Chapter 3

Compact and Connected Sets

3.1 Compactness（緊緻性）

Definition 3.1. Let (M, d) be a metric space. A subset $K \subseteq M$ is called ***sequentially compact*** if every sequence in K has a subsequence that converges to a point in K.

Example 3.2. Any closed and bounded set in $(\mathbb{R}, |\cdot|)$ is sequentially compact.

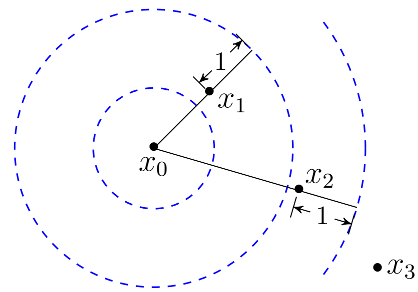
Proof. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in a closed and bounded set S. Then $\{x_k\}_{k=1}^{\infty}$ is also bounded; thus by Bolzano-Weierstrass property of $\mathbb{R}$, there exists a subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ converging to a point $x \in \mathbb{R}$. Since S is closed, $x \in S$; thus S is sequentially compact. □

Proposition 3.3. *Let (M, d) be a metric space, and $K \subseteq M$ be sequentially compact. Then K is closed and bounded.*

Proof. For closedness, assume that $\{x_k\}_{k=1}^{\infty} \subseteq K$ and $x_k \to x$ as $k \to \infty$. By the definition of sequential compactness, there exists $\{x_{k_j}\}_{j=1}^{\infty}$ converging to a point $y \in K$. By Proposition 2.72, $x = y$; thus $x \in K$.

For boundedness, assume the contrary that $\forall (x_0, B) \in M \times \mathbb{R}^+$, there exists $y \in K$ such that $d(x_0, y) > B$. In particular, there exists

$$x_k \in K, d(x_k, x_0) > 1 + d(x_{k-1}, x_0) \quad \forall k \in \mathbb{N}.$$



Then any subsequence of $\{x_k\}_{k=1}^{\infty}$ cannot be Cauchy since $d(x_k, x_\ell) > 1$ for all $k, \ell \in \mathbb{N}$; thus $\{x_k\}_{k=1}^{\infty}$ has no convergent subsequence, a contradiction. □

Remark 3.4. Example 3.2 and Proposition 3.3 together suggest that in $(\mathbb{R}, |\cdot|)$,

$$\text{sequentially compact} \Leftrightarrow \text{closed and bounded}.$$

Corollary 3.5. *If $K \subseteq \mathbb{R}$ is sequentially compact, then $\inf K \in K$ and $\sup K \in K$.*

Proof. By Proposition 3.3, K must be closed and bounded. Therefore, $\inf K \in \mathbb{R}$. Then for each $n \in \mathbb{N}$, there exists $x_n \in K$ such that $\inf K \leqslant x_n < \inf K + \frac{1}{n}$. Since $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$, the Bolzano-Weierstrass theorem (Theorem 1.100) implies that there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and $x \in \mathbb{R}$ such that $\lim_{k\to\infty} x_{n_k} = x$. Note that $x = \inf K$, and by the closedness of K, $x \in K$. The proof of $\sup K \in K$ is similar. □

Definition 3.6. Let (M, d) be a metric space, and $A \subseteq M$. A ***cover*** of A is a collection of sets $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ whose union contains A; that is,

$$A \subseteq \bigcup_{\alpha\in I} \mathcal{U}_\alpha .$$

It is an ***open cover*** of A if $\mathcal{U}_\alpha$ is open for all $\alpha \in I$. A ***subcover*** of a given cover is a sub-collection $\{\mathcal{U}_\alpha\}_{\alpha\in J}$ of $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ whose union also contains A; that is,

$$A \subseteq \bigcup_{\alpha\in J} \mathcal{U}_\alpha , \quad J \subseteq I .$$

It is a ***finite subcover*** if $\#J < \infty$.

Definition 3.7. Let (M, d) be a metric space. A subset $K \subseteq M$ is called ***compact*** if every open cover of K possesses a finite subcover; that is, $K \subseteq M$ is compact if

$$\forall \text{ open cover } \{\mathcal{U}_\alpha\}_{\alpha\in I} \text{ of } K, \exists\, J \subseteq I, \#J < \infty \ni K \subseteq \bigcup_{\alpha\in J} \mathcal{U}_\alpha .$$

Example 3.8. Consider $\mathbb{R} \times \{0\}$ in the normed space $(\mathbb{R}^2, \|\cdot\|_2)$. For $x \in \mathbb{R}$, then $\{D\big((x,0),1\big)\}_{x\in\mathbb{R}}$ is an open cover of $\mathbb{R} \times \{0\}$; that is,

$$\mathbb{R} \times \{0\} \subseteq \bigcup_{x\in\mathbb{R}} D\big((x,0),1\big)\,.$$

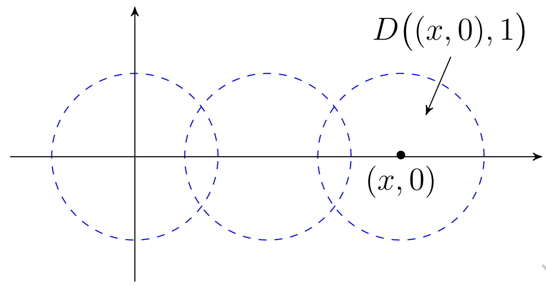


Figure 3.1: An open cover of the x-axis

However, there is no finite subcover; thus $\mathbb{R} \times \{0\}$ is not compact.

Example 3.9. Consider $(0,1]$ in the normed space $(\mathbb{R}, |\cdot|)$. Let $I_k = \big(\frac{1}{k}, 2\big)$. Then $\{I_k\}_{k=1}^{\infty}$ is an open cover of $(0,1]$; that is,

$$(0,1] \subseteq \bigcup_{k=1}^{\infty} \big(\frac{1}{k}, 2\big)\,.$$

However, there is no finite subcover since

$$\frac{1}{N+1} \notin \bigcup_{k=1}^{N} \big(\frac{1}{k}, 2\big)\,.$$

Therefore, $(0,1]$ is not compact.

Lemma 3.10. *Let (M,d) be a metric space, and $K \subseteq M$ be compact. Then K is closed. In other words, compact subsets of metric spaces are closed.*

Proof. Suppose the contrary that $\exists\, \{x_k\}_{k=1}^{\infty} \subseteq K$, $x_k \to x$ as $k \to \infty$, but $x \notin K$. For $y \in K$, define the open ball $\mathcal{U}_y$ by

$$\mathcal{U}_y = D\big(y, \frac{1}{2}d(x,y)\big).$$

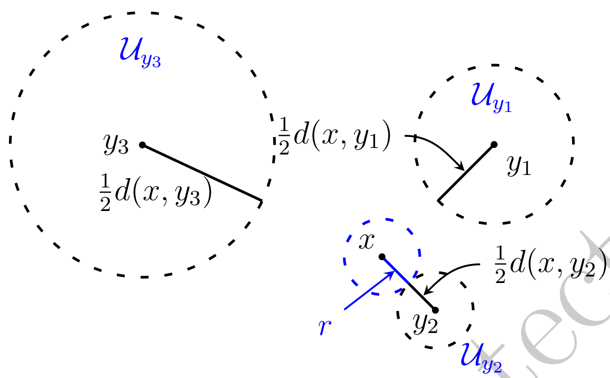
Then $\{\mathcal{U}_y\}_{y\in K}$ is an open cover of K; that is, $K \subseteq \bigcup_{y\in K} \mathcal{U}_y$. Since K is compact, there exist $\{y_1, \cdots, y_n\} \subseteq K$ such that

$$K \subseteq \bigcup_{i=1}^{n} \mathcal{U}_{y_i} = \bigcup_{i=1}^{n} D\big(y_i, \frac{1}{2}d(x,y_i)\big)\,.$$

Let $r = \frac{1}{2}\min\{d(x,y_1),\cdots,d(x,y_n)\} > 0$. Then if $d(x,z) < r$,

$$d(z,y_i) \geqslant d(x,y_i) - d(x,z) > d(x,y_i) - r > d(x,y_i) - \frac{1}{2}d(x,y_i) = \frac{1}{2}d(x,y_i)$$

which implies that $D(x,r) \cap \mathcal{U}_{y_i} = \varnothing$ for all $i = 1,\cdots,n$.



