Chapter 2

Point-Set Topology of Metric spaces

2.1 Open Sets and the Interior of Sets

Definition 2.1. Let (M, d) be a metric space. For each $x \in M$ and $\varepsilon > 0$, the set

$$D(x, \varepsilon) = \left\{ y \in M \,\middle|\, d(x, y) < \varepsilon \right\}$$

is called the ε***-disk*** (ε***-ball***) about x or the disk/ball centered at x with radius ε.

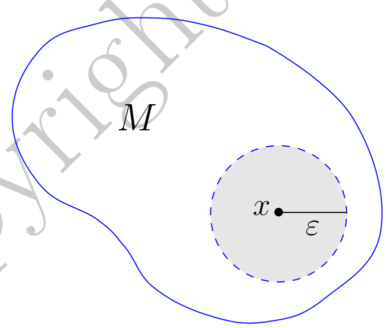


Figure 2.1: The ε-ball about x in a metric space

Example 2.2. $(\mathbb{R}^2, \|\cdot\|_p)$ is a normed vector space. Consider $x = 0$, $\varepsilon = 1$ and $p = 1,\ p = 2$ and $p = \infty$ respectively.

1. $p = 1$: $\|x\|_1 = |x_1| + |x_2|$, $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$.

2. $p = 2$: $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

3. $p = \infty$: $\|x\|_\infty = \max\left\{|x_1|, |x_2|\right\}$, $d(x, y) = \max\left\{|x_1 - y_1|, |x_2 - y_2|\right\}$.

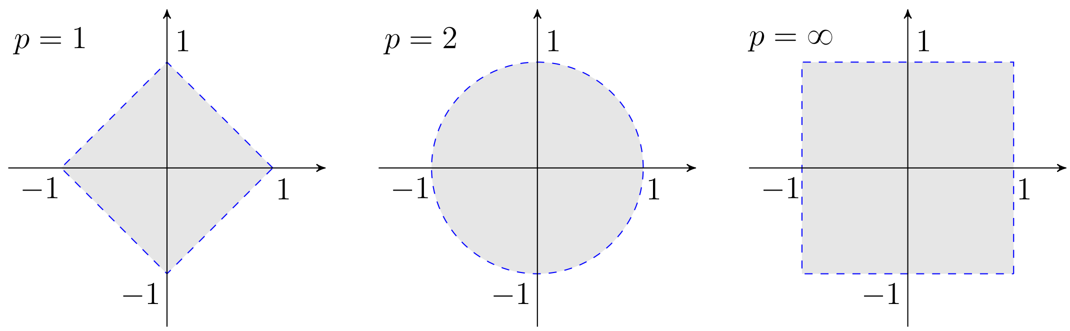


Figure 2.2: The 1-ball about 0 in $\mathbb{R}^2$ with different p

Example 2.3. Let (M, d) be a metric space with discrete metric; that is,

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Then $D(x, \varepsilon) = \begin{cases} \{x\} & \text{if } 0 < \varepsilon \leqslant 1, \\ M & \text{if } \varepsilon > 1. \end{cases}$

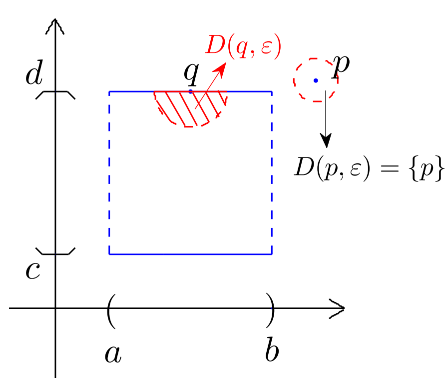
Definition 2.4. Let (M, d) be a metric space. A set $\mathcal{U} \subseteq M$ is said to be ***open*** (in M) if

$$\forall\, x \in \mathcal{U},\ \exists\, \varepsilon > 0 \ni D(x, \varepsilon) \subseteq \mathcal{U}\,.$$

Example 2.5. The set $A = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < x < 1\}$ is open: given $(x, y) \in A$, take $\varepsilon = \min\{1 - x, x\}$, then $D(x, \varepsilon) \subseteq A$.

Example 2.6. $A = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < x \leqslant 1\}$ is not open: let $u = (1, 0)$, then $\forall\, \varepsilon > 0 \ni D(u, \varepsilon) \nsubseteq A$ (since $\big(1 + \frac{\varepsilon}{2}, 0\big) \in D(u, \varepsilon)$ but $\big(1 + \frac{\varepsilon}{2}, 0\big) \notin A$).

Example 2.7. $M \equiv (a, b) \times [c, d] \cup \{p\}$, $p \notin [a, b] \times [c, d]$, and for two points (x_1, y_1) and (x_2, y_2) in M, we define the metric as $d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Then $D(p, \varepsilon) = \{p\}$ if $\varepsilon \ll 1$. Let $q = \big(\frac{a + b}{2}, d\big)$, $\varepsilon < \min\big\{\frac{b - a}{2}, d - c\big\}$. Then $D(q, \varepsilon)$ is the shaded region in red color shown in the figure below.



Proposition 2.8. *Let (M, d) be a metric space. Then every ε-disk is open.*

Proof. Let $D(x, \varepsilon)$ be an ε-disk. We would like to show that $\forall\, y \in D(x, \varepsilon)$, $\exists\, \delta > 0 \ni D(y, \delta) \subseteq D(x, \varepsilon)$. Let $\delta = \varepsilon - d(x, y) > 0$. Then if $z \in D(y, \delta)$, we have

$$d(z, x) \leqslant d(z, y) + d(x, y) < \delta + d(x, y) = \varepsilon\,;$$

thus $z \in D(x, \varepsilon)$. □

Proposition 2.9. *Let (M, d) be a metric space.*

1. *The intersection of finitely many open sets is open.*
2. *The union of arbitrary family of open sets is open.*
3. *The empty set $\varnothing$ and the universal set M are open.*

Proof. 1. Let $U_1, U_2, \cdots, U_k$ be open sets in M, and $U \equiv \bigcap\limits_{i=1}^{k} U_i$. If $y \in U$, then $y \in U_i$ for all $1 \leqslant i \leqslant k$. Since U_i is open, $\exists\, \delta_i > 0 \ni D(y, \delta_i) \subseteq U_i$. Let $\delta = \min\{\delta_1, \cdots, \delta_k\}$.

