1)\big) + \big(U(g,\mathcal{P}_2) - L(g,\mathcal{P}_2)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.\end{aligned}$$

By Proposition 4.80, $f+g$ is Riemann integrable over $[a,b]$.

To see $\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$, we note that by Proposition 4.78,

$$\begin{aligned} U(f,\mathcal{P}) &\leqslant L(f,\mathcal{P}) + U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) < L(f,\mathcal{P}) + \frac{\varepsilon}{2} \\ &\leqslant \underline{\int_a^b} f(x)dx + \frac{\varepsilon}{2} = \int_a^b f(x)dx + \frac{\varepsilon}{2} \end{aligned}$$

and similarly, $U(g,\mathcal{P}) < \int_a^b g(x)dx + \frac{\varepsilon}{2}$. Therefore, by (4.7.3),

$$\begin{aligned} \int_a^b (f+g)(x)dx &= \overline{\int_a^b} (f+g)(x)dx \leqslant U(f+g,\mathcal{P}) \\ &\leqslant U(f,\mathcal{P}) + U(g,\mathcal{P}) < \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon \,. \end{aligned} \tag{4.7.4}$$

On the other hand,

$$L(f,\mathcal{P}) > U(f,\mathcal{P}) - \frac{\varepsilon}{2} \geqslant \int_a^b f(x)dx - \frac{\varepsilon}{2}$$

and

$$L(g,\mathcal{P}) > U(g,\mathcal{P}) - \frac{\varepsilon}{2} \geqslant \int_a^b g(x)dx - \frac{\varepsilon}{2}\,;$$

hence by (4.7.3),

$$\begin{aligned} \int_a^b (f+g)(x)dx &= \underline{\int_a^b} (f+g)(x)dx \geqslant L(f+g,\mathcal{P}) \geqslant L(f,\mathcal{P}) + L(g,\mathcal{P}) \\ &> \int_a^b f(x)dx + \int_a^b g(x)dx - \varepsilon \,. \end{aligned} \tag{4.7.5}$$

By (4.7.4) and (4.7.5),

$$\int_a^b f(x)dx + \int_a^b g(x)dx - \varepsilon < \int_a^b (f+g)(x)dx < \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon \,.$$

Since $\varepsilon > 0$ is arbitrary, $\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

3. Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$. Define

$$m_i(f) = \inf_{x\in[x_{i-1},x_i]} f(x) \quad \text{and} \quad m_i(g) = \inf_{x\in[x_{i-1},x_i]} g(x)\,.$$

Since $f(x) \leqslant g(x)$ on $[a,b]$, $m_i(f) \leqslant m_i(g)$. As a consequence, for any partition $\mathcal{P}$,

$$L(f,\mathcal{P}) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) \leqslant \sum_{i=1}^{n} m_i(g)(x_i - x_{i-1}) = L(g,\mathcal{P});$$

thus taking the infimum over all partition $\mathcal{P}$,

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \sup_{\mathcal{P}} L(f,\mathcal{P}) \leqslant \sup_{\mathcal{P}} L(g,\mathcal{P}) = \underline{\int_a^b} g(x)dx = \int_a^b g(x)dx\,.$$

4. Let $\varepsilon > 0$ be given. Since f is Riemann integrable of $[a,b]$ and $[b,c]$, there exist a partition $\mathcal{P}_1$ over $[a,b]$ and a partition $\mathcal{P}_2$ of $[b,c]$ such that

$$U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) < \frac{\varepsilon}{2} \quad \text{and} \quad U(f,\mathcal{P}_2) - L(f,\mathcal{P}_2) < \frac{\varepsilon}{2}\,.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $\mathcal{P}$ is a partition of $[a,c]$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) = U(f,\mathcal{P}_1) + U(f,\mathcal{P}_2) - L(f,\mathcal{P}_1) - L(f,\mathcal{P}_2) < \varepsilon\,.$$

Therefore, Proposition 4.80 implies that f is Riemann integrable over $[a,c]$.

Now we show that $\displaystyle\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$. To simplify the notation, we let

$$A = \int_a^c f(x)dx, \quad B = \int_a^b f(x)dx, \quad C = \int_b^c f(x)dx\,.$$

Let $\varepsilon > 0$ be given. Then $\exists$ partition $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ of $[a,c]$ such that

$$A \leqslant U(f,\mathcal{P}) < A + \varepsilon\,.$$

Let $\mathcal{P}' = \mathcal{P} \cup \{b\}$. Then $\mathcal{P}'$ is a refinement of $\mathcal{P}$. Moreover,

$$U(f,\mathcal{P}') = U(f,\mathcal{P}_1) + U(f,\mathcal{P}_2)\,,$$

where $\mathcal{P}_1 = \mathcal{P}' \cap [a,b]$ and $\mathcal{P}_2 = \mathcal{P}' \cap [b,c]$ are partitions of $[a,b]$ and $[b,c]$ whose union is $\mathcal{P}$. Therefore,

$$B + C \leqslant U(f,\mathcal{P}_1) + U(f,\mathcal{P}_2) = U(f,\mathcal{P}') \leqslant U(f,\mathcal{P}) < A + \varepsilon\,.$$

On the other hand, $\exists$ partition $\mathcal{P}_1$ of $[a,b]$ and partition $\mathcal{P}_2$ of $[b,c]$ such that

$$B \leqslant U(f,\mathcal{P}_1) < B + \frac{\varepsilon}{2} \quad \text{and} \quad C \leqslant U(f,\mathcal{P}_2) < C + \frac{\varepsilon}{2}\,.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $\mathcal{P}$ is a partition of $[a,c]$. Therefore,

$$A \leqslant U(f,\mathcal{P}) = U(f,\mathcal{P}_1) + U(f,\mathcal{P}_2) < B + C + \varepsilon\,.$$

Therefore, $\forall\, \varepsilon > 0$, $B + C < A + \varepsilon$ and $A < B + C + \varepsilon$; thus $A = B + C$.