On the other hand, since $x_k \to x$ as $k \to \infty$, $\exists\, N > 0$ such that

$$d(x_k,x) < r \quad \forall\, k \geqslant N\,.$$

In particular, $x_N \in D(x,r) \cap K$; thus $x_N \notin \mathcal{U}_{y_i}$ for all $i = 1,\cdots,n$, which contradicts to that $\{\mathcal{U}_{y_i}\}_{i=1}^n$ is a cover of K. □

Lemma 3.11. *Let (M,d) be a metric space, and $K \subseteq M$ be compact. If $F \subseteq K$ is closed, then F is compact. In other words, closed subsets of compact sets are compact.*

Proof. Let $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ be an open cover of F. Then $\{\mathcal{U}_\alpha\}_{\alpha\in I} \cup \{F^\complement\}$ is an open cover of K; thus possessing a finite subcover of K. Therefore, we must have

$$K \subseteq \bigcup_{i=1}^n \mathcal{U}_{\alpha_i} \cup F^\complement$$

for some $\alpha_i \in I$. In particular, $F \subseteq \bigcup_{i=1}^n \mathcal{U}_{\alpha_i} \cup F^\complement$, so $F \subseteq \bigcup_{i=1}^n \mathcal{U}_{\alpha_i}$. □

Definition 3.12. Let (M,d) be a metric space. A subset $A \subseteq M$ is called ***totally bounded*** if for each $r > 0$, there exists $\{x_1,\cdots,x_N\} \subseteq M$ such that

$$A \subseteq \bigcup_{i=1}^N D(x_i,r)\,.$$

Proposition 3.13. *Let (M,d) be a metric space, and $A \subseteq M$ be totally bounded. Then A is bounded. In other words, totally bounded sets are bounded.*

Proof. By total boundedness, there exists $\{y_1,\cdots,y_N\} \subseteq M$ such that $A \subseteq \bigcup_{i=1}^{N} D(y_i,1)$. Let $x_0 = y_1$ and $R = \max\{d(x_0,y_2),\cdots,d(x_0,y_N)\} + 1$. Then if $z \in A$, $z \in D(y_j,1)$ for some $j = 1,\cdots,N$, and

$$d(z,x_0) \leqslant d(z,y_j) + d(y_j,x_0) < 1 + d(x_0,y_j) \leqslant R$$

which implies that $A \subseteq D(x_0,R)$. Therefore, A is bounded. □

Example 3.14. In a general metric space (M,d), a bounded set might not be totally bounded. For example, consider the metric space (M,d) with the discrete metric, and $A \subseteq M$ be a set having infinitely many points. Then A is bounded since $A \subseteq D(x,2)$ for any $x \in M$; however, A is not totally bounded since A cannot be covered by finitely many balls with radius $\frac{1}{2}$.

Example 3.15. Every bounded set in $(\mathbb{R}^n, \|\cdot\|_2)$ is totally bounded **(Check!)**. In particular, the set $\{1\}\times[1,2]$ in $(\mathbb{R}^2,\|\cdot\|)$ is totally bounded.

On the other hand, let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(x,y) = \begin{cases} |x_1 - y_1| & \text{if } x_2 = y_2\,, \\ |x_1-y_1| + |x_2-y_2| + 1 & \text{if } x_2 \neq y_2\,. \end{cases} \quad \text{where } x=(x_1,x_2) \text{ and } y=(y_1,y_2).$$

Then $(\mathbb{R}^2,d)$ is also a metric space (exercise). The set $\{1\}\times[1,2]$ is not totally bounded. In fact, consider open ball with radius $\frac{1}{2}$:

$$\begin{aligned} y \in D\big(x,\tfrac{1}{2}\big) &\Leftrightarrow d(x,y) < \tfrac{1}{2} \Leftrightarrow |x_1-y_1| < \tfrac{1}{2} \text{ and } x_2 = y_2 \\ &\Leftrightarrow y_1 \in \big(x_1 - \tfrac{1}{2}, x_1 + \tfrac{1}{2}\big) \text{ and } x_2 = y_2\,. \end{aligned}$$

In other words,

$$D\big(x,\tfrac{1}{2}\big) = \big(x_1-\tfrac{1}{2}, x_1+\tfrac{1}{2}\big) \times \{x_2\}\,;$$

thus one cannot cover $\{1\}\times[1,2]$ by the union of finitely many balls with radius $\frac{1}{2}$.

Proposition 3.16. *Let (M,d) be a metric space, and $T \subseteq M$ be totally bounded. If $S \subseteq T$, then S is totally bounded. In other words, subsets of totally bounded sets are totally bounded.*

Proof. Let $r > 0$ be given. By the total boundedness of T, there exists $\{x_1, \cdots, x_N\} \subseteq M$ such that
$$S \subseteq T \subseteq \bigcup_{i=1}^{N} D(x_i, r) .$$
□

Proposition 3.17. *Let (M, d) be a metric space, and $A \subseteq M$. Then A is totally bounded if and only if $\forall\, r > 0$, $\exists \{y_1, \cdots, y_N\} \subseteq A$ such that $A \subseteq \bigcup_{i=1}^{N} D(y_i, r)$.*

Proof. It suffices to show the "only if" part. Let $r > 0$ be given. Since A is totally bounded,
$$\exists\, \{y_1, \cdots, y_N\} \subseteq M \ni A \subseteq \bigcup_{i=1}^{N} D\big(y_i, \frac{r}{2}\big) .$$
W.L.O.G., we may assume that for each $i = 1, \cdots, N$, $D\big(y_i, \frac{r}{2}\big) \cap A \neq \varnothing$. Then for each $i = 1, \cdots, N$, there exists $x_i \in D\big(y_i, \frac{r}{2}\big) \cap A$ which suggests that
$$A \subseteq \bigcup_{i=1}^{N} D\big(y_i, \frac{r}{2}\big) \subseteq \bigcup_{i=1}^{N} D(x_i, r)$$
since $D\big(y_i, \frac{r}{2}\big) \subseteq D(x_i, r)$ for all $i = 1, \cdots, N$. □

Lemma 3.18. *Let (M, d) be a metric space, and $K \subseteq M$. If K is either compact or sequentially compact, then K is totally bounded..*

Proof. Suppose first that K is compact. Let $r > 0$ be given, then $\big\{D(x, r)\big\}_{x \in K}$ is an open cover of K. Since K is compact, there exists a finite subcover; thus $\exists\, \{x_1, \cdots, x_N\} \subseteq K$ such that
$$K \subseteq \bigcup_{i=1}^{N} D(x_i, r) .$$
Therefore, K is totally bounded.