Claim: $D(y, \delta) \subseteq U$.

Proof of claim: Let $z \in D(y, \delta)$. Then $d(y, z) < \delta \leqslant \delta_i$ if $i = 1, 2, \cdots, k$.

$\Rightarrow z \in D(y, \delta_i)\ \forall\, i = 1, 2, \cdots, k \Rightarrow z \in U_i$ if $i = 1, 2, \cdots, k \Rightarrow z \in \bigcap\limits_{i=1}^{k} U_i \equiv U$.

2. Let $\mathscr{F} = \big\{U_\alpha \,\big|\, U_\alpha \text{ open in } M,\ \alpha \in I\big\}$ be a family of open sets, and $U \equiv \bigcup\limits_{\alpha \in I} U_\alpha$. If $y \in U$, then $y \in U_\beta$ for some $\beta \in I$. Since U_β is open, $\exists\, \delta > 0 \ni D(y, \delta) \subseteq U_\beta$; thus $D(y, \delta) \subseteq \bigcup\limits_{\alpha \in I} U_\alpha \equiv U$.
3. $\varnothing$ is trivially an open set. Moreover, if $y \in M$, then $D(y, 1) \subseteq M$ (by definition). □

Corollary 2.10. *Let (M, d_0) be a metric space with discrete metric. Then every subset of M is open.*

Proof. $\forall\, y \in M, \{y\} = D\big(y, \frac{1}{2}\big)$ is an open set in M. If $A \subseteq M$, $A \neq \varnothing$, then $A = \bigcup\limits_{y \in A} D\big(y, \frac{1}{2}\big)$ which suggests that A is open since it is an arbitrary union of open sets. □

Remark 2.11. Infinite intersection of open sets need not be open:

1. Take $A_n = \big(-\frac{1}{n}, \frac{1}{n}\big)$, then $\bigcap\limits_{n=1}^{\infty} A_n = \{0\}$ which is not open.

2. Let $U_k = (-2 - \dfrac{1}{k}, 2 + \dfrac{1}{k}) \subseteq \mathbb{R}$. Then $A = \bigcap\limits_{k=1}^{\infty} U_k \supseteq [-2, 2]$. Moreover, if $x \notin [-2, 2]$, then $\exists\, k \in \mathbb{N} \ni x \notin U_k$ $\big($If $x > 2$, $\dfrac{1}{k} < \dfrac{x-2}{2}$. If $x < -2$, $\dfrac{1}{k} < \dfrac{-x-2}{2}\big)$. Therefore, $\bigcap\limits_{k=1}^{\infty} U_k = [-2, 2]$.

Example 2.12. Let $A \subseteq \mathbb{R}^n$ be open, and $B \subseteq \mathbb{R}^n$. Then $A + B = \{a + b \,|\, a \in A, b \in B\}$ is open.

Proof. Let $y \in A + B$. Then $y = a + b$ for some $a \in A, b \in B$. Since A is open, $\exists\, \delta > 0 \ni D(a, \delta) \subseteq A$.
Claim: $D(y, \delta) \subseteq A + B$.
Proof of claim: Let $z \in D(y, \delta)$. Then $\|z - y\|_2 < \delta$. Since $z = b + (z - b)$, if we can show that $z - b \in A$, then $z \in A + B$. Nevertheless, we have

$$\|(z - b) - a\|_2 = \|z - a - b\|_2 = \|z - y\|_x < \delta$$

which implies that $z - b \in D(a, \delta) \subseteq A$. □

Definition 2.13. Let (M, d) be a metric space, and $A \subseteq M$ be a subset of M. A point $x \in A$ is called an ***interior point*** of A if $\exists\, \varepsilon > 0 \ni D(x, \varepsilon) \subseteq A$. The ***interior*** of A is the collection of all interior points of A, and is denoted by $\text{int}(A)$ or $\mathring{A}$.

Example 2.14. Let $M = \mathbb{R}$ with $d(x, y) = |x - y|$, and $A = [0, 1)$, $B = \big\{1, \dfrac{1}{2}, \dfrac{1}{3}, \cdots, \dfrac{1}{n}, \cdots\big\} \cup \{0\} = \big\{\dfrac{1}{n}\big\}_{n=1}^{\infty} \cup \{0\}$. Then $\mathring{A} = (0.1)$ and $\mathring{B} = \varnothing$ since

1. If $x \in (0, 1)$, then $\exists\, \varepsilon > 0 \ni D(x, \varepsilon) \subseteq (0, 1) \subseteq A$.

2. 0 is not an interior point since $(-\varepsilon, \varepsilon) \cap [0, 1)^{\complement} \neq \phi \;\forall\, \varepsilon > 0$.

Remark 2.15. $\mathring{A}$ might be empty.

Theorem 2.16. *Let (M, d) be a metric space, and $A \subseteq M$ be a subset of M. The interior of A is the largest open set contained in A. In other words, if $U \subseteq A$ is open, then $U \subseteq \text{int}(A)$.*

Proof. Let $z \in U$. Since U is open, $\exists\, \delta > 0 \ni D(z, \delta) \subseteq U \subseteq A \Rightarrow z \in \mathring{A} \Rightarrow U \subseteq \mathring{A}$.
To show that $\mathring{A}$ is open, we observe that $\mathring{A} = \bigcup\limits_{x \in \mathring{A}} D(x, \varepsilon_x)$, where $\varepsilon_x > 0$ is chosen so that $D(x, \varepsilon_x) \subseteq \mathring{A}$ if $x \in \mathring{A}$, for the following reason:

1. "$\subseteq$": trivial.

2. "$\supseteq$": Let $y \in \bigcup_{x\in \mathring{A}} D(x,\varepsilon_x) \Rightarrow \exists\, x \in \mathring{A} \ \ni y \in D(x,\varepsilon_x)$. Then if $\delta = \varepsilon_x - d(x,y)$,
$$D(y,\delta) \subseteq D(x,\varepsilon_x) \subseteq A \Rightarrow y \in \mathring{A}.$$ □

Theorem 2.17. *Let (M,d) be a metric space. A set $A \subseteq M$ is open if and only if $A = \mathring{A}$.*

Example 2.18. Let (M,d) be a metric space, and A and B be two subsets of M.

1. $\text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B)$.