5. Note that for any interval $[\alpha,\beta]$,

$$\sup_{x\in[\alpha,\beta]}|f(x)| - \inf_{x\in[\alpha,\beta]}|f(x)| \leqslant \sup_{x\in[\alpha,\beta]} f(x) - \inf_{x\in[\alpha,\beta]} f(x)\,; \quad (\textbf{Check!})$$

thus for any partition $\mathcal{P}$ of $[a,b]$,

$$U(|f|,\mathcal{P}) - L(|f|,\mathcal{P}) \leqslant U(f,\mathcal{P}) - L(f,\mathcal{P})\,.$$

Therefore, Proposition 4.80 implies that $|f|$ is Riemann integrable over $[a,b]$. Moreover, since $-|f(x)| \leqslant f(x) \leqslant |f(x)|$ for all $x\in[a,b]$, by 3 we have

$$-\int_a^b |f(x)|dx \leqslant \int_a^b f(x)dx \leqslant \int_a^b |f(x)|dx\,. \qquad \square$$

Remark 4.82. The proof of 4 in Proposition 4.81 in fact also shows that if $a<b<c$, then

$$\overline{\int_a^c} f(x)dx = \overline{\int_a^b} f(x)dx + \overline{\int_b^c} f(x)dx\,.$$

Similar proof also implies that

$$\underline{\int_a^c} f(x)dx = \underline{\int_a^b} f(x)dx + \underline{\int_b^c} f(x)dx\,.$$

Remark 4.83. If $a<b$, we let the number $\int_b^a f(x)dx$ denote the number $-\int_a^b f(x)dx$. Then (4.7.2) holds for all $a,b,c\in\mathbb{R}$.

Example 4.84. Let $f:[0,1]\to\mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x\in\mathbb{Q}, \\ -1 & \text{if } x\in\mathbb{Q}^\complement. \end{cases}$$

Then $f(x)$ is not Riemann integrable over $[0,1]$ since $U(f,P)=1$ and $L(f,P)=-1$. However $|f(x)|\equiv 1$, thus $|f|$ is Riemann integrable. In other words, if $|f|$ is integrable, we cannot know whether f is integrable or not.

Theorem 4.85. *If $f:[a,b]\to\mathbb{R}$ is continuous, then f is Riemann integrable.*

Proof. Let $\varepsilon>0$ be given. Theorem 4.52 implies that

$$\exists\,\delta>0 \ni |f(x)-f(y)| < \frac{\varepsilon}{2(b-a)} \text{ whenever } |x-y|<\delta \text{ and } x,y\in[a,b]\,.$$

Let $\mathcal{P}$ be a partition with mesh size less than δ. Then

$$\begin{aligned} U(f,\mathcal{P}) - L(f,\mathcal{P}) &= \sum_{k=1}^{n} \Big(\sup_{x\in[x_{k-1},x_k]} f(x) - \inf_{x\in[x_{k-1},x_k]} f(x)\Big)(x_k - x_{k-1}) \\ &\leqslant \frac{\varepsilon}{2(b-a)} \sum_{k=1}^{n}(x_k - x_{k-1}) = \frac{\varepsilon}{2(b-a)}(x_n - x_0) < \varepsilon\,; \end{aligned}$$

thus by Proposition 4.80 f is Riemann integrable over $[a,b]$. □

Corollary 4.86. *If $f : (a,b) \to \mathbb{R}$ is continuous and f is bounded on $[a,b]$, then f is Riemann integrable over $[a,b]$.*

Proof. Let $|f(x)| \leqslant M$ for all $x \in [a,b]$, and $\varepsilon > 0$ be given. Since $f : \big[a+\frac{\varepsilon}{8M}, b-\frac{\varepsilon}{8M}\big] \to \mathbb{R}$ is continuous, by Theorem 4.85 f is Riemann integrable; thus

$$\exists\, \mathcal{P}' \text{: partition of } \Big[a + \frac{\varepsilon}{8M}, b - \frac{\varepsilon}{8M}\Big] \ni U(f,\mathcal{P}') - L(f,\mathcal{P}') < \frac{\varepsilon}{2}\,.$$

Let $\mathcal{P} = \mathcal{P}' \cup \{a,b\}$. Then

$$\begin{aligned} &U(f,\mathcal{P}) - L(f,\mathcal{P}) \\ &\quad < \Big(\sup_{x\in[a,a+\frac{\varepsilon}{8M}]} f(x) - \inf_{x\in[a,a+\frac{\varepsilon}{8M}]} f(x)\Big)\frac{\varepsilon}{8M} + \frac{\varepsilon}{2} + \Big(\sup_{x\in[b-\frac{\varepsilon}{8M},b]} f(x) - \inf_{x\in[b-\frac{\varepsilon}{8M},b]} f(x)\Big)\frac{\varepsilon}{8M} \\ &\quad \leqslant 2M\cdot\frac{\varepsilon}{8M} + \frac{\varepsilon}{2} + 2M\cdot\frac{\varepsilon}{8M} = \varepsilon\,; \end{aligned}$$

thus Proposition 4.80 implies that f is Riemann integrable over $[a,b]$. □

Corollary 4.87. *If $f : [a,b] \to \mathbb{R}$ is bounded and is continuous at all but finitely many points of $[a,b]$, then f is Riemann integrable.*

Proof. Let $\{c_1, \cdots, c_N\}$ be the collection of all discontinuities of f in (a,b) such that $c_1 < c_2 < \cdots < c_N$. Let $a = c_0$ and $b = c_{N+1}$. Then for all $k = 0,1,\cdots,N$, $f : (c_k, c_{k+1})$ is continuous and $f : [c_k, c_{k+1}]$ is bounded; thus f is Riemann integrable by Corollary 4.87. Finally, 4 of Proposition 4.81 implies that f is Riemann integrable over $[a,b]$. □

Theorem 4.88. *Any increasing or decreasing function on $[a,b]$ is Riemann integrable.*

Proof. Let $f : [a,b] \to \mathbb{R}$ be a monotone function, and $\varepsilon > 0$ be given. W.L.O.G. we may assume that $f(b) \neq f(a)$. Let $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a,b]$ with mesh size less than $\dfrac{\varepsilon}{|f(b)-f(a)|}$. Then

$$\begin{aligned} U(f,\mathcal{P}) - L(f,\mathcal{P}) &= \sum_{k=1}^{n} \Big(\sup_{x\in[x_{k-1},x_k]} f(x) - \inf_{x\in[x_{k-1},x_k]} f(x) \Big)(x_k - x_{k-1}) \\ &< \sum_{k=1}^{n} \big|f(x_k) - f(x_{k-1})\big| \frac{\varepsilon}{|f(b)-f(a)|} = \big|f(b)-f(a)\big| \frac{\varepsilon}{|f(b)-f(a)|} = \varepsilon\,; \end{aligned}$$

thus Proposition 4.80 implies that f is Riemann integrable over $[a,b]$. ▫

Definition 4.89. A continuous function $F : [a.b] \to \mathbb{R}$ is called an **anti-derivative**（反導函數）of $f : [a,b] \to \mathbb{R}$ if F is differentiable on (a,b) and $F'(x) = f(x)$ for all $x \in (a,b)$.