Now we assume that K is sequentially compact. Suppose the contrary that there is an $r > 0$ such that any finite set $\{y_1, \cdots, y_n\} \subseteq K$, $K \nsubseteq \bigcup_{i=1}^{n} D(y_i, r)$. This implies that we can choose a sequence $\{x_k\}_{k=1}^{\infty} \subseteq K$ such that
$$x_{k+1} \in K \backslash \bigcup_{i=1}^{k} D(x_i, r) .$$
Then $\{x_k\}_{k=1}^{\infty}$ is a sequence in K without convergent subsequence since $d(x_k, x_\ell) > r$ for all $k, \ell \in \mathbb{N}$. □

Theorem 3.19. *Let (M, d) be a metric space, and $K \subseteq M$. Then the following three statements are equivalent:*

1. *K is compact.*

2. *K is sequentially compact.*

3. *K is totally bounded and (K, d) is complete.*

Proof. We show that $1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$ to conclude the theorem.

"$1 \Rightarrow 3$": By Lemma 3.18, it suffices to show the completeness of (K, d). Let $\{x_k\}_{k=1}^\infty$ be a Cauchy sequence in K. Suppose that $\{x_k\}_{k=1}^\infty$ does not converge in K. Then

$$\forall\, y \in K, \exists\, \delta_y > 0 \ni \#\big\{k \in \mathbb{N} \,|\, x_k \in D(y, \delta_y)\big\} < \infty \tag{3.1.1}$$

for otherwise there is a subsequence of $\{x_k\}_{k=1}^\infty$ that converges to x which will suggests the convergence of the Cauchy sequence. The collection $\big\{D(y, \delta_y)\big\}_{y \in K}$ then is an open cover of K; thus possesses a finite subcover $\big\{D\big(y_i, \delta_{y_i}\big)\big\}_{i=1}^N$. In particular, $\{x_k\}_{k=1}^\infty \subseteq \bigcup\limits_{i=1}^N D\big(y_i, \delta_{x_i}\big)$ or

$$\#\Big\{k \in \mathbb{N} \,\Big|\, x_k \in \bigcup_{i=1}^N D(y_i, \delta_{y_i})\Big\} = \infty$$

which contradicts to (3.1.1).

"$3 \Rightarrow 2$": The proof of this step is similar to the proof of the Bolzano-Weierstrass Theorem in $\mathbb{R}$ (Theorem 1.100) that we proceed as follows. Let $\{x_k\}_{k=1}^\infty$ be a sequence in $T_0 \equiv K$. Since K is totally bound, there exist $\big\{y_1^{(1)}, \cdots y_{N_1}^{(1)}\big\} \subseteq K$ such that

$$T_0 \equiv K \subseteq \bigcup_{i=1}^{N_1} D(y_i^{(1)}, 1)\,.$$

One of these $D(y_i^{(1)}, 1)$'s must contain infinitely many x_k's; that is, $\exists\, 1 \leqslant \ell_1 \leqslant N_1$ such that $\#\big\{k \in \mathbb{N} \,\big|\, x_k \in D(y_{\ell_1}^{(1)}, 1)\big\} = \infty$. Define $T_1 = K \cap D(y_{\ell_1}^{(1)}, 1)$. Then T_1 is also totally bounded by Proposition 3.16, so there exist $\big\{y_1^{(2)}, \cdots, y_{N_2}^{(2)}\big\} \subseteq T_1$ such that

$$T_1 \subseteq \bigcup_{i=1}^{N_2} D\big(y_i^{(2)}, \frac{1}{2}\big)\,.$$

Suppose that $\#\big\{k\in\mathbb{N}\,\big|\,x_k\in D\big(y^{(2)}_{\ell_2},\frac{1}{2}\big)\big\}=\infty$ for some $1\leqslant \ell_2\leqslant N_2$. Define $T_2=T_1\cap D\big(y^{(2)}_{\ell_2},\frac{1}{2}\big)$. We continue this process, and obtain that for all $n\in\mathbb{N}$,

(1) $\exists\,\big\{y^{(n)}_1,\cdots,y^{(n)}_{N_n}\big\}\subseteq T_{n-1}$ such that

$$T_{n-1}\subseteq\bigcup_{i=1}^{N_n}D\big(y^{(n)}_i,\frac{1}{n}\big)\,.$$

(2) $T_n=T_{n-1}\cap D(y^{(n)}_{\ell_n},\frac{1}{n})$, where $1\leqslant \ell_n\leqslant N_n$ is chosen so that

$$\#\big\{k\in\mathbb{N}\,\big|\,x_k\in D\big(y^{(n)}_{\ell_n},\frac{1}{n}\big)\big\}=\infty\,. \tag{3.1.2}$$

Pick an $k_1\in\big\{k\in\mathbb{N}\,\big|\,x_k\in D\big(y^{(1)}_{\ell_1},1\big)\big\}$, and $k_j\in\big\{k\in\mathbb{N}\,\big|\,x_k\in D\big(y^{(j)}_{\ell_j},\frac{1}{j}\big)\big\}$ such that $k_j>k_{j-1}$ for all $j\geqslant 2$. We note such k_j always exists because of (3.1.2). Then $\big\{x_{k_j}\big\}_{j=1}^{\infty}$ is a subsequence of $\{x_k\}_{k=1}^{\infty}$, and $x_{k_j}\in T_j\subseteq K$ for all $j\in\mathbb{N}$.

Claim: $\big\{x_{k_j}\big\}_{j=1}^{\infty}$ is a Cauchy sequence.

Proof of claim: Let $\varepsilon>0$ be given, and $N>0$ be large enough so that $\frac{1}{N}<\frac{\varepsilon}{2}$. Since if $j\geqslant N$, we must have $x_{k_j}\in D\big(y^{(N)}_{\ell_N},\frac{1}{N}\big)$, we conclude that if $n,m\geqslant N$, by triangle inequality

$$d\big(x_{k_n},x_{k_m}\big)\leqslant d\big(x_{k_n},y^{(N)}_{\ell_N}\big)+d\big(x_{k_m},y^{(N)}_{\ell_N}\big)<\frac{1}{N}+\frac{1}{N}<\varepsilon.$$

Since (K,d) is complete, the Cauchy sequence $\big\{x_{k_j}\big\}_{j=1}^{\infty}$ converges to a point in K.

"$2\Rightarrow 1$": Let $\big\{\mathcal{U}_\alpha\big\}_{\alpha\in I}$ be an open cover of K.

Claim: there exists $r>0$ such that for each $x\in K$, $D(x,r)\subseteq\mathcal{U}_\alpha$ for some $\alpha\in I$.

Proof of claim: Suppose the contrary that for all $k>0$, there exists $x_k\in K$ such that $D\big(x_k,\frac{1}{k}\big)\nsubseteq\mathcal{U}_\alpha$ for all $\alpha\in I$. Then $\{x_k\}_{k=1}^{\infty}$ is a sequence in K; thus by the assumption of sequential compactness, there exists a subsequence $\big\{x_{k_j}\big\}_{j=1}^{\infty}$ converging in K. Suppose that $x_{k_j}\to x$ as $j\to\infty$, and $x\in\mathcal{U}_\beta$ for some $\beta\in I$. Then

(1) there is $r>0$ such that $D(x,r)\subseteq\mathcal{U}_\beta$ since $\mathcal{U}_\beta$ is open.

(2) there exists $N>0$ such that $d(x_{k_j},x)<\frac{r}{2}$ for all $j\geqslant N$.

Choose $j \geqslant N$ such that $\dfrac{1}{k_j} < \dfrac{r}{2}$. Then $D\big(x_{k_j}, \dfrac{1}{k_j}\big) \subseteq D(x,r) \subseteq \mathcal{U}_\beta$, a contradiction.