   Proof. Let $x \in \text{int}(A) \cup \text{int}(B)$. W.L.O.G. Assume $x \in \text{int}(A)$, then $\exists\, r > 0$ such that $D(x,r) \subseteq A$. Therefore, $x \in D(x,r) \subseteq A \cup B$, so $x \in \text{int}(A \cup B)$. □

2. Strict containment might happen because of the following example:

   Take $A = [0,1]$, $B = [1,2]$, then $\text{int}(A) = (0,1)$, $\text{int}(B) = (1,2)$.

   Sine $A \cup B = [0,2]$, $\text{int}(A \cup B) = (0,2)$; however, $\text{int}(A) \cup \text{int}(B) = (0,2)\backslash\{1\}$.

   Hence, $\text{int}(A) \cup \text{int}(B) \neq \text{int}(A \cup B)$.

   Another example is stated as follows: Let $A = \mathbb{Q} \cap [0,1]$ and $B = \mathbb{Q}^\complement \cap [0,1]$. Then
$$(0,1) = \text{int}([0,1]) = \text{int}(A \cup B) \supsetneq \text{int}(A) \cup \text{int}(B) = \varnothing.$$

Example 2.19. In a metric space (M,d), it is not always true that $\text{int}\big(\{y \in M \,|\, d(x,y) \leqslant R\}\big) = \{y \in M \,|\, d(x,y) < R\}$. To see this, we consider the discrete metric
$$d_0(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = 0. \end{cases}$$

Let $R = 1$, and fix $x_0 \in M \neq \varnothing$. Then
$$\{y \in M \,|\, d_0(y,x_0) \leqslant 1\} = M \Rightarrow \text{int}\big(\{y \in M \,|\, d_0(y,x_0) \leqslant 1\}\big) = \text{int}(M) = M\,.$$

Now $\{y \in M \,|\, d_0(y,x_0) < 1\} = \{x_0\}$. As long as M has more than one point, we have $\text{int}\big(\{y \in M \,|\, d_0(y,x_0) \leqslant 1\}\big) = M \neq \{x_0\} = D(x_0,1)$.

2.2 Closed Sets, the Closure of Sets, and the Boundary of Sets

Definition 2.20. Let (M,d) be a metric space. A set $F \subseteq M$ is said to be ***closed*** if $F^{\complement} = M \backslash F$ is open. In other words,

$$F \text{ is closed} \Leftrightarrow \forall\, x \in F^{\complement}, \exists\, \varepsilon > 0 \ni D(x,\varepsilon) \subseteq F^{\complement}\,.$$

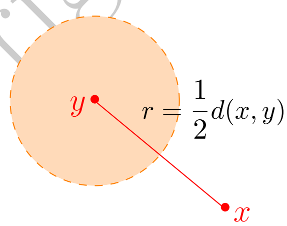
Example 2.21. The set $[0,1] \subseteq \mathbb{R}$ is closed, and the set $(0,1] \subseteq \mathbb{R}$ is not open and not closed.

Example 2.22. Let $S = \{(x,y) \mid x^2 + y^2 \leqslant 1\}$. Take $z \in \mathbb{R}^2 \backslash S$, then $D(z, \|z\|_2 - 1) \subseteq \mathbb{R}^2 \backslash S$. As a consequence, $\mathbb{R}^2 \backslash S$ is open; thus S is closed.

Example 2.23. Let $S = \big\{(x,y) \,\big|\, 0 < x \leqslant 1, 0 \leqslant y \leqslant 1\big\}$. Since $\mathbb{R}^2 \backslash S$ is not open, S is not closed.

Proposition 2.24. *Any point in a metric space is closed; that is, if (M,d) is a metric space and $A = \{x\}$ for some $x \in M$, then A is closed.*

Proof. We show that $M \backslash \{x\}$ is open. Let $y \in M \backslash \{x\}$. Pick $r = \frac{1}{2} d(x,y) > 0$.
Claim: $D(y,r) \subseteq M \backslash \{x\}$.



Proof of claim: Let $z \in D(y,r)$. Then $d(z,y) < r = \frac{1}{2} d(x,y)$. Then

$$d(z,x) \geqslant d(x,y) - d(y,z) \geqslant d(x,y) - \frac{1}{2} d(x,y) = \frac{1}{2} d(x,y) > 0 \Rightarrow z \neq x\,. \qquad \square$$

Proposition 2.25. *Let (M,d) be a metric space.*

1. *The union of finitely many closed sets is closed.*

2. *The intersection of arbitrary family of closed sets is closed.*

3. *The universal set M and the empty set $\varnothing$ are closed.*

Proof. 1. Let $F_1, \cdots, F_k$ be closed sets, and $F = \bigcup_{j=1}^{k} F_j$. Then by De Morgan's law,

$$F^{\complement} = M \backslash F = M \backslash \bigcup_{j=1}^{k} F_j = \bigcap_{j=1}^{k} (M \backslash F_j) = \bigcap_{j=1}^{k} {F_j}^{\complement} .$$

Since F_j is closed, ${F_j}^{\complement}$ is open. By Proposition 2.9, $\bigcap_{j=1}^{k} {F_j}^{\complement}$ is open.

2. Let $\mathscr{F} = \{F_\alpha \,|\, F_\alpha \text{ closed in } M, \alpha \in I\}$ be a family of closed sets, and $F \equiv \bigcap_{\alpha \in I} F_\alpha$. Then by De Morgan's law,

$$F^{\complement} = M \backslash \bigcap_{\alpha \in I} F_\alpha = \bigcup_{\alpha \in I} (M \backslash F_\alpha) = \bigcup_{\alpha \in I} F_\alpha^{\complement}$$

which suggests that $F^{\complement}$ is the union of open sets $\{F_\alpha^{\complement}\}_{\alpha \in I}$. By Proposition 2.9 we conclude that $F^{\complement}$ is open or equivalently, F is closed.

3. $M^{\complement} = \varnothing, \varnothing^{\complement} = M$ are both open. ▫

Corollary 2.26. *Any set consists of finitely many points of a metric space is closed.*

Example 2.27. Let $F_k = \big[-2 + \frac{1}{k}, 2 - \frac{1}{k}\big] \subseteq \mathbb{R}$. Then $B = \bigcup_{k=1}^{\infty} F_k \subseteq (-2, 2)$. Moreover, if $x \in (-2, 2)$, then $\exists\, k > 0, \ni x \in F_k$ $\big($If $x \leqslant 0$, $\frac{1}{k} < \frac{x+2}{2}$. If $x > 0$, $\frac{1}{k} < \frac{2-x}{2}\big)$. Therefore, $\bigcup_{k=1}^{\infty} F_k = (-2, 2)$. This example suggests that an arbitrary union of closed sets might not be closed.