Theorem 4.90 (Fundamental Theorem of Calculus（微積分基本定理）)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f has an anti-derivative F, and*

$$\int_a^b f(x)dx = F(b) - F(a)\,.$$

Moreover, if G is any other anti-derivative of f, we also have $\displaystyle\int_a^b f(x)dx = G(b) - G(a)$.

Proof. Define $F(x) = \displaystyle\int_a^x f(y)dy$, where the integral of f over $[a,x]$ is well-defined because of continuity of f on $[a,x]$. We first show that F is differentiable on (a,b).

Let $x_0 \in (a,b)$ and $\varepsilon > 0$ be given. Since $[a,b]$ is compact,

$$\exists\, \delta_1 > 0 \ni \big|f(x) - f(y)\big| < \frac{\varepsilon}{2} \text{ whenever } |x-y| < \delta_1 \text{ and } x,y \in [a,b]\,.$$

Let $\delta = \min\{\delta_1, x_0 - a, b - x_0\}$. By 4 of Proposition 4.81, if $x, x_0 \in (a,b)$,

$$\int_{x_0}^x f(y)dy = \int_a^x f(y)dy - \int_a^{x_0} f(y)dy = F(x) - F(x_0)\,;$$

thus if $0 < |x - x_0| < \delta$,

$$\begin{aligned} \Big|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\Big| &= \Big|\frac{1}{x-x_0}\int_{x_0}^x f(y)dx - f(x_0)\Big| = \Big|\frac{1}{x-x_0}\int_{x_0}^x \big(f(y)-f(x_0)\big)dy\Big| \\ &\leqslant \frac{1}{|x-x_0|}\int_{\min\{x_0,x\}}^{\max\{x_0,x\}} \big|f(y)-f(x_0)\big|dy \leqslant \frac{1}{|x-x_0|}\int_{\min\{x_0,x\}}^{\max\{x_0,x\}} \frac{\varepsilon}{2}dy < \varepsilon\,. \end{aligned}$$

Therefore, $\displaystyle\lim_{x\to x_0} \frac{F(x)-F(x_0)}{x-x_0} = f(x_0)$ for all $x_0 \in (a,b)$, so $F'(x) = f(x)$ for all $x \in (a,b)$.

Next we show that F is continuous at $x = a$ and $x = b$. This is simply because of the boundedness of f on $[a, b]$ which implies that

$$\limsup_{x\to a^+} \big|F(x) - F(a)\big| = \limsup_{x\to a^+} \Big|\int_a^x f(t)dt\Big| \leqslant \max_{x\in[a,b]} |f(x)| \cdot \limsup_{x\to a^+} \int_a^x 1dt = 0$$

and

$$\limsup_{x\to b^-} \big|F(x) - F(b)\big| = \limsup_{x\to b^-} \Big|\int_x^b f(t)dt\Big| \leqslant \max_{x\in[a,b]} |f(x)| \cdot \limsup_{x\to b^-} \int_x^b 1dt = 0\,.$$

Therefore, F is an anti-derivative of f.

Now suppose that G is another anti-derivative of f. Then $(G - F)'(x) = 0$ for all $x \in (a, b)$. By Corollary 4.66, $(G - F)(x) = (G - F)(a)$ for all $x \in [a, b]$; thus $G(b) - G(a) = F(b) - F(a)$. ▫

Example 4.91. If f is only integrable but not continuous, then the function

$$F(x) = \int_a^x f(t)dt$$

is not necessarily differentiable. For example, consider

$$f(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant 1, \\ 1 & \text{if } 1 < x \leqslant 2. \end{cases}$$

Then

$$F(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant 1, \\ x - 1 & \text{if } 1 \leqslant x \leqslant 2. \end{cases}$$

so F is continuous on $[0, 2]$ but not differentiable at $x = 1$.

Theorem 4.92. *Let $f : [a, b] \to \mathbb{R}$ be differentiable. If f' is Riemann integrable over $[a, b]$, then $\int_a^b f'(x)dx = f(b) - f(a)$.*

Proof. Let $\mathcal{P} = \{x_0, x_1, \cdots x_n\}$ be a partition of $[a, b]$. Since $f : [a, b] \to \mathbb{R}$ is differentiable, by the Mean Value Theorem there exists $\{\xi_1, \cdots, \xi_n\}$ with the property that $x_k < \xi_{k+1} < x_{k+1}$ for all $k = 0, 1 \cdots, n - 1$ such that

$$f'(\xi_{k+1})(x_{k+1} - x_k) = f(x_{k+1}) - f(x_k) \qquad \forall\, k = 0, 1, \cdots, n - 1\,.$$

Therefore,

$$\sum_{k=0}^{n-1} \inf_{x\in[x_k,x_{k+1}]} f'(x)(x_{k+1} - x_k) \leqslant \sum_{k=0}^{n-1} f'(\xi_{k+1})(x_{k+1} - x_k) \leqslant \sum_{k=0}^{n-1} \sup_{x\in[x_k,x_{k+1}]} f'(x)(x_{k+1} - x_k)\,.$$

Since $\sum_{k=0}^{n-1} f'(\xi_{k+1})(x_{k+1} - x_k) = \sum_{k=0}^{n-1} \big(f(x_{k+1}) - f(x_k)\big) = f(b) - f(a)$, the inequality above implies that

$$L(f', \mathcal{P}) \leqslant f(b) - f(a) \leqslant U(f', \mathcal{P}) \text{ for all partitions } \mathcal{P} \text{ of } [a, b]\,;$$

thus by the definition of the upper and the lower integrals,

$$\underline{\int_a^b} f'(x)dx \leqslant f(b) - f(a) \leqslant \overline{\int_a^b} f'(x)dx\,.$$

We then conclude the theorem by the identity

$$\underline{\int_a^b} f'(x)dx = \overline{\int_a^b} f'(x)dx = \int_a^b f'(x)dx$$

since f' is Riemann integrable. □

Definition 4.93. Let $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ be a partition of a bounded set $A \subseteq \mathbb{R}$. A collection of points $\{\xi_1, \cdots, \xi_n\}$ is called a **sample set** for the partition $\mathcal{P}$ if $\xi_k \in [x_{k-1}, x_k]$ for all $k = 1, \cdots, n$.