By Lemma 3.18, there exists $\{x_1, \cdots, x_N\} \subseteq K$ such that $K \subseteq \bigcup\limits_{i=1}^{N} D(x_i, r)$. For each $1 \leqslant i \leqslant N$, the claim above implies that there exists $\alpha_i \in I$ such that $D(x_i, r) \subseteq \mathcal{U}_{\alpha_i}$. Then $\bigcup\limits_{i=1}^{N} D(x_i, r) \subseteq \bigcup\limits_{i=1}^{N} \mathcal{U}_{\alpha_i}$ which suggests that

$$K \subseteq \bigcup_{i=1}^{N} \mathcal{U}_{\alpha_i} . \qquad \square$$

Remark 3.20.

1. The equivalency between 1 and 2 is sometimes called the Bolzano-Weistrass Theorem.
2. A number $r > 0$ satisfying the claim in the step "2 ⇒ 1" is called a Lebesgue number for the cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$. The supremum of all such r is called the ***Lebesgue number*** for the cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$.

Alternative Proof of Theorem 3.19. In this proof we show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$ to conclude the theorem.

"1 ⇒ 2": Assume the contrary that K is not sequentially compact. Then there is a sequence $\{x_k\}_{k=1}^{\infty} \subseteq K$ that does not have a convergent subsequence with a limit in K. Therefore, for each $x \in K$, there exists $\delta_x > 0$ such that

$$\#\big\{k \in \mathbb{N} \,\big|\, x_k \in D(x, \delta_x)\big\} < \infty$$

for otherwise x is a cluster point of $\{x_k\}_{k=1}^{\infty}$ so Proposition 2.72 guarantees the existence of a subsequence of $\{x_k\}_{k=1}^{\infty}$ converging to x. Since $\big\{D(x, \delta_x)\big\}_{x \in K}$ is an open cover of K, by the compactness of K there exists $\{y_1, \cdots, y_N\} \subseteq K$ such that

$$\{x_k\}_{k=1}^{\infty} \subseteq K \subseteq \bigcup_{i=1}^{N} D\big(y_i, \delta_{y_i}\big)$$

while this is impossible since $\#\big\{k \in \mathbb{N} \,\big|\, x_k \in D\big(y_i, \delta_{y_i}\big)\big\} < \infty$ for all $i = 1, \cdots N$.

"2 ⇒ 3": By Lemma 3.18, it suffices to show that (K, d) is complete. Let $\{x_k\}_{k=1}^{\infty} \subseteq K$ be a Cauchy sequence. By sequential compactness of K, there is a subsequence $\big\{x_{k_j}\big\}_{j=1}^{\infty}$ converging to a point $x \in K$. By Proposition 2.81, $\{x_k\}_{k=1}^{\infty}$ also converges to x; thus every Cauchy sequence in (K, d) converges to a point in K.

"3 ⇒ 1": We first prove the following

Claim: If $\{\mathcal{V}_\alpha\}_{\alpha\in I}$ is an open cover of a totally bounded set A such that there is no finite subcover, then for all $r > 0$, there exists $x \in A$ such that $A \cap D(x,r)$ does not admit a finite subcover.

Proof of claim: Let $r > 0$ be given. Since A is totally bounded, by Proposition 3.17 there exists $\{a_1, \cdots, a_N\} \subseteq A$ such that $A \subseteq \bigcup_{j=1}^{N} D(a_j, r)$. If for each $j = 1, \cdots, N$, $A \cap D(a_j, r)$ can be covered by finitely many $\mathcal{V}_\alpha$'s, then A itself can be covered by finitely many $\mathcal{V}_\alpha$'s, a contradiction. Therefore, at least one $A \cap D(a_j, r)$ does not admit a finite subcover.

Now assume the contrary that there exists an open cover $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ of K such that there is no finite subcover. Let $\varepsilon_n = 2^{-n}$. Since K is totally bounded, by the claim there exists $x_1 \in K$ such that $K \cap D(x_1, \varepsilon_1)$ which does not admit a finite subcover. By Proposition 3.16, $K \cap D(x_1, \varepsilon_1)$ is totally bounded, so there must be an $x_2 \in K \cap D(x_1, \varepsilon_1)$ such that $K \cap D(x_1, \varepsilon_1) \cap D(x_2, \varepsilon_2)$ cannot be covered by the union of finitely many $\mathcal{U}_\alpha$. We continuous this process, and obtain a sequence $\{x_k\}_{k=1}^{\infty}$ such that

(1) $x_{k+1} \in K \cap \bigcap_{i=1}^{k} D(x_i, \varepsilon_i)$ (which implies that $d(x_{k+1}, x_k) < \varepsilon_k$);

(2) $K \cap \bigcap_{i=1}^{k} D(x_i, \varepsilon_i)$ cannot be covered by the union of finitely many $\mathcal{U}_\alpha$.

Then similar to Example 1.105, we find that $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence in (K, d). By the completeness of K, $x_k \to x$ as $k \to \infty$ for some $x \in K$.

Since $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ is an open cover of K, $x \in \mathcal{U}_\beta$ for some $\beta \in I$. Since $\mathcal{U}_\beta$ is open, $\exists\, r > 0$ such that $D(x,r) \subseteq \mathcal{U}_\beta$. For this particular r, there exists $N > 0$ such that $d(x_k, x) < \frac{r}{2}$. Therefore, if $k \geqslant N$ such that $\varepsilon_k < \frac{r}{2}$,

$$D(x_k, \varepsilon_k) \subseteq D(x, r) \subseteq \mathcal{U}_\beta$$

which contradicts to (2). □

Example 3.21. Let (M, d) be a metric space, and $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence with limit x. Let $A = \{x_1, x_2, \cdots\} \cup \{x\}$. Then A is compact.

Definition 3.22. Let (M, d) be a metric space. A subset $A \subseteq M$ is called ***pre-compact*** if $\bar{A}$ is compact. Let $\mathcal{U} \subseteq M$ be an open set, a subset A of $\mathcal{U}$ is said to be ***compactly contained*** in $\mathcal{U}$, denoted by $A \subset\subset \mathcal{U}$, if A is pre-compact and $\bar{A} \subseteq \mathcal{U}$.

Example 3.23. Let (M, d) be a complete metric space, and $A \subseteq M$ be totally bounded. Then $\bar{A}$ is compact. In other words, in a complete metric space, totally bounded sets are pre-compact.
(Hint: Use the total boundedness equivalence to show compactness.)

Definition 3.24. Let (M, d) be a metric space, and $A \subseteq M$. A collection of closed sets $\{F_\alpha\}_{\alpha \in I}$ is said to have the ***finite intersection property*** for the set A if the intersection of any finite number of F_α with A is non-empty; that is, $\{F_\alpha\}_{\alpha \in I}$ has the finite intersection property for A if

$$A \cap \bigcap_{\alpha \in J} F_\alpha \neq \varnothing \text{ for all } J \subseteq I \text{ and } \#J < \infty.$$

Theorem 3.25. *Let (M, d) be a metric space, and $K \subseteq M$. The K is compact if and only if every collection of closed sets with the finite intersection property for K has non-empty intersection with K; that is,*

$$K \cap \bigcap_{\alpha \in I} F_\alpha \neq \varnothing \textit{ for all } \{F_\alpha\}_{\alpha \in I} \textit{ having the finite intersection property for } K.$$

Proof. It can be proved by contradiction, and is left as an exercise. ▫

Example 3.26. Let $A = (0, 1) \subseteq \mathbb{R}$, and $K_j = \big[-1, \frac{1}{j}\big]$. Take $K_{j_1}, K_{j_2}, \cdots, K_{j_n}$, where $j_1 < j_2 < \cdots < j_n$. Then $\bigcap\limits_{\ell=1}^{n} K_{j_\ell} \cap A = \big[-1, \frac{1}{j_n}\big] \cap (0, 1) \neq \varnothing$. However $x \in \bigcap\limits_{j=1}^{\infty} K_j \Leftrightarrow -1 \leqslant x \leqslant \frac{1}{j}$ for all $j \in \mathbb{N}$. So $\bigcap\limits_{j=1}^{\infty} K_j = [-1, 0]$; thus $\bigcap\limits_{j=1}^{\infty} K_j \cap A = \varnothing$. Therefore, $(0, 1)$ is not compact.