Example 2.28. Let (M, d) be a metric space, and $A = \{y_1, y_2, \ldots, y_k\} \subseteq M$. Define $B = \{x \in M \,|\, d(x, y_i) \leqslant 1 \text{ for some } y_i \in A\} = \bigcup_{i=1}^{k} \{x \in M \,|\, d(x, y_i) \leqslant 1\}$. Then B is closed.

Proof. It suffices to show $B_i = \{x \in M \,|\, d(x, y_i) \leqslant 1\}$ is closed for $i = 1, 2, \ldots, k$ since $B = \bigcup_{i=1}^{n} F_i$.. Take $z \in M \backslash B_i$ (if $M \backslash B_i = \varnothing$, then $B_i = M$ and B_i is closed). Let $N = \{u \in M \,|\, d(u, z) < d(z, y_i) - 1\}$.
Claim: $N \subseteq M \backslash B_i$; that is, $M \backslash B_i$ is open.
Proof of claim: Take $u \in N$ and compute $d(u, y_i) \geqslant d(y_i, z) - d(u, z) > d(z, y_i) - (d(z, y_i) - 1) = 1$. Hence $u \notin B_i \Rightarrow u \in M \backslash B_i$. So $N \subseteq M \backslash B_i$. ▫

Example 2.29. Let (M, d) be a metric space, $A \subseteq M$ be closed, and $B \subseteq M$ be finite ($\#(B) < \infty$). Then $A + B$ is closed.

Proof. Left as an exercise. □

Definition 2.30. Let (M, d) be a metric space, and $A \subseteq M$.

1. A point $x \in M$ is called an ***accumulation point*** of A if $\forall\, \varepsilon > 0, D(x, \varepsilon)$ contains points in A other than x; that is, $\forall\, \varepsilon > 0, D(x, \varepsilon) \cap (A \backslash \{x\}) = (D(x, \varepsilon) \backslash \{x\}) \cap A \neq \varnothing$.

2. A point $x \in M$ is called a ***limit point*** of A if $\forall\, \varepsilon > 0, D(x, \varepsilon)$ contains points in A; that is, $\forall\, \varepsilon > 0, D(x, \varepsilon) \cap A \neq \varnothing$.

3. A point $x \in A$ is called an ***isolated point*** (孤立點) if $\exists\, \varepsilon > 0 \ni D(x, \varepsilon) \cap A = \{x\}$.

4. The ***derived set*** of A is the collection of all accumulation points of A, and is denoted by A'.

5. The collection of all limit points of A is denoted by $\bar{A}$.

Remark 2.31. 1. An accumulation point of A needs not to be in A.

2. If $A = \{x\}$ (that is, a single point), then A has no accumulation point; that is, $A' = \varnothing$.

3. Accumulation points are called cluster points in some books.

4. If $x \in A'$, then x is a limit point of A. In other words, $A' \subseteq \bar{A}$.

5. If $x \in A$, then x is a limit point of A. In other words, $A \subseteq \bar{A}$.

Example 2.32. Let $A = (0, 1) \subseteq \mathbb{R}$, then $A' = [0, 1]$ and $\bar{A} = [0, 1]$.

Example 2.33. Let $A = (0, 1) \cup \{2\} \subseteq \mathbb{R}$. Then

1. for any $x \in [0, 1], x \in A'$;

2. $2 \notin A'$, but 2 is a limit point of A;

3. if $x \notin [0, 1] \cup \{2\}$ then $x \notin A'$.

Therefore, $A' = [0, 1]$. Note that $\sup A = 2$; thus $\sup A$ might not belong to A'.

Example 2.34. Let $A = \{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ consists of a bounded sequence of distinct points. Then $A' \neq \varnothing$.

Proof. By Bolzano-Weierstrass property (Theorem 1.100), A has a convergent subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ converging to $x \in \mathbb{R}$.
Claim: $x \in A'$.
Proof of claim: $\forall\, \varepsilon > 0, \exists\, K \in \mathbb{N} \ni \left|x_{n_j} - x\right| < \varepsilon$ for $j \geqslant K$. Moreover, $x_{n_j} \in A$. □

Example 2.35. In a metric space (M, d), let $B(x,r) = \{y \in M \,|\, d(x,y) \leqslant r\}$. Is it true that $B(x,r) \subseteq D(x,r)'$; that is, every point of $B(x,r)$ is an accumulation point of $D(x,r)$? Answer: No, take a metric space with discrete metric

$$d_0(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = 0. \end{cases}$$

and M has more than one point. We have $D(x,1) = \{x\}$, then $D(x,1)' = \varnothing$. Also, $B(x,1) = M \nsubseteq \varnothing = D(x,1)'$.

Proposition 2.36. *If $A \subseteq B$, then $A' \subseteq B'$.*

Proof. Let $x \in A'$. Then $\forall\, \varepsilon > 0, \exists\, y \in A, y \neq x \ni y \in D(x,\varepsilon) \cap A$. Since $A \subseteq B$, $y \in B$. Therefore, $\forall\, \varepsilon > 0, \exists\, y \in B, y \neq x \ni y \in D(x,\varepsilon) \cap B \Leftrightarrow x \in B'$. □

Example 2.37. Let A be a subset of $\mathbb{R}^n$. An interior point of A is an accumulation point of A ($\mathring{A} \subseteq A'$ if $A \subseteq \mathbb{R}^n$).

Proof. If $x \in \mathring{A}$, then $\exists\, r > 0, \ni D(x,r) \subseteq A$. Let $\varepsilon > 0$ be given.

1. $\varepsilon \geqslant r, D(x,\varepsilon) \cap (A \backslash \{x\}) \supseteq D(x,r) \cap (A \backslash \{x\}) \neq \varnothing$.

2. $\varepsilon < r, D(x,\varepsilon) \subseteq D(x,r) \subseteq A \Rightarrow D(x,\varepsilon) \cap (A \backslash \{x\}) \neq \varnothing$.