Let $f : A \to \mathbb{R}$ be a bounded function with extension $\bar{f}$ given by (4.7.1). A **Riemann sum** of f for the partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of A is a sum which takes the form

$$\sum_{k=0}^{n-1} \bar{f}(\xi_k)(x_{k+1} - x_k)\,,$$

where the set $\Xi = \{\xi_0, \xi_1, \cdots, \xi_{n-1}\}$ is a sample set for $\mathcal{P}$.

Theorem 4.94 (Darboux). *Let $f : A \to \mathbb{R}$ be a bounded function with extension $\bar{f}$ given by (4.7.1). Then f is Riemann integrable over A if and only if there exists $\mathrm{I} \in \mathbb{R}$ such that for every given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of A satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sum of f for the partition $\mathcal{P}$ lies in the interval $(\mathrm{I} - \varepsilon, \mathrm{I} + \varepsilon)$. In other words, f is Riemann integrable over A if and only if for every given $\varepsilon > 0$, there exists $\delta > 0$ such that there exists $\mathrm{I} \in \mathbb{R}$ such that*

$$\Big|\sum_{k=0}^{n-1} \bar{f}(\xi_{k+1})(x_{k+1} - x_k) - \mathrm{I}\Big| < \varepsilon \tag{4.7.6}$$

whenever $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is a partition of A satisfying $\|\mathcal{P}\| < \delta$ and $\{\xi_1, \xi_2, \cdots, \xi_N\}$ is a sample set for $\mathcal{P}$.

Proof. "$\Leftarrow$" Suppose the right-hand side statement is true. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of A satisfying $\|\mathcal{P}\| < \delta$, then for all sets of sample points $\{\xi_1, \cdots, \xi_n\}$ with respect to $\mathcal{P}$, we must have

$$\Big|\sum_{k=0}^{n-1} \bar{f}(\xi_{k+1})(x_{k+1} - x_k) - \mathrm{I}\Big| < \frac{\varepsilon}{4}.$$

Let $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ be a partition of A with $\|\mathcal{P}\| < \delta$. Choose sets of sample points $\{\xi_1, \cdots, \xi_n\}$ and $\{\eta_1, \cdots, \eta_n\}$ with respect to $\mathcal{P}$ such that

(a) $\displaystyle\sup_{x\in[x_k,x_{k+1}]} \bar{f}(x) - \frac{\varepsilon}{4(x_n - x_0)} < \bar{f}(\xi_{k+1}) \leqslant \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)$;

(b) $\displaystyle\inf_{x\in[x_k,x_{k+1}]} \bar{f}(x) + \frac{\varepsilon}{4(x_n - x_0)} > \bar{f}(\eta_{k+1}) \geqslant \inf_{x\in[x_k,x_{k+1}]} \bar{f}(x)$.

Then

$$\begin{aligned} U(f,\mathcal{P}) &= \sum_{k=0}^{n-1} \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k) < \sum_{k=0}^{n-1} \big[\bar{f}(\xi_{k+1}) + \frac{\varepsilon}{4(x_n - x_0)}\big](x_{k+1} - x_k) \\ &= \sum_{k=0}^{n-1} \bar{f}(\xi_{k+1})(x_{k+1} - x_k) + \frac{\varepsilon}{4(x_n - x_0)} \sum_{k=0}^{n-1}(x_{k+1} - x_k) < \mathrm{I} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \mathrm{I} + \frac{\varepsilon}{2} \end{aligned}$$

and

$$\begin{aligned} L(f,\mathcal{P}) &= \sum_{k=0}^{n-1} \inf_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1} - x_k) > \sum_{k=0}^{n-1} \big[\bar{f}(\eta_{k+1}) - \frac{\varepsilon}{4(x_n - x_0)}\big](x_{k+1} - x_k) \\ &= \sum_{k=0}^{n-1} \bar{f}(\eta_{k+1})(x_{k+1} - x_k) - \frac{\varepsilon}{4(x_n - x_0)} \sum_{k=0}^{n-1}(x_{k+1} - x_k) > \mathrm{I} - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = \mathrm{I} - \frac{\varepsilon}{2}. \end{aligned}$$

As a consequence, $\mathrm{I} - \frac{\varepsilon}{2} < L(f,\mathcal{P}) \leqslant U(f,\mathcal{P}) < \mathrm{I} + \frac{\varepsilon}{2}$; thus $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$.

"$\Rightarrow$" Let $\varepsilon > 0$ be given, and $\mathrm{I} = \displaystyle\int_A \bar{f}(x)dx$. Since f is Riemann integrable over A, there exists a partition $\mathcal{P}_1 = \{y_0, y_1, \cdots, y_m\}$ of A such that $U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) < \frac{\varepsilon}{2}$. Define

$$\delta = \min\Big\{|y_1 - y_0|, |y_2 - y_1|, \cdots, |y_m - y_{m-1}|, \frac{\varepsilon}{4m\big(\sup f(A) - \inf f(A) + 1\big)}\Big\}.$$

If $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is a partition of A with $\|\mathcal{P}\| < \delta$, then at most $2m$ intervals of the form $[x_k, x_{k+1}]$ contains one of these y_j's, and each such interval $[x_k, x_{k+1}]$ can only contain one of these y_j's. Let $\mathcal{P}' = \mathcal{P} \cup \mathcal{P}_1$.

Claim: $U(f,\mathcal{P}) - U(f,\mathcal{P}') < \dfrac{\varepsilon}{2}$.

Proof of claim: We note that

$$\begin{aligned} U(f,\mathcal{P}) &= \sum_{k=0}^{n-1} \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1}-x_k) \\ &= \sum_{\substack{0\leqslant k\leqslant n-1 \text{ with} \\ \mathcal{P}_1\cap[x_k,x_{k+1}]=\varnothing}} \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1}-x_k) + \sum_{\substack{0\leqslant k\leqslant n-1 \text{ with} \\ \mathcal{P}_1\cap[x_k,x_{k+1}]\neq\varnothing}} \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1}-x_k) \end{aligned}$$

and

$$\begin{aligned} U(f,\mathcal{P}') = &\sum_{\substack{0\leqslant k\leqslant n-1 \text{ with} \\ \mathcal{P}_1\cap[x_k,x_{k+1}]=\varnothing}} \sup_{x\in[x_k,x_{k+1}]} \bar{f}(x)(x_{k+1}-x_k) \\ &+ \sum_{\substack{0\leqslant k\leqslant n-1 \text{ with} \\ \mathcal{P}_1\cap[x_k,x_{k+1}]=y_j}} \Big[\sup_{x\in[x_k,y_j]} \bar{f}(x)(y_j-x_k) + \sup_{x\in[y_j,x_{k+1}]} \bar{f}(x)(x_{k+1}-y_j)\Big]. \end{aligned}$$