Example 3.27. Let X be the collection of all bounded real sequences; that is,

$$X = \Big\{\{x_k\}_{k=1}^{\infty} \subseteq \mathbb{R} \,\Big|\, \text{for some } M > 0, |x_k| \leqslant M \text{ for all } k\Big\}.$$

The number $\sup\limits_{k \geqslant 1} |x_k| \equiv \sup\{|x_1|, |x_2|, \cdots, |x_k|, \cdots\} < \infty$ is denoted by $\big\|\{x_k\}_{k=1}^{\infty}\big\|$. For example, if $x_k = \frac{(-1)^k}{k}$, then $\big\|\{x_k\}_{k=1}^{\infty}\big\| = 1$. Then $(X, \|\cdot\|)$ is a complete normed space (left as an exercise). Define

$$A = \Big\{\{x_k\}_{k=1}^\infty \in X \,\Big|\, |x_k| \leqslant \frac{1}{k}\Big\},$$
$$B = \Big\{\{x_k\}_{k=1}^\infty \in X \,\Big|\, x_k \to 0 \text{ as } k \to \infty\Big\},$$
$$C = \Big\{\{x_k\}_{k=1}^\infty \in X \,\Big|\, \text{the sequence } \{x_k\}_{k=1}^\infty \text{ converges}\Big\},$$
$$D = \Big\{\{x_k\}_{k=1}^\infty \in X \,\Big|\, \sup_{k\geqslant 1} |x_k| = 1\Big\} \quad (\text{the unit sphere in } (X, \|\cdot\|)).$$

The closedness of A (which implies the completeness of $(A, \|\cdot\|)$) is left as an exercise. We show that A is totally bounded.

Let $r > 0$ be given. Then $\exists\, N > 0 \ni \dfrac{1}{N} < r$. Define

$$E = \Big\{\{x_k\}_{k=1}^\infty \,\Big|\, x_1 = \frac{i_1}{N+1}, x_2 = \frac{i_2}{N+1}, \cdots, x_{N-1} = \frac{i_{N-1}}{N+1} \text{ for some } i_1, \cdots, i_{N-1} = -N, -N+1, \cdots, N-1, N, \text{ and } x_k = 0 \text{ if } k \geqslant N+1\Big\}.$$

Then

1. $\#E < \infty$. In fact, $\#E = (2N+1)^{N-1} < \infty$.
2. $A \subseteq \displaystyle\bigcup_{\{x_k\}_{k=1}^\infty \in E} D\big(\{x_k\}_{k=1}^\infty, \frac{1}{N}\big) \subseteq \bigcup_{\{x_k\}_{k=1}^\infty \in E} D\big(\{x_k\}_{k=1}^\infty, r\big)$.

Therefore, A is totally bounded.

On the other hand, B and C are not compact since they are not bounded; thus not totally bounded by Proposition 3.13. D is bounded but not totally bounded. In fact, D cannot be covered by the union of finitely many balls with radius $\dfrac{1}{2}$ since each ball with radius $\dfrac{1}{2}$ contains at most one of the points from the subset $\Big\{\{x_j^{(k)}\}_{j=1}^\infty\Big\}_{k=1}^\infty \subseteq D$, where for each k

$$\{x_j^{(k)}\}_{j=1}^\infty = \{\, \underbrace{0, \cdots, 0,}_{(k-1) \text{ terms}}\, 1, 0, \cdots \};$$

that is, $x_j^{(k)} = \delta_{kj}$, the kronecker delta.

3.1.1 The Heine-Borel theorem

Theorem 3.28. *In the Euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$, a subset K is compact if and only if it is closed and bounded.*

Proof. By Proposition 3.13 and Theorem 3.19, it is clear that K is closed and bounded if K is compact (in any metric space). It remains to show the direction "$\Leftarrow$". Nevertheless, by Theorem 2.83 closed subsets of a complete metric space must be complete, so it suffices to show that a bounded set in $(\mathbb{R}^n, \|\cdot\|_2)$ is totally bounded.

Let $r > 0$ be given. By the boundedness of K, for some $M > 0$ we have $\|x\|_2 \leqslant M$ for all $x \in K$; thus $K \subseteq [-M, M]^n$. Choose $N > 0$ so that $\dfrac{\sqrt{n}M}{N} < r$, and define

$$E = \Big\{ \big(\frac{Mi_1}{N}, \cdots, \frac{Mi_n}{N}\big) \,\Big|\, i_1, i_2, \cdots, i_n \in \big\{-N, -N+1, \cdots, N-1, N\big\} \Big\}.$$

Then $\#E = (2N+1)^n < \infty$, and

$$K \subseteq [-M, M]^n \subseteq \bigcup_{x \in E} D(x, r). \qquad \square$$

Alternative Proof of "$\Leftarrow$". Let $\{x_k\}_{k=1}^\infty \subseteq K$ be a sequence. Since $K \subseteq \mathbb{R}^n$, we can write $x_k = (x_k^{(1)}, x_k^{(2)}, \cdots, x_k^{(n)}) \in \mathbb{R}^n$. Since K is bounded, then all the sequence $\{x_k^{(j)}\}_{k=1}^\infty$, $j = 1, 2, \cdots, n$, are bounded; that is, $-M_j \leqslant x_k^{(j)} \leqslant M_j$ for all $k \in \mathbb{N}$. Applying the Bolzano-Weierstrass property (Theorem 1.100) to the sequence $\{x_k^{(1)}\}_{k=1}^\infty$, we obtain a sequence $\{x_{k_j}^{(1)}\}_{j=1}^\infty$ with $x_{k_j}^{(1)} \to y^{(1)}$ as $j \to \infty$. Now $\{x_{k_j}^{(2)}\}_{j=1}^\infty$ has a subsequence $\{x_{k_{j_\ell}}^{(2)}\}_{\ell=1}^\infty$ converges, say $x_{k_{j_\ell}}^{(2)} \to y^{(2)}$ as $\ell \to \infty$.

Continuing in this way, we obtain a subsequence of $\{x_k\}_{k=1}^\infty$ that converges to $y = (y^{(1)}, y^{(2)}, \cdots, y^{(n)})$. Since K is close, $y \in K$; thus K is sequentially compact which is equivalent to the compactness of K. $\square$

Corollary 3.29. *A bounded set A in the Euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$ is pre-compact. In particular, if $\{x_k\}_{k=1}^\infty$ is a bounded sequence in $\mathbb{R}^n$, there exists a convergent subsequence $\{x_{k_j}\}_{j=1}^\infty$ (the sentence in blue color is again called the* ***Bolzano-Weierstrass theorem****).*

Example 3.30. Let $A = \{0\} \cup \big\{1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\big\}$. Then A is compact in $(\mathbb{R}, |\cdot|)$.

Example 3.31. Let $A = [0, 1] \cup (2, 3] \subseteq (\mathbb{R}, |\cdot|)$. Since A is not closed, A is not compact.