Then for all $\varepsilon > 0$, $D(x,\varepsilon) \cap (A \backslash \{x\}) \neq \varnothing \Rightarrow x \in A'$. □

Theorem 2.38. *Let (M,d) be a metric space and $A \subseteq M$, then A is closed if and only if $A = \bar{A}$.*（一個集合是閉集合若且唯若該集合包含了它所有的 *limit points*）

Proof. A is closed $\Leftrightarrow \forall\, y \in A^{\complement}, \exists\, r > 0 \ni D(y,r) \subseteq A^{\complement}$ (or $D(y,r) \cap A = \varnothing$).
$\Leftrightarrow \forall\, y \in A^{\complement}, y \notin \bar{A}$ (or $y \in \bar{A}^{\complement}$).
$\Leftrightarrow$ if $y \in \bar{A}$, then $y \in A$. □

Theorem 2.39. *Let* (M,d) *be a metric space and* $A \subseteq M$. *Then* $\bar{A} = A \cup A'$ $\big(= (A\backslash A') \cup A'\big)$.

Proof. By definition, $x \in \bar{A} \Leftrightarrow \forall\, \varepsilon > 0, D(x,\varepsilon) \cap A \neq \varnothing$.

$$\Rightarrow \text{If } x \in \bar{A}\backslash A, \text{ then } \forall\, \varepsilon > 0, D(x,\varepsilon) \cap (A\backslash\{x\}) \neq \varnothing.$$
$$\Rightarrow \text{If } x \in \bar{A}\backslash A, \text{ then } x \in A'.$$

Therefore, $\bar{A}\backslash A \subseteq A' \Rightarrow \bar{A} \subseteq A \cup A'$. On the other hand, we also have (1) $A \subseteq \bar{A}$ and (2) $A' \subseteq \bar{A}$; thus $A \cup A' \subseteq \bar{A}$. □

Corollary 2.40. *Let* (M,d) *be a metric space, and* $A \subseteq B \subseteq M$. *Then* $\bar{A} \subseteq \bar{B}$. *In particular, if* $A \subseteq B$ *and* B *is closed, then* $\bar{A} \subseteq B$.

Proposition 2.41. *Let* (M,d) *be a metric space, and* $A \subseteq M$. *Then* $A\backslash A'$ *is the collection of all isolated points of* A.

Proof. Let $x \in A\backslash A'$. Then $x \in A$, but $\exists\, \varepsilon > 0 \ni D(x,\varepsilon) \cap (A\backslash\{x\}) = \varnothing$. Therefore, $D(x,\varepsilon) \cap A = \{x\}$ which implies that x is an isolated point. □

Theorem 2.42. *Let* (M,d) *be a metric space, and* $A \subseteq M$. *Then* A' *is closed; that is,* $\forall\, y \notin A', \exists\, r > 0 \ni D(y,r) \cap A' = \varnothing$.

Proof. Let $y \notin A'$. Then $\exists\, \varepsilon > 0 \ni D(y,\varepsilon) \cap (A\backslash\{y\}) = (D(y,\varepsilon)\backslash\{y\}) \cap A = \varnothing$. Then

$$A \subseteq \big(D(y,\varepsilon)\backslash\{y\}\big)^{\complement}.$$

Since $D(y,\varepsilon)\backslash\{y\} = D(y,\varepsilon) \cap \{y\}^{\complement}$ is open, $\big(D(y,\varepsilon)\backslash\{y\}\big)^{\complement}$ is closed; thus Corollary 2.40 implies that

$$\bar{A} \subseteq \big(D(y,\varepsilon)\backslash\{y\}\big)^{\complement} \quad \text{or equivalently,} \quad \bar{A} \cap D(y,\varepsilon)\backslash\{y\} = \varnothing\,.$$

Since $\bar{A} = A \cup A'$, the equality above suggests that

$$A' \cap D(y,\varepsilon)\backslash\{y\} = \varnothing\,;$$

thus the fact that $y \notin A'$ implies that $D(y,\varepsilon) \cap A' = \varnothing$. □

Definition 2.43. Let (M,d) be a metric space and $A \subseteq M$. The ***closure*** of A is the intersection of closed sets containing A, and is denoted by $\text{cl}(A)$. In other word, $\text{cl}(A) = \bigcap\limits_{\substack{F \text{ closed.}\\ A \subseteq F}} F$ (thus $\text{cl}(A)$ is the smallest closed set containing A).

Proposition 2.44. *Let (M,d) be a metric space, and $A \subseteq M$.*

1. $A \subseteq \mathrm{cl}(A)$ ($x \in A \Rightarrow$ *if $F \supseteq A$ is closed, then $x \in F$*).
2. *A is closed if and only if $A = \mathrm{cl}(A)$.*

Proposition 2.45. *Let (M,d) be a metric space, and $A \subseteq M$. Then* $\mathrm{cl}(A) = \bar{A}(= A \cup A')$.

Proof. Since $A \subseteq \mathrm{cl}(A)$ and $\mathrm{cl}(A)$ is closed, Corollary 2.40 implies that $\bar{A} \subseteq \mathrm{cl}(A)$.

On the other hand, if $x \notin A \cup A' = \bar{A}$, then $\exists\, r > 0 \ni D(x,r) \cap A = \varnothing$ or in other words, $A \subseteq D(x,r)^{\complement}$. By the definition of the closure of sets, $\mathrm{cl}(A) \subseteq D(x,r)^{\complement}$ or equivalently, $D(x,r) \subseteq \mathrm{cl}(A)^{\complement}$; thus $x \notin \mathrm{cl}(A)$. Therefore, $\mathrm{cl}(A) \subseteq \bar{A}$. ▫

Example 2.46. Let $A = [0.1) \cup \{2\} \subseteq \mathbb{R}$. Find $\mathrm{cl}(A)$.
Answer: $A' = [0.1]$, $\mathrm{cl}(A) = A \cup A' = [0,1] \cup \{2\}$.

Example 2.47. $\mathrm{cl}(A \cap B) \overset{?}{=} \mathrm{cl}(A) \cap \mathrm{cl}(B)$.
Answer: No. Take $A = [0,1], B = (1,2]$. Since A is closed, then $\mathrm{cl}(A) = A$. Since $\mathrm{cl}(B) = [1,2]$, $A \cap B = \varnothing$. So $\mathrm{cl}(A \cap B) = \varnothing \neq \{1\} = \mathrm{cl}(A) \cap \mathrm{cl}(B)$; thus $\mathrm{cl}(A \cap B) \subsetneq \mathrm{cl}(A) \cap \mathrm{cl}(B)$.