Therefore,

$$\begin{aligned} U(f,\mathcal{P}) - U(f,\mathcal{P}') &\leqslant \big(\sup f(A) - \inf f(A)\big) \sum_{\substack{0\leqslant k\leqslant n-1 \text{ with} \\ \mathcal{P}_1\cap[x_k,x_{k+1}]\neq\varnothing}} (x_{k+1}-x_k) \\ &< 2m\big(\sup f(A) - \inf f(A)\big)\delta \leqslant \frac{\varepsilon}{2}\,. \end{aligned}$$

On the other hand, the inequality $U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) < \dfrac{\varepsilon}{2}$ implies that

$$U(f,\mathcal{P}_1) - \mathrm{I} < \frac{\varepsilon}{2}\,.$$

As a consequence,

$$U(f,\mathcal{P}) - \mathrm{I} \leqslant U(f,\mathcal{P}) - \mathrm{I} + U(f,\mathcal{P}_1) - U(f,\mathcal{P}') < \varepsilon\,.$$

Therefore, for any sample set $\{\xi_1,\cdots,\xi_n\}$ with respect to $\mathcal{P}$,

$$\sum_{k=0}^{n-1} \bar{f}(\xi_{k+1})(x_{k+1}-x_k) \leqslant U(f,\mathcal{P}) < \mathrm{I} + \varepsilon\,.$$

Similar argument can be used to show that

$$\sum_{k=0}^{n-1} \bar{f}(\xi_{k+1})(x_{k+1}-x_k) \geqslant L(f,\mathcal{P}) > \mathrm{I} - \varepsilon\,;$$

thus (4.7.6) is established. □

Theorem 4.95 (Change of Variable Formula)**.** *Let $g : [a, b] \to \mathbb{R}$ be a one-to-one continuously differentiable function, and $f : g([a, b]) \to \mathbb{R}$ be Riemann integrable. Then $(f \circ g)g'$ is also Riemann integrable, and*

$$\int_{g([a,b])} f(y)\, dy = \int_a^b f(g(x))|g'(x)|\, dx\,.$$

Proof. We only prove the case that f is continuous on $g([a, b])$, and the general case is covered by Theorem 8.65 (which will be proved in detail).

W.L.O.G. we can assume that $g'(x) \geqslant 0$ for all $x \in [a, b]$ so that $g([a, b]) = [g(a), g(b)]$. Let F be an anti-derivative of f. Then F is differentiable, and the chain rule implies that

$$\frac{d}{dx}(F \circ g)(x) = (F' \circ g)(x)g'(x) = (f \circ g)(x)g'(x)\,.$$

Therefore, the fundamental theorem of Calculus implies that

$$\begin{aligned}\int_{g([a,b])} f(y)dy = \int_{g(a)}^{g(b)} f(y)dy = F(g(b)) - F(g(a)) = \int_a^b \frac{d}{dx}(F \circ g)(x)dx \\ = \int_a^b (f \circ g)(x)g'(x)dx\,.\end{aligned}$$

□

4.8 Exercises

§4.1 Continuity

Started from this section, for all $n \in \mathbb{N}$ $\mathbb{R}^n$ always denotes the normed space $(\mathbb{R}^n, \|\cdot\|_2)$.

Problem 4.1. Use whatever methods you know to find the following limits:

1. $\lim\limits_{x\to 0^+} (1 + \sin 2x)^{\frac{1}{x}}$;
2. $\lim\limits_{x\to -\infty} \left(\sqrt{1 + x + x^2} - \sqrt{1 - x + x^2}\right)$;
3. $\lim\limits_{x\to 1} (2 - x)^{\sec\frac{\pi x}{2}}$;
4. $\lim\limits_{x\to\infty} x\left(\frac{\pi}{2} - \sin^{-1}\frac{x}{\sqrt{x^2 + 1}}\right)$;
5. $\lim\limits_{x\to\infty} x\left(e^{-1} - \left(\frac{x}{x + 1}\right)^x\right)$;
6. $\lim\limits_{x\to\infty} \left(\frac{a^x - 1}{x(a - 1)}\right)^{\frac{1}{x}}$, where $a > 0$ and $a \neq 1$.

Problem 4.2. Complete the following.

1. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{x\to 0}\lim_{y\to 0} f(x, y) \quad \text{and} \quad \lim_{y\to 0}\lim_{x\to 0} f(x, y)$$

exist but are not equal.

2. Find a function $f:\mathbb{R}^2\to\mathbb{R}$ such that the two limits above exist and are equal but f is not continuous.

3. Find a function $f:\mathbb{R}^2\to\mathbb{R}$ that is continuous on every line through the origin but is not continuous.

Problem 4.3. Complete the following.

1. Show that the projection map $f: \begin{array}{rcl}\mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x\end{array}$ is continuous.

2. Show that if $\mathcal{U}\subseteq\mathbb{R}$ is open, then $A=\left\{(x,y)\in\mathbb{R}^2 \,\middle|\, x\in\mathcal{U}\right\}$ is open.

3. Give an example of a continuous function $f:\mathbb{R}\to\mathbb{R}$ and an open set $\mathcal{U}\subseteq\mathbb{R}$ such that $f(\mathcal{U})$ is not open.

Problem 4.4. Show that $f:A\to\mathbb{R}^m$, where $A\subseteq\mathbb{R}^n$, is continuous if and only if for every $B\subseteq A$,

$$f(\mathrm{cl}(B)\cap A)\subseteq \mathrm{cl}(f(B))\,.$$

Problem 4.5. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, and $f:\mathbb{R}^n\to\mathbb{R}$ be defined by $f(x)=\|x\|$. Show that f is continuous on $(\mathbb{R}^n,\|\cdot\|_2)$.
Hint: Show that $|f(x)-f(y)|\leqslant C\|x-y\|_2$ for some fixed constant $C>0$.

Problem 4.6. Let $T:\mathbb{R}^n\to\mathbb{R}^m$ satisfy $T(x+y)=T(x)+T(y)$ for all $x,y\in\mathbb{R}^n$.

1. Show that $T(rx)=rT(x)$ for all $r\in\mathbb{Q}$ and $x\in\mathbb{R}^n$.

2. Suppose that T is continuous on $\mathbb{R}^n$. Show that T is linear; that is, $T(cx+y)=cT(x)+T(y)$ for all $c\in\mathbb{R}$, $x,y\in\mathbb{R}^n$.