3.1.2 The nested set property

Theorem 3.32. *Let $\{K_n\}_{n=1}^\infty$ be a sequence of non-empty compact sets in a metric space (M, d) such that $K_n \supseteq K_{n+1}$ for all $n \in \mathbb{N}$. Then there is at least one point in $\bigcap_{n=1}^\infty K_n$; that is,*

$$\bigcap_{n=1}^\infty K_n \neq \varnothing.$$

Proof. Assume the contrary that $\bigcap_{n=1}^{\infty} K_n = \varnothing$. Then $\bigcup_{n=1}^{\infty} K_n^{\complement} = \big(\bigcap_{n=1}^{\infty} K_n\big)^{\complement} = M$. Since $K_n^{\complement}$ is open, $\{K_n^{\complement}\}_{n=1}^{\infty}$ is an open cover of K_1; thus by compactness of K_1, there exists $J \subseteq \mathbb{N}$, $\#J < \infty$ such that

$$K_1 \subseteq \bigcup_{n\in J} K_n^{\complement} = \big(\bigcap_{n\in J} K_n\big)^{\complement}.$$

Therefore, $K_1 \cap \bigcap_{n\in J} K_n = \varnothing$ which implies that $K_{\max J} = \varnothing$, a contradiction. □

Alternative Proof. By assumption, $\{K_n\}_{n=2}^{\infty}$ has the finite intersection property for K_1. Since K_1 is compact, by Theorem 3.25,

$$K_1 \cap \bigcap_{n=2}^{\infty} K_n \neq \varnothing\,.$$ □

Corollary 3.33. *Let $\{\mathcal{U}_k\}_{k=1}^{\infty}$ be a collection of open sets in a metric space (M,d) such that $\mathcal{U}_k \subseteq \mathcal{U}_{k+1}$ for all $k \in \mathbb{N}$ and $\mathcal{U}_k^{\complement}$ is compact. Then $\bigcup_{k=1}^{\infty} \mathcal{U}_k \neq M$.*

Proof. This is proved by letting $K_n = \mathcal{U}_n^{\complement}$, and applying Theorem 3.32. □

Remark 3.34. If the compactness is removed from the condition, then the intersection might be empty. Suppose that the metric space under consideration is $(\mathbb{R}, |\cdot|)$.

1. If the closedness condition is removed, then $\mathcal{U}_k = \big(0, \frac{1}{k}\big)$ has empty intersection.
2. If the boundedness condition is removed, then $F_k = [k,\infty)$ has empty intersection.

3.2 Connectedness（連通性）

Definition 3.35. Let (M,d) be a metric space, and $A \subseteq M$. Two non-empty open sets $\mathcal{U}$ and $\mathcal{V}$ are said to separate A if

1. $A \cap \mathcal{U} \cap \mathcal{V} = \varnothing$; 2. $A \cap \mathcal{U} \neq \varnothing$; 3. $A \cap \mathcal{V} \neq \varnothing$; 4. $A \subseteq \mathcal{U} \cup \mathcal{V}$.

We say that A is ***disconnected*** or ***separated*** if such separation exists, and A is ***connected*** if no such separation exists.

Proposition 3.36. *Let (M,d) be a metric space. A subset $A \subseteq M$ is disconnected if and only if $A = A_1 \cup A_2$ with $A_1 \cap \bar{A}_2 = \bar{A}_1 \cap A_2 = \varnothing$ for some non-empty A_1 and A_2.*

Proof. "⇒" Suppose that there exist $\mathcal{U}$, $\mathcal{V}$ non-empty open sets such that 1-4 in Definition 3.35 hold. Let $A_1 = A \cap \mathcal{U}$ and $A_2 = A \cap \mathcal{V}$. By 1, $A_1 \subseteq \mathcal{V}^{\complement}$; thus by the definition of the closure of sets, $\bar{A}_1 \subseteq \mathcal{V}^{\complement}$. This implies that $\bar{A}_1 \cap A_2 = \varnothing$. Similarly, $\bar{A}_2 \cap A_1 = \varnothing$.

"⇐" Let $\mathcal{U} = \bar{A}_2^{\complement}$ and $\mathcal{V} = \bar{A}_1^{\complement}$ be two open sets. Then $\mathcal{V} \cap A_1 = \mathcal{U} \cap A_2 = \varnothing$; thus

$$A \cap \mathcal{U} \cap \mathcal{V} = (A_1 \cup A_2) \cap \mathcal{U} \cap \mathcal{V} = (A_1 \cap \mathcal{U}) \cap \mathcal{V} = \mathcal{U} \cap (A_1 \cap \mathcal{V}) = \varnothing\,.$$

Moreover, 2-4 in Definition 3.35 also hold since $A_1 \subseteq \mathcal{U}$ and $A_2 \subseteq \mathcal{V}$. □

Corollary 3.37. *Let (M, d) be a metric space. Suppose that a subset $A \subseteq M$ is connected, and $A = A_1 \cup A_2$, where $A_1 \cap \bar{A}_2 = \bar{A}_1 \cap A_2 = \varnothing$. Then A_1 or A_2 is empty.*

Theorem 3.38. *A subset A of the Euclidean space $(\mathbb{R}, |\cdot|)$ is connected if and only if it has the property that if $x, y \in A$ and $x < z < y$, then $z \in A$.*

Proof. "⇒" Suppose that there exist $x, y \in A$, $x < z < y$ but $z \notin A$. Then $A = A_1 \cup A_2$, where

$$A_1 = A \cap (-\infty, z) \quad \text{and} \quad A_2 = A \cap (z, \infty)\,.$$

Since $x \in A_1$ and $y \in A_2$, A_1 and A_2 are non-empty. Moreover, $\bar{A}_1 \cap A_2 = A_1 \cap \bar{A}_2 = \varnothing$; thus by Proposition 3.36, A is disconnected, a contradiction.

"⇐" Suppose that A is not connected. Then there exist non-empty sets A_1 and A_2 such that $A = A_1 \cup A_2$ with $\bar{A}_1 \cap A_2 = A_1 \cap \bar{A}_2 = \varnothing$. Pick $x \in A_1$ and $y \in A_2$. W.L.O.G., we may assume that $x < y$. Define $z = \sup(A_1 \cap [x, y])$.

Claim: $z \in \bar{A}_1$.

Proof of claim: By definition, for any $n > 0$ there exists $x_n \in A_1 \cap [x, y]$ such that $z - \dfrac{1}{n} < x_n \leqslant z$. Therefore, $x_n \to z$ as $n \to \infty$ which implies that $z \in \bar{A}_1$.

Since $z \in \bar{A}_1$, $z \notin A_2$. In particular, $x \leqslant z < y$.

(a) If $z \notin A_1$, then $x < z < y$ and $z \notin A$, a contradiction.

(b) If $z \in A_1$, then $z \notin \bar{A}_2$; thus $\exists\, r > 0$ such that $(z - r, z + r) \subseteq \bar{A}_2^{\complement}$. Then for all $z_1 \in (z, z + r)$, $z < z_1 < y$ and $z_1 \notin A_2$. Then $x < z_1 < y$ and $z_1 \notin A$, a contradiction. □

3.3 Subspace Topology

Let (M, d) be a metric space, and $N \subseteq M$ be a subset. Then (N, d) is a metric space, and the topology of (N, d) is called the ***subspace topology*** of (N, d).

Remark 3.39. The topology of a metric is the collection of all open sets of that metric space.