Example 2.48. In a metric space (M,d),

$$x \in \mathrm{cl}(A) \text{ if and only if } d(x,A) \equiv \inf\big\{d(x,y) \,\big|\, y \in A\big\} = 0.$$

Proof. "$\Leftarrow$" Suppose $d(x,A) = 0$. If $x \in A$, then $x \in A \cup A' = \mathrm{cl}(A)$. If $x \notin A$, since $d(x,A) = 0$, $\forall\, \varepsilon > 0\ \exists\, y \in A \ni d(x,y) < d(x,A) + \varepsilon = \varepsilon$. In other words, $(D(x,\varepsilon)\backslash\{x\}) \cap A \neq \varnothing$. Therefore, $x \in A'$; thus $x \in A \cup A' = \mathrm{cl}(A)$.

"$\Rightarrow$" Suppose $x \in \mathrm{cl}(A)$. Since $\bar{A} = \mathrm{cl}(A)$, $\forall\, \varepsilon > 0$, $D(x,\varepsilon) \cap A \neq \varnothing$. In other words,

$$\forall\, \varepsilon > 0,\ \exists\, y \in A \ \ni d(x,y) < \varepsilon.$$

Therefore, $d(x,A) < \varepsilon$ for all $\varepsilon > 0$ which implies that $d(x,A) = 0$. ▫

Example 2.49. $A = \big\{\frac{1}{n} \,\big|\, n = 1,2,\cdots\big\}$. Find $\mathrm{cl}(A)$.
Answer: $A' = \{0\} \Rightarrow \mathrm{cl}(A) = A \cup A' = \big\{\frac{1}{n} \,\big|\, n = 1,2,\cdots\big\} \cup \{0\}$.

Example 2.50. $A = \big\{(x,y) \,\big|\, x \in \mathbb{Q}\big\}$. Find $\mathrm{cl}(A)$.
Answer: $A' = \mathbb{R}^2 \Rightarrow \mathrm{cl}(A) = \mathbb{R}^2$.

Definition 2.51. Let (M, d) be a metric space. A subset $A \subseteq M$ is said to be ***dense***（稠密）in another subset $B \subseteq M$ if $A \subseteq B \subseteq \text{cl}(A)$.

Example 2.52. The rational numbers $\mathbb{Q}$ is dense in the real number system $\mathbb{R}$.

Definition 2.53. Let (M, d) be a metric space, and $A \subseteq M$. The ***boundary*** of A, denoted by $\text{bd}(A)$ or ∂A, is the intersection of $\bar{A}$ and $\overline{A^\complement}$ ($\partial A = \bar{A} \cap \overline{A^\complement}$).

Remark 2.54. 1. ∂A is closed since the closure of a set is closed.

2. By the definition of limit points of a set, we find that $x \in \partial A \Leftrightarrow \forall\, \varepsilon > 0, D(x, \varepsilon) \cap A \neq \varnothing$ and $D(x, \varepsilon) \cap A^\complement \neq \varnothing$.

3. $\partial A = \partial(A^\complement)$.

Proposition 2.55. *Let (M, d) be a metric space, and $A \subseteq M$. Then $\partial A = \bar{A} \backslash \mathring{A}$.*

Proof. If $x \in \partial A$, then $\forall\, \varepsilon > 0, D(x, \varepsilon) \cap A^\complement \neq \varnothing$. Therefore, $x \notin \mathring{A}$ which implies that $\partial A \subseteq \bar{A} \backslash \mathring{A}$.

On the other hand, if $x \in \bar{A} \backslash \mathring{A}$, then $\forall\, \varepsilon > 0, D(x, \varepsilon) \nsubseteq A$. As a consequence, $\forall\, \varepsilon > 0, D(x, \varepsilon) \cap A^\complement \neq \varnothing$; thus $x \in \overline{A^\complement}$ and this further implies that $x \in \bar{A} \cap \overline{A^\complement} = \partial A$. □

Example 2.56. Let $M = \mathbb{R}$, $d(x, y) = |x - y|$, and $A = [0, 1] \cap \mathbb{Q}$. Then

1. $A' = [0, 1]$.
   $\big(r \in A, r + \dfrac{1}{n} \in A, r + \dfrac{1}{n} \to r \Rightarrow r \in A'$.
   If $s \in [0, 1] \cap \mathbb{Q}^\complement$. $\exists\, s_n \in A, s_n \to s \Rightarrow s \in A'$.
   If $t \notin [0, 1]$, $\exists \varepsilon > 0 \ni D(t, \varepsilon) \cap [0, 1] = \varnothing \Rightarrow t \notin A'\big)$.

2. $\bar{A} = [0, 1] (= A \cup A')$. 3. $\mathring{A} = \varnothing$. 4. $\partial A = [0, 1]$.

Example 2.57. Let (M, d) be a metric space with discrete metric, and $A \subseteq M$. Recall that every point is an open set.

1. A is open. 2. A is also closed since $A^\complement$ is open. 3. $\mathring{A} = A$. 4. $A' = \varnothing$.

5. $\text{cl}(A) = \bar{A} = A$. 6. $\partial A = \varnothing$.

Remark 2.58. If $A \subseteq B$, then $\partial A \nsubseteq \partial B$. For example, let $A = \mathbb{Q} \cap [0, 1]$ and $B = [0, 1]$. Then $A \subseteq B$ but $\partial A = [0, 1]$, $\partial B = \{0, 1\}$.

Example 2.59. $\partial A \nsubseteq A'$: take $A = \{0\}$, then $A' = \varnothing$, $\partial A = \{0\}$.

Example 2.60. It is not always true that $\partial A = \partial(\text{int}(A))$. For example, take $A = [0,1] \cup \{2\}$, then $\partial A = \{0,1,2\}, \text{int}(A) = (0,1), \partial(\text{int}(A)) = \{0,1\}$, so $\partial A \neq \partial(\text{int}(A))$.