3. Suppose that T is continuous at some point x_0 in $\mathbb{R}^n$. Show that T is continuous on $\mathbb{R}^n$.

4. Suppose that T is bounded on some open subset of $\mathbb{R}^n$. Show that T is continuous on $\mathbb{R}^n$.

5. Suppose that T is bounded from above (or below) on some open subset of $\mathbb{R}^n$. Show that T is continuous on $\mathbb{R}^n$.

6. Construct a $T : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every point of $\mathbb{R}$, but $T(x+y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}$.

Problem 4.7. Let (M, d) be a metric space, $A \subseteq M$, and $f : A \to \mathbb{R}$. For $a \in A'$, define

$$\liminf_{x\to a} f(x) = \lim_{r\to 0^+} \inf\big\{f(x) \,\big|\, x \in D(a,r) \cap A\backslash\{a\}\big\},$$
$$\limsup_{x\to a} f(x) = \lim_{r\to 0^+} \sup\big\{f(x) \,\big|\, x \in D(a,r) \cap A\backslash\{a\}\big\}.$$

Complete the following.

1. Show that both $\liminf\limits_{x\to a} f(x)$ and $\limsup\limits_{x\to a} f(x)$ exist (which may be $\pm\infty$), and

$$\liminf_{x\to a} f(x) \leqslant \limsup_{x\to a} f(x).$$

Furthermore, there exist sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty \subseteq A\backslash\{a\}$ such that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ both converge to a, and

$$\lim_{n\to\infty} f(x_n) = \liminf_{x\to a} f(x) \qquad \text{and} \qquad \lim_{n\to\infty} f(y_n) = \limsup_{x\to a} f(x).$$

2. Let $\{x_n\}_{n=1}^\infty \subseteq A\backslash\{a\}$ be a convergent sequence with limit a. Show that

$$\liminf_{x\to a} f(x) \leqslant \liminf_{n\to\infty} f(x_n) \leqslant \limsup_{n\to\infty} f(y_n) \leqslant \limsup_{x\to a} f(x).$$

3. Show that $\lim\limits_{x\to a} f(x) = \ell$ if and only if

$$\liminf_{x\to a} f(x) = \limsup_{x\to a} f(x) = \ell.$$

4. Show that $\liminf\limits_{x\to a} f(x) = \ell \in \mathbb{R}$ if and only if the following two conditions hold:

   (a) for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\ell - \varepsilon < f(x)$ for all $x \in D(a,\delta) \cap A\backslash\{a\}$;

   (b) for all $\varepsilon > 0$ and $\delta > 0$, there exists $x \in D(a,\delta) \cap A\backslash\{a\}$ such that $f(x) < \ell + \varepsilon$.

   Formulate a similar criterion for limsup and for the case that $\ell = \pm\infty$.

5. Compute the liminf and limsup of the following functions at any point of $\mathbb{R}$.

   (a) $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}^\complement, \\ \dfrac{1}{p} & \text{if } x = \dfrac{q}{p} \text{ with } (p,q) = 1,\ q > 0,\ p \neq 0. \end{cases}$

(b) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}\,, \\ -x & \text{if } x \in \mathbb{Q}^{\complement}\,. \end{cases}$

Problem 4.8. Let (M, d) be a metric space, and $A \subseteq M$. A function $f : A \to \mathbb{R}$ is called $\dfrac{\textbf{lower semi-continuous}}{\textbf{upper semi-continuous}}$ at $a \in A$ if $\begin{array}{l} \liminf\limits_{x\to a} f(x) \geqslant f(a)\,, \\ \limsup\limits_{x\to a} f(x) \leqslant f(a)\,, \end{array}$ and is called lower/upper semi-continuous on A if f is lower/uppser semi-continuous at a for all $a \in A$.

1. Show that if $f : A \to \mathbb{R}$ is lower semi-continuous on A, then $f^{-1}((-\infty, r])$ is closed relative to A. Also show that if $f : A \to \mathbb{R}$ is upper semi-continuous on A, then $f^{-1}([r, \infty))$ is closed relative to A.

2. Show that f is lower semi-continuous at a if and only if for all convergent sequences $\{x_n\}_{n=1}^{\infty} \subseteq A$ and $\{r_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ satisfying $f(x_n) \leqslant r_n$ for all $n \in \mathbb{N}$, we have
$$f\big(\lim_{n\to\infty} x_n\big) \leqslant \lim_{n\to\infty} r_n\,.$$

3. Let $\{f_\alpha\}_{\alpha\in\mathcal{I}}$ be a family of lower semi-continuous functions on A. Prove that $f(x) = \sup\limits_{\alpha\in\mathcal{I}} f_\alpha(x)$ is lower semi-continuous on A.

4. Let $f : A \to \mathbb{R}$ be given. Define
$$f^*(x) = \limsup_{y\to x} f(y) \qquad \text{and} \qquad f_*(x) = \liminf_{y\to x} f(y)\,.$$
Show that f^* is upper semi-continuous and f_* is lower semi-continuous, and $f_*(x) \leqslant f(x) \leqslant f^*(x)$ for all $x \in A$. Moreover, if g is a lowe semi-continuous function on A such that $g(x) \leqslant f(x)$ for all $x \in A$, then $g \leqslant f_*$.

§4.2 Operations on Continuous Maps

Problem 4.9.

Problem 4.10.

§4.3 Images of Compact Sets under Continuous Maps

Problem 4.11. Complete the following.

1. Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, and $B \subseteq \mathbb{R}^n$ is bounded, then $f(B)$ is bounded.

2. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $K \subseteq \mathbb{R}$ is compact, is $f^{-1}(K)$ necessarily compact?

3. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $C \subseteq \mathbb{R}$ is connected, is $f^{-1}(C)$ necessarily connected?

Problem 4.12. Consider a compact set $K \subseteq \mathbb{R}^n$ and let $f : K \to \mathbb{R}^m$ be continuous and one-to-one. Show that the inverse function $f^{-1} : f(K) \to K$ is continuous. How about if K is not compact but connected?

Problem 4.13. Let (M, d) be a metric space, $K \subseteq M$ be compact, and $f : K \to \mathbb{R}$ be lower semi-continuous (see Problem 4.8 for the definition). Show that f attains its minimum on K.