Proposition 3.40. *Let (M, d) be a metric space, and $N \subseteq M$. A subset $\mathcal{V} \subseteq N$ is open in (N, d) if and only if $\mathcal{V} = \mathcal{U} \cap N$ for some open set $\mathcal{U}$ in (M, d).*

Proof. "$\Rightarrow$" Let $\mathcal{V} \subseteq N$ be open in (N, d). Then $\forall\, x \in \mathcal{V}$, $\exists\, r_x > 0$ such that

$$D_N(x, r_x) \equiv \{y \in N \,|\, d(x, y) < r_x\} \subseteq \mathcal{V}.$$

In particular, $\mathcal{V} = \bigcup\limits_{x \in \mathcal{V}} D_N(x, r_x)$. Note that $D_N(x, r) = D(x, r) \cap N$; thus if $\mathcal{U} = \bigcup\limits_{x \in \mathcal{V}} D(x, r_x)$, then $\mathcal{U}$ is open in (M, d), and

$$\mathcal{V} = \bigcup_{x \in \mathcal{V}} D(x, r_x) \cap N = \mathcal{U} \cap N\,.$$

"$\Leftarrow$" Suppose that $\mathcal{V} = \mathcal{U} \cap N$ for some open set $\mathcal{U}$ in (M, d). Let $x \in \mathcal{V}$. Then $x \in \mathcal{U}$; thus $\exists\, r > 0$ such that $D(x, r) \subseteq \mathcal{U}$. Therefore,

$$D_N(x, r) \equiv \{y \in N \,|\, d(x, y) < r\} = D(x, r) \cap N \subseteq \mathcal{U} \cap N = \mathcal{V}\,;$$

hence $\mathcal{V}$ is open in (N, d). □

Corollary 3.41. *Let (M, d) be a metric space, and $N \subseteq M$. Let (M, d) be a metric space, and $N \subseteq M$. A subset $E \subseteq N$ is closed in (N, d) if and only if $E = F \cap N$ for some closed set F in (M, d).*

Definition 3.42. Let (M, d) be a metric space, and $N \subseteq M$. A subset A is said to be $\begin{matrix}\text{open}\\\text{closed}\\\text{compact}\end{matrix}$ relative to N if $A \cap N$ is $\begin{matrix}\text{open}\\\text{closed}\\\text{compact}\end{matrix}$ in the metric space (N, d).

Theorem 3.43. *Let (M, d) be a metric space, and $K \subseteq N \subseteq M$. Then K is compact in (M, d) if and only if K is compact in (N, d).*

Proof. "⇒" Let $\{\mathcal{V}_\alpha\}_{\alpha\in I}$ be an open cover of K in (N,d). By Proposition 3.40, there are open sets $\mathcal{U}_\alpha$ in (M,d) such that $\mathcal{V}_\alpha = \mathcal{U}_\alpha \cap N$ for all $\alpha \in I$. Then $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ is also an open cover of K; thus possesses a finite subcover; that is, $\exists\, J \subseteq I$, $\#J < \infty$ such that $K \subseteq \bigcup\limits_{\alpha\in J} \mathcal{U}_\alpha$ which, together with the fact that $K \subseteq N$, implies that

$$K \subseteq \Big(\bigcup_{\alpha\in J} \mathcal{U}_\alpha\Big) \cap N = \bigcup_{\alpha\in J} (\mathcal{U}_\alpha \cap N) = \bigcup_{\alpha\in J} \mathcal{V}_\alpha \,.$$

"⇐" Let $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ be an open cover of K in (M,d). Letting $\mathcal{V}_\alpha = \mathcal{U}_\alpha \cap N$, by Proposition 3.40 we find that $\{\mathcal{V}_\alpha\}_{\alpha\in I}$ is an open cover of K in (N,d). Since K is compact in (N,d), there exists $J \subseteq I$, $\#J < \infty$ such that $K \subseteq \bigcup\limits_{\alpha\in J} \mathcal{V}_\alpha$; thus

$$K \subseteq \bigcup_{\alpha\in J} \mathcal{U}_\alpha \,.$$ □

Remark 3.44. Another way to look at Theorem 3.43 is using the sequential compactness equivalence. Let $\{x_k\}_{k=1}^\infty \subseteq K$ be a sequence. By sequential compactness of K in either (M,d) or (N,d), there exists $\{x_{k_j}\}_{j=1}^\infty$ and $x \in K$ such that $x_{k_j} \to x$ as $j \to \infty$. As long as the metric d used in different space are identical, the concept of convergence of a sequence are the same; thus compactness in (M,d) or (N,d) are the same.

Example 3.45. Let (M,d) be $(\mathbb{R}, |\cdot|)$, and $N = \mathbb{Q}$. Consider the set $F = [0,1] \cap \mathbb{Q}$. By Corollary 3.41 F is closed in $(\mathbb{Q}, |\cdot|)$. However, F is not compact in $(\mathbb{Q}, |\cdot|)$ since F is not complete. We can also apply Theorem 3.43 to see this: if $F \subseteq \mathbb{Q}$ is compact in $(\mathbb{Q}, |\cdot|)$, then F is compact in $(\mathbb{R}, |\cdot|)$ which is clearly not the case since F is not closed in $(\mathbb{R}, |\cdot|)$.

Remark 3.46. Let (M,d) be a metric space. By Proposition 3.36 a subset $A \subseteq M$ is disconnected if and only if there exist two subsets $\mathcal{U}_1, \mathcal{U}_2$ of A, open relative to A, such that $A = \mathcal{U}_1 \cup \mathcal{U}_2$ and $\mathcal{U}_1 \cap \mathcal{U}_2 = \varnothing$ (one choice of $(\mathcal{U}_1, \mathcal{U}_2)$ is $\mathcal{U}_1 = A \backslash \bar{A}_1$ and $\mathcal{U}_2 = A \backslash \bar{A}_2$, where A_1 and A_2 are given by Proposition 3.36). Note that $\mathcal{U}_1$ and $\mathcal{U}_2$ are also closed relative to A.

Given the observation above, if A is a connected set and E is a subset of A such that E is closed and open relative to A, then $E = \varnothing$ or $E = A$.

3.4 Exercises

§3.1 Compactness

Problem 3.1. Let (M, d) be a metric space.

1. Show that the union of a finite number of compact subsets of M is compact.
2. Show that the intersection of an arbitrary collection of compact subsets of M is compact.

Problem 3.2. A metric space (M, d) is said to be separable if there is a countable subset A which is dense in M. Show that every compact set is separable.

Problem 3.3. Given $\{a_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ a bounded sequence. Define

$$A = \left\{ x \in \mathbb{R} \,\middle|\, \text{there exists a subsequence } \left\{a_{k_j}\right\}_{j=1}^{\infty} \text{ such that } \lim_{j\to\infty} a_{k_j} = x \right\}.$$

Show that A is a non-empty compact set in $\mathbb{R}$. Furthermore , $\limsup_{k\to\infty} a_k = \sup A$ and $\liminf_{k\to\infty} a_k = \inf A$.

Problem 3.4. Let (M, d) be a compact metric space; that is, M itself is a compact set. If $\{F_k\}_{k=1}^{\infty}$ is a sequence of closed sets such that $\operatorname{int}(F_k) = \varnothing$, then $M \setminus \bigcup_{k=1}^{\infty} F_k \neq \varnothing$.

Problem 3.5. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(x, y) = \begin{cases} |x_1 - y_1| & \text{if } x_2 = y_2\,, \\ |x_1 - y_1| + |x_2 - y_2| + 1 & \text{if } x_2 \neq y_2\,. \end{cases} \quad \text{where } x = (x_1, x_2) \text{ and } y = (y_1, y_2).$$

1. Show that d is a metric on $\mathbb{R}^2$. In other words, $(\mathbb{R}^2, d)$ is a metric space.
2. Find $D(x, r)$ with $r < 1$, $r = 1$ and $r > 1$.
3. Show that the set $\{c\} \times [a, b] \subseteq (\mathbb{R}^2, d)$ is closed and bounded.
4. Examine whether the set $\{c\} \times [a, b] \subseteq (\mathbb{R}^2, d)$ is compact or not.