Example 2.61. Let (M,d) be a metric space, and $A, B \subseteq M$. Then

$$\partial(A \cup B) \subseteq \partial A \cup \partial B \qquad \text{and} \qquad \partial(A \cap B) \subseteq \partial A \cup \partial B$$

since

$$\begin{aligned} x \in \partial(A \cup B) &\Leftrightarrow \forall\, r > 0, D(x,r) \cap (A \cup B) \neq \varnothing \text{ and } D(x,r) \cap (A^{\complement} \cap B^{\complement}) \neq \varnothing \\ &\Rightarrow \forall\, r > 0, D(x,r) \cap A^{\complement} \neq \varnothing, D(x,r) \cap B^{\complement} \neq \varnothing, \text{ and one of the following} \\ &\quad\ \text{holds: } D(x,r) \cap A \neq \varnothing \text{ or } D(x,r) \cap B \neq \varnothing \\ &\Rightarrow x \in \bar{A} \cap \overline{A^{\complement}} \text{ or } x \in \bar{B} \cap \overline{B^{\complement}}, \end{aligned}$$

and with $A^{\complement}, B^{\complement}$ replacing A, B in the inclusion we just arrive,

$$\partial(A \cap B) = \partial(A \cap B)^{\complement} = \partial(A^{\complement} \cup B^{\complement}) \subseteq \partial A^{\complement} \cup \partial B^{\complement} = \partial A \cup \partial B\,.$$

2.3 Sequences and Completeness（完備性）

Definition 2.62. Let (M,d) be a metric space. A sequence in (M,d) is a function $f : \mathbb{N} \to M$, and is denoted by $\{f(n)\}_{n=1}^{\infty}$. Write x_n for $f(n)$. A sequence $\{x_n\}_{n=1}^{\infty}$ in M is said to converge to x if

$$\begin{aligned} &\forall\, \varepsilon > 0, \exists\, N > 0 \ \ni d(x_n, x) < \varepsilon \text{ whenever } n \geqslant N. \\ \Leftrightarrow\ &\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, d(x_n, x) \geqslant \varepsilon\big\} < \infty. \\ \Leftrightarrow\ &\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin D(x, \varepsilon)\big\} < \infty. \end{aligned}$$

As Definition 1.46, one writes $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$ to denote that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x.

Remark 2.63. Let (M,d) be a metric space, $A \subseteq M$ be a subset.

$$\begin{aligned} x \text{ is a limit point of } A &\Leftrightarrow \forall\, \varepsilon > 0,\ D(x, \varepsilon) \cap A \neq \varnothing. \\ &\Leftrightarrow \forall\, n > 0,\ \exists\, x_n \in A,\ x_n \in D(x, \frac{1}{n}). \\ &\Leftrightarrow \forall\, n > 0,\ \exists\, x_n \in A, d(x_n, x) < \frac{1}{n}. \\ &\Leftrightarrow \exists\, \{x_n\}_{n=1}^{\infty} \subseteq A \ni x_n \to x \text{ as } n \to \infty \end{aligned}$$

and

$$\begin{aligned} y \text{ is an accumulation point of } A &\Leftrightarrow \forall\, \varepsilon > 0,\ D(y,\varepsilon) \cap (A\backslash\{y\}) \neq \varnothing. \\ &\Leftrightarrow \forall\, n > 0,\ \exists\, y_n \neq y,\ y_n \in A,\ y_n \in D(y, \frac{1}{n}). \\ &\Leftrightarrow \forall\, n > 0,\ \exists\, y_n \neq y,\ y_n \in A,\ d(y_n, y) < \frac{1}{n}. \\ &\Leftrightarrow \exists\, \{y_n\}_{n=1}^{\infty} \subseteq A\backslash\{y\} \ni y_n \to y \text{ as } n \to \infty. \end{aligned}$$

Remark 2.64. A is closed $\Leftrightarrow$ $A = \mathrm{cl}(A) = \bar{A}$ $\Leftrightarrow$ If $\{x_n\}_{n=1}^{\infty} \subseteq A$ and $x_n \to x$ as $n \to \infty$, then $x \in A$.

Remark 2.65. The sequence $\{x_k\}_{k=1}^{\infty}$ does not converge to x as $k \to \infty$ if

$$\begin{aligned} \exists\, \varepsilon > 0 \ni \forall\, N > 0,\ \exists\, k \geqslant N \ni d(x_k, x) \geqslant \varepsilon &\Leftrightarrow \exists\, \varepsilon > 0 \ni \#\big\{n \in \mathbb{N} \,\big|\, d(x_n, x) \geqslant \varepsilon\big\} = \infty \\ &\Leftrightarrow \exists\, \varepsilon > 0 \ni \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin D(x,\varepsilon)\big\} = \infty. \end{aligned}$$

Proposition 2.66. *In $\mathbb{R}^n$, a sequence of vectors converges if and only if every component of the vectors converges. In other words, in $\mathbb{R}^n$*

$$\textit{Componentwise convergence} \ \Leftrightarrow\ \textit{Convergence.}$$

Proof. Let $\{v_k\}_{k=1}^{\infty}$, $v_k = (v_k^{(1)}, v_k^{(2)}, \cdots, v_k^{(n)})$, be a sequence of vectors in $\mathbb{R}^n$.

"$\Rightarrow$" Suppose $v_k \to v = (v^{(1)}, \cdots, v^{(n)})$ as $k \to \infty$. Then

$$\forall\, \varepsilon > 0,\ \exists\, N > 0 \ni \|v_k - v\|_2 < \varepsilon \text{ whenever } k \geqslant N;$$

thus if $k \geqslant N$,

$$|v_k^{(i)} - v^{(i)}| \leqslant \|v_k - v\|_2 = \sqrt{(v_k^{(1)} - v^{(1)})^2 + \cdots + (v_k^{(n)} - v^{(n)})^2} < \varepsilon.$$

"$\Leftarrow$" Assume that $v_k^{(i)} \to u_i$ as $k \to \infty$ for $i = 1, 2, \cdots, n$. Then

$$\forall\, \varepsilon > 0, \exists\, N_i > 0,\ \ni |v_k^{(i)} - u_i| < \frac{\varepsilon}{\sqrt{n}} \text{ whenever } k \geqslant N_i.$$

Let $N = \max\{N_1, N_2, \cdots, N_n\}$. Then if $k \geqslant N$,

$$\|v_k - u\|_2 = \sqrt{(v_k^{(1)} - u_1)^2 + \cdots + (v_k^{(n)} - u_n)^2} < \sqrt{\frac{\varepsilon^2}{n} + \cdots + \frac{\varepsilon^2}{n}} = \varepsilon.$$

□

Example 2.67. Let $v_k = \big(\frac{1}{k}, \frac{1}{k^2}\big) \in \mathbb{R}^2$. Then $v_k \to (0,0)$ as $k \to \infty$ since

$$\sqrt{(\frac{1}{k} - 0)^2 + (\frac{1}{k^2} - 0)^2} = \frac{1}{k^2}\sqrt{k^2+1} \to 0 \text{ as } k \to \infty\,.$$

Proposition 2.68. *Suppose that $\{v_k\}_{k=1}^{\infty}$ and $\{w_k\}_{k=1}^{\infty}$ are sequences of vectors in a normed space $(\mathcal{V}, \|\cdot\|)$, λ_k is a sequence in $\mathbb{R}$, and $v_k \to v,\ w_k \to w$ in $\mathcal{V}$, $\lambda_k \to \lambda$ in $\mathbb{R}$ as $k \to \infty$. Then*

1. $v_k + w_k \to v + w$ *as* $k \to \infty$.