§4.4 Images of Connected and Path Connected Sets under Continuous Maps

Problem 4.14. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open connected set, where $n > 1$. If a, b and c are any three points in $\mathcal{D}$, show that there is a path in G which connects a and b without passing through c. In particular, this shows that $\mathcal{D}$ is path connected and $\mathcal{D}$ is not homeomorphic to any subset of $\mathbb{R}$.

Problem 4.15.

§4.5 Uniform Continuity

Problem 4.16. Check if the following functions on uniformly continuous.

1. $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \sin \log x$.

2. $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = x \sin \dfrac{1}{x}$.

3. $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$.

4. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos(x^2)$.

5. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos^3 x$.

6. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x \sin x$.

Problem 4.17. Find all positive numbers a and b such that the function $f(x) = \dfrac{\sin(x^a)}{1 + x^b}$ is uniformly continuous on $[0, \infty)$.

Problem 4.18. Find all positive numbers a and b such that the function $f(x, y) = |x|^a |y|^b$ is uniformly continuous on $\mathbb{R}^2$.

Problem 4.19. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous, and $\lim\limits_{|x|\to\infty} f(x) = b$ exists for some $b \in \mathbb{R}^m$. Show that f is uniformly continuous on $\mathbb{R}^n$.

Problem 4.20. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous. Show that there exists $a > 0$ and $b > 0$ such that $\|f(x)\|_{\mathbb{R}^m} \leqslant a\|x\|_{\mathbb{R}^n} + b$.

Problem 4.21. Let $f(x) = \dfrac{q(x)}{p(x)}$ be a rational function define on $\mathbb{R}$, where p and q are two polynomials. Show that f is uniformly continuous on $\mathbb{R}$ if and only if the degree of q is not more than the degree of p plus 1.

Problem 4.22. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function; that is, $\exists\, p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$ (and f is continuous). Show that f is uniformly continuous on $\mathbb{R}$.

Problem 4.23. Let $(a, b) \subseteq \mathbb{R}$ be an open interval, and $f : (a, b) \to \mathbb{R}^m$ be a function. Show that the following three statements are equivalent.

1. f is uniformly continuous on (a, b).
2. f is continuous on (a, b), and both limits $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} f(x)$ exist.
3. For all $\varepsilon > 0$, there exists $N > 0$ such that $\big|f(x) - f(y)\big| < \varepsilon$ whenever $\left|\dfrac{f(x) - f(y)}{x - y}\right| > N$.

Problem 4.24. Suppose that $f : [a, b] \to \mathbb{R}$ is <u>**Hölder continuous with exponent** α</u>; that is, there exist $M > 0$ and $\alpha \in (0, 1]$ such that

$$|f(x_1) - f(x_2)| \leqslant M|x_1 - x_2|^\alpha \qquad \forall\, x_1, x_2 \in [a, b]\,.$$

Show that f is uniformly continuous on $[a, b]$. Show that $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is Hölder continuous with exponent $\dfrac{1}{2}$.

Problem 4.25. A function $f : A \times B \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}^p$, is said to be separately continuous if for each $x_0 \in A$, the map $g(y) = f(x_0, y)$ is continuous and for

$y_0 \in B$, $h(x) = f(x, y_0)$ is continuous. f is said to be continuous on A uniformly with respect to B if

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni \big\|f(x,y) - f(x_0,y)\big\|_2 < \varepsilon \text{ whenever } \|x - x_0\|_2 < \delta \text{ and } y \in B\,.$$

Show that if f is separately continuous and is continuous on A uniformly with respect to B, then f is continuous on $A \times B$.

Problem 4.26. Let (M, d) be a metric space, $A \subseteq M$, and $f, g : A \to \mathbb{R}$ be uniformly continuous on A. Show that if f and g are bounded, then fg is uniformly continuous on A. Does the conclusion still hold if f or g is not bounded?

§4.6 Differentiation of Functions of One Variable

Problem 4.27. Show that $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$ if and only if there exists $m \in \mathbb{R}$, denoted by $f'(x_0)$, such that

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni \big|f(x) - f(x_0) - f'(x_0)(x - x_0)\big| \leqslant \varepsilon|x - x_0| \quad \text{whenever} \quad |x - x_0| < \delta\,.$$

Problem 4.28. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable, and $f \geqslant 0$. Find $\dfrac{d}{dx} f(x)^{g(x)}$.

Problem 4.29. Suppose α and β are real numbers, $\beta > 0$ and $f : [-1, 1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x^\alpha \sin(x^{-\beta}) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Prove the following statements.

1. f is continuous if and only if $\alpha > 0$.

2. $f'(0)$ exists if and only if $\alpha > 1$.

3. f' is bounded if and only if $\alpha \geqslant 1 + \beta$.

4. f' is continuous if and only if $\alpha > 1 + \beta$.

5. $f''(0)$ exists if and only if $\alpha > 2 + \beta$.

6. f'' is bounded if and only if $\alpha \geqslant 2 + 2\beta$.

7. f'' is continuous if and only if $\alpha > 2 + 2\beta$.

Problem 4.30 (The inverse statement of the chain rule)**.** Let $f : (a, b) \to \mathbb{R}$ be continuous and $g : (c, d) \to \mathbb{R}$ be differentiable at $y_0 = f(x_0) \in (c, d)$. Show that if $(g \circ f)$ is differentiable at x_0 and $g'(y_0) \neq 0$, then f is differentiable at x_0.

Problem 4.31. Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial, and f has a double root at a and b. Show that $f'(x)$ has at least three roots in $[a, b]$.

Problem 4.32. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Assume that for all $x \in \mathbb{R}$, $0 \leqslant f'(x) \leqslant f(x)$. Show that $g(x) = e^{-x} f(x)$ is decreasing. If f vanishes at some point, conclude that f is zero.

Problem 4.33. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable. Suppose that $f(x + h) - f(x) = hf'(x + \theta h)$ for all $x, h \in \mathbb{R}$, where θ is independent of h. Show that f is a quadratic polynomial.

Problem 4.34. Let f be a differentiable function defined on some interval I of $\mathbb{R}$. Prove that f' maps connected subsets of I into connected set; that is, f' has the intermediate value property.

Problem 4.35. Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial, and f has a double root at a and b. Show that $f'(x)$ has at least three roots in $[a, b]$.

Problem 4.36. Let $f : [-1, 1] \to \mathbb{R}$ be a function such that $x^2 + f(x)^2 = 1$ for all $|x| \leqslant 1$. Define $C = \big\{x \,\big|\, |x| \leqslant 1, f \text{ is continuous at } x\big\}$. Show that C contains at least 2 points and $C \cap (-1, 1)$ is an open set. Hence if f is continuous at more than 2 points, it is continuous at uncountably many points.