Problem 3.6. Let (M, d) be a complete metric space, and $A \subseteq M$ be totally bounded. Show that $\operatorname{cl}(A)$ is compact.

Problem 3.7. Let $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence in a metric space, and $x_k \to x$ as $k \to \infty$. Show that the set $A \equiv \{x_1, x_2, \cdots, \} \cup \{x\}$ is compact by

1. showing that A is sequentially compact; and

2. showing that every open cover of A has a finite subcover; and

3. showing that A is totally bounded and complete.

Problem 3.8. Let Y be the collection of all sequences $\{y_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ such that $\sum\limits_{k=1}^{\infty} |y_k|^2 < \infty$. In other words,

$$Y = \Big\{ \{y_k\}_{k=1}^{\infty} \,\Big|\, y_k \in \mathbb{R} \text{ for all } k \in \mathbb{N}, \sum_{k=1}^{\infty} |y_k|^2 < \infty \Big\}.$$

Define $\|\cdot\| : Y \to \mathbb{R}$ by

$$\big\| \{y_k\}_{k=1}^{\infty} \big\| = \Big(\sum_{k=1}^{\infty} |y_k|^2 \Big)^{\frac{1}{2}}.$$

1. Show that $\|\cdot\|$ is a norm on Y. The normed space $(Y, \|\cdot\|)$ usually is denoted by ℓ^2.

2. Show that $\|\cdot\|$ is induced by an inner product.

3. Show that $(Y, \|\cdot\|)$ is complete.

4. Let $B = \big\{ y \in Y \,\big|\, \|y\| \leqslant 1 \big\}$. Is E compact or not?

Problem 3.9. Let A, B be two non-empty subsets in $\mathbb{R}^n$. Define

$$d(A, B) = \inf\big\{ \|x - y\|_2 \,\big|\, x \in A, y \in B \big\}$$

to be the distance between A and B. When $A = \{x\}$ is a point, we write $d(A, B)$ as $d(x, B)$.

(1) Prove that $d(A, B) = \inf\big\{ d(x, B) \,\big|\, x \in A \big\}$.

(2) Show that $\big| d(x_1, B) - d(x_2, B) \big| \leqslant \|x_1 - x_2\|_2$ for all $x_1, x_2 \in \mathbb{R}^n$.

(3) Define $B_\varepsilon = \big\{ x \in \mathbb{R}^n \,\big|\, d(x, B) < \varepsilon \big\}$ be the collection of all points whose distance from B is less than ε. Show that B_ε is open and $\bigcap\limits_{\varepsilon > 0} B_\varepsilon = \text{cl}(B)$.

(4) If A is compact, show that there exists $x \in A$ such that $d(A, B) = d(x, B)$.

(5) If A is closed and B is compact, show that there exists $x \in A$ and $y \in B$ such that $d(A, B) = d(x, y)$.

(6) If A and B are both closed, does the conclusion of (5) hold?

Problem 3.10. Let $\mathcal{K}(n)$ denote the collection of all non-empty compact sets in $\mathbb{R}^n$. Define the Hausdorff distance of $K_1, K_2 \in \mathcal{K}(n)$ by

$$d^H(K_1, K_2) = \max\Big\{\sup_{x\in K_2} d(x, K_1), \sup_{x\in K_1} d(x, K_2)\Big\}\,,$$

in which $d(x, K)$ is the distance between x and K given in Problem 3.9. Show that $(\mathcal{K}(n), d^H)$ is a metric space.

Problem 3.11. Let $M = \big\{(x, y) \in \mathbb{R}^2 \,\big|\, x^2 + y^2 \leqslant 1\big\}$ with the standard metric $\|\cdot\|_2$. Show that $A \subseteq M$ is compact if and only if A is closed.

Problem 3.12. 1. Let $\{x_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ be a sequence in $(\mathbb{R}, |\cdot|)$ that converges to x and let $A_k = \{x_k, x_{k+1}, \cdots\}$. Show that $\{x\} = \bigcap\limits_{k=1}^{\infty} \overline{A_k}$. Is this true in any metric space?

2. Suppose that $\{K_j\}_{j=1}^{\infty}$ is a sequence of comapct non-empty sets satisfying the nested set property; that is, $K_j \supseteq K_{j+1}$, and diameter$(K_j) \to 0$ as $j \to \infty$, where

$$\text{diameter}(K_j) = \sup\big\{d(x, y) \,\big|\, x, y \in K_j\big\}\,.$$

Show that there is exactly one point in $\bigcap\limits_{j=1}^{\infty} K_j$.

§3.2 Connectedness

Problem 3.13. Let (M, d) be a metric space, and $A \subseteq M$. Show that A is disconnected (not connected) if and only if there exist non-empty closed set F_1 and F_2 such that

1. $A \cap F_1 \cap F_2 = \varnothing$; 2. $A \cap F_1 \neq \varnothing$; 3. $A \cap F_2 \neq \varnothing$; 4. $A \subseteq F_1 \cup F_2$.

Problem 3.14. Prove that if A is connected in a metric space (M, d) and $A \subseteq B \subseteq \bar{A}$, then B is connected.

Problem 3.15. Let (M, d) be a metric space, and $A \subseteq M$ be a subset. Suppose that A is connected and contain more than one point. Show that $A \subseteq A'$.

Problem 3.16. Show that the Cantor set C defined in Problem 2.11 is totally disconnected; that is, if $x, y \in C$, and $x \neq y$, then $x \in \mathcal{U}$ and $y \in \mathcal{V}$ for some open sets $\mathcal{U}$, $\mathcal{V}$ separate C.

Problem 3.17. Let F_k be a nest of connected compact sets (that is, $F_{k+1} \subseteq F_k$ and F_k is connected for all $k \in \mathbb{N}$). Show that $\bigcap_{k=1}^{\infty} F_k$ is connected. Give an example to show that compactness is an essential condition and we cannot just assume that F_k is a nest of closed connected sets.

Problem 3.18. Let $\{A_k\}_{k=1}^{\infty}$ be a family of connected subsets of M, and suppose that A is a connected subset of M such that $A_k \cap A \neq \varnothing$ for all $k \in \mathbb{N}$. Show that the union $\big(\bigcup_{k\in\mathbb{N}} A_k\big) \cup A$ is also connected.

Problem 3.19. Let $A, B \subseteq M$ and A is connected. Suppose that $A \cap B \neq \varnothing$ and $A \cap B^{\complement} \neq \varnothing$. Show that $A \cap \partial B \neq \varnothing$.

Problem 3.20. Given (M, d) a metric space and $A \subseteq M$ a non-empty subset. A maximal connected subset of A is called a ***connected component*** of A.

1. Let $a \in A$. Show that there is a unique connected components of A containing a.
2. Show that any two distinct connected components of A are disjoint. Therefore, A is the disjoint union of its connected components.
3. Show that every connected component of A is a closed subset of A.
4. If A is open, prove that every connected component of A is also open. Therefore, when $M = \mathbb{R}^n$, show that A has at most countable infinite connected components.
5. Find the connected components of the set of rational numbers or the set of irrational numbers in $\mathbb{R}$.

Problem 3.21 (**True or False).** Determine whether the following statements are true or false. If it is true, prove it. Otherwise, give a counter-example.

1. There exists a non-zero dimensional normed vector space in which some compact non-zero dimensional linear subspace exists.
2. There exists a set $A \subseteq (0, 1]$ which is compact in $(0, 1]$ (in the sense of subspace topology), but A is not compact in $\mathbb{R}$.
3. Let $A \subseteq \mathbb{R}^n$ be a non-empty set. Then a subset B of A is compact in A if and only if B is closed and bounded in A.