Problem 4.37. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Suppose that $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = 0$, $g'(x) \neq 0$ for all $x \in \mathbb{R}$, and the limit $\lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)}$ exists. Show that the limit $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)}$ also exists, and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \,.$$

Problem 4.38. Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions. Show that if $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = \infty$, $g'(x) \neq 0$ for all $x \in (a, b)$, and the limit $\lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x\to a^+} \dfrac{f(x)}{g(x)}$

exists and

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}\,. \qquad (\star)$$

Hint: Let $L = \lim\limits_{x\to a^+}\dfrac{f'(x)}{g'(x)}$ and $\epsilon > 0$ be given. Choose $c \in (a,b)$ such that

$$\Big|\frac{f'(x)}{g'(x)} - L\Big| < \frac{\epsilon}{2} \qquad \forall\, a < x < c\,.$$

Then for $a < x < c$, the Cauchy mean value theorem implies that for some $\xi \in (x,c)$ such that

$$\frac{f(x)-f(c)}{g(x)-g(c)} = \frac{f'(\xi)}{g'(\xi)}\,.$$

Show that there exists $\delta > 0$ such that $a+\delta < c$ and

$$\Big|\frac{f(x)-f(c)}{g(x)-g(c)} - \frac{f(x)}{g(x)}\Big| < \frac{\epsilon}{2} \qquad \forall\, x \in (a, a+\delta)$$

and then conclude $(\star)$.

Problem 4.39. Let $f : (a,b) \to \mathbb{R}$ be k-times differentiable, and $c \in (a,b)$. Let $h_k : (a,b) \to \mathbb{R}$ be given by

$$h_k(x) = f(x) - \sum_{j=0}^{k}\frac{f^{(j)}(c)}{j!}(x-c)^j\,.$$

Show that $\lim\limits_{x\to c}\dfrac{h_k(x)}{(x-c)^k} = 0$.

Problem 4.40. Two metric spaces (M,d) and (N,ρ) are called **homeomorphics** if there exists a continous map $f : M \to N$, called a **homeomorphism** between M and N, such that f is one-to-one and onto, and its inverse f^{-1} is also continuous. Homeomorphic metric spaces have the same topological properties. In the following problems, (M,d) and (N,ρ) are two metric spaces.

1. Suppose that M is compact, and $f : M \to N$ is one-to-one and onto. Show that f is a homeomorphism between M and N.

2. Suppose that f is a homeomorphism between M and N. Show that the restriction of f to any subset $A \subseteq M$ establishes a homeomorphism between A and $f(A)$.

3. Determine which of the following pairs of metric spaces is homeomorphic.

(a) $M = (a, b] \subseteq \mathbb{R}$ and $N = \mathbb{R}$.

(b) M is an open ball in $\mathbb{R}^n$ and $N = \mathbb{R}^n$.

(c) $M = \mathbb{R}$ and $N = \mathbb{R}^n$.

(d) $M = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$ and $N = [0, 1] \subseteq \mathbb{R}$.

(e) $M = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$ and $N = [0, 1] \subseteq \mathbb{R}$.

(f) $M = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$ and $N = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + xy + y^2 = 1\}$.

(g) $M = \mathbb{R}^2$ and $N = \mathbb{R}^3$.

4. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a one-to-one continuous function. Show that f must be strictly monotonic in I and f is a homeomorphism between I and $f(I)$.

   If $I \subseteq \mathbb{R}^n$ for $n > 1$ and $f : I \to \mathbb{R}^n$ is continuous and one-to-one, can we still assert that f is homeomorphism between I and $f(I)$?

§4.7 Integration of Functions of One Variable

Problem 4.41. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and $\mathcal{P}_n$ denote the division of $[a, b]$ into 2^n equal sub-intervals. Show that f is Riemann integrable over $[a, b]$ if and only if

$$\lim_{n\to\infty} U(f, \mathcal{P}_n) = \lim_{n\to\infty} L(f, \mathcal{P}_n)\,.$$

Problem 4.42. Let $f, g : [a, b] \to \mathbb{R}$ be functions, where g is continuous, and f be non-negative, bounded, Riemann integrable over $[a, b]$. Show that

1. fg is Riemann integrable.

2. $\exists\, x_0 \in (a, b)$ such that

$$\int_a^b f(x)g(x)dx = g(x_0)\int_a^b f(x)dx\,.$$

Problem 4.43. Let $f : [a, b] \to \mathbb{R}$ be differentiable and assume that f' is Riemann integrable. Prove that $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$.

Hint: Use the Mean Value Theorem.

Problem 4.44. Suppose that $f:[a,b]\to\mathbb{R}$ is Riemann integrable, $m\leqslant f(x)\leqslant M$ for all $x\in[a,b]$, and $\varphi:[m,M]\to\mathbb{R}$ is continuous. Show that $\varphi\circ f$ is Riemann integrable on $[a,b]$.

Problem 4.45 (True or False). Determine whether the following statements are true or false. If it is true, prove it. Otherwise, give a counter-example.

1. Let $f:\mathbb{R}^2\backslash\{(0,0)\}\to\mathbb{R}$ satisfy $\lim\limits_{x\to 0} f(x,ax^n)=0$ for all $a\in\mathbb{R}$, $n\in\mathbb{N}$ and $\lim\limits_{y\to 0} f(0,y)=0$. Then $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$.
2. There exists a function $f:\mathbb{R}\to\mathbb{R}$ which is continuous only at three points of $\mathbb{R}$.
3. Let $f:\mathbb{R}\to\mathbb{R}$. Then f is continuous on $\mathbb{R}$ if and only if its graph $\{(x,f(x))\,|\,x\in\mathbb{R}\}$ is closed in $\mathbb{R}^2$.
4. Let I_1 and I_2 be open intervals in $\mathbb{R}$. Then $f:I_1\to I_2$ is a diffeomorphism if and only if f is differentiable and $f'(x)\neq 0$ for all $x\in I_1$.
5. Let $f:[a,b]\to\mathbb{R}$ be a function. If f^2 is Riemann integrable, then f is Riemann integrable.
6. Let $f:[a,b]\to\mathbb{R}$ be a function. If f is Riemann integrable, then $\sqrt[3]{f}$ is Riemann integrable.
7. Let $f(x)=\sin\dfrac{1}{x}$ be defined on $(0,1]$. Then no matter how we define $f(0)$, f is always Riemann integrable on $[0,1]$.