2. $\lambda_k v_k \to \lambda v$ *as* $k \to \infty$.

3. $\dfrac{1}{\lambda_k} v_k \to \dfrac{1}{\lambda} v$ *as* $k \to \infty$ *if* $\lambda_k \neq 0,\ \lambda \neq 0$.

Proposition 2.69. *Let (M, d) be a metric space.*

1. *A set $A \subseteq M$ is closed if and only if every convergent sequence $\{x_k\}_{k=1}^{\infty} \subseteq A$ converges to a limit in A.*

2. $x \in \bar{A}$ *if and only if there is a sequence* $\{x_k\}_{k=1}^{\infty} \subseteq A,\ \ni x_k \to x$ *as* $k \to \infty$

Proof. "⇒" Since A is closed, Theorem 2.38 implies that $A = \bar{A}$. Let $\{x_k\}_{k=1}^{\infty} \subseteq A$ be a convergent sequence with limit x. Then

$$\forall\, \varepsilon > 0,\ \exists\, N > 0 \ni d(x_k, x) < \varepsilon \text{ whenever } k \geqslant N.$$

Therefore,

$$\forall\, \varepsilon > 0,\ D(x, \varepsilon) \cap A \supseteq \{x_k\}_{k=N}^{\infty} \neq \varnothing$$

which implies that $x \in \bar{A} (= A)$.

"⇐" Assume the contrary that A is not closed. Then

$$\exists\, x \in A^{\complement} \ \ni \forall\, \varepsilon > 0,\ D(x, \varepsilon) \nsubseteq A^{\complement}.$$

Let $\varepsilon = \frac{1}{n},\ x_n \in D\big(x, \frac{1}{n}\big) \cap A$. Then $\{x_n\}_{n=1}^{\infty} \subseteq A$ and $x_n \to x$ as $n \to \infty$; thus we obtain a sequence $\{x_n\}_{n=1}^{\infty}$ which converges to a point $x \notin A$, a contradiction. □

Example 2.70. Suppose $\{x_k\} \subseteq \mathbb{R}^n$ is such that (i) $\|x_k\| \leqslant 1$ (ii) $x_k \to x$ as $k \to \infty$.
Question 1: $\|x\| \leqslant 1$?
Question 2: Can $\leqslant$ be replaced by $<$; that is, is it true that $\|x_k\| < 1$, $x_k \to x$ as $k \to \infty$, then $\|x\| < 1$?
Answer to Question 1: Yes, consider $B(0,1) = \{x \in \mathbb{R}^n \,|\, \|x\| \leqslant 1\}$. Then B is closed since if $x \in B^{\complement}$, $\exists\, \varepsilon = \|x\|_2 - 1 > 0 \ni D(x,\varepsilon) \subseteq B^{\complement}$. Since $\{x_k\}_{k=1}^{\infty} \subseteq B$ and $x_k \to x$ as $k \to \infty$, by Proposition 2.69 $x \in B$; thus $\|x\| \leqslant 1$.

On the other hand, we can obtain the inequality above by the triangle inequality:

$$\|x\|_2 \leqslant \|x_k - x\|_2 + \|x_k\|_2 \leqslant \|x_k - x\|_2 + 1 \;\forall\, k > 0 \Rightarrow \|x\|_2 \leqslant \lim_{k\to\infty} \|x_k - x\|_2 + 1 = 1.$$

Answer to Question 2: No. For example, consider the case $n = 1$, and take $x_k = 1 - \dfrac{1}{n}$. Then $|x_k| < 1$ and $x_k \to x = 1$ as $k \to \infty$. However, $|x| = 1 \nless 1$.

Definition 2.71. A point x in a metric space is said to be a ***cluster point*** of a sequence $\{x_n\}_{n=1}^{\infty}$ if

$$\forall\, \varepsilon > 0, \#\{n \in \mathbb{N} \,|\, x_n \in D(x,\varepsilon)\} = \infty\,.$$

Proposition 2.72. *If $\{x_n\}_{n=1}^{\infty}$ is a sequence in a metric space (M,d), then*

1. *x is a cluster point of $\{x_n\}_{n=1}^{\infty}$ if and only if $\forall\, \varepsilon > 0$ and $N > 0, \exists\, n \geqslant N \ni d(x_n, x) < \varepsilon$.*
2. *x is a cluster point of $\{x_n\}_{n=1}^{\infty}$ if and only if $\exists\, \{x_{n_j}\}_{j=1}^{\infty} \ni x_{n_j} \to x$ as $j \to \infty$.*
3. *$x_n \to x$ as $n \to \infty$ if and only if every subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to x.*
4. *$x_n \to x$ as $n \to \infty$ if and only if every proper subsequence of $\{x_n\}_{n=1}^{\infty}$ has a further subsequence that converges to x.*

Proof. See the proof of Proposition 1.109 by changing $|\cdot - \cdot|$ to $d(\cdot,\cdot)$. □

Theorem 2.73. *The collection of cluster points of a sequence is closed.*

Proof. Let $\{x_k\}_{k=1}^{\infty} \subseteq M$ be a sequence, and A be the collection of cluster points of $\{x_k\}_{k=1}^{\infty}$. If $y \in A^{\complement}$, then y is not a cluster point of $\{x_k\}_{k=1}^{\infty}$; thus

$$\exists\, \varepsilon > 0 \ni \#\{n \in \mathbb{N} \,|\, x_n \in D(y,\varepsilon)\} < \infty\,.$$

If $z \in D(y,\varepsilon)$, let $r = \varepsilon - d(y,z) > 0$, then $D(z,r) \subseteq D(y,\varepsilon)$ **(Check!)**. As a consequence, $\#\{n \in \mathbb{N} \,|\, x_n \in D(z,r)\} < \infty$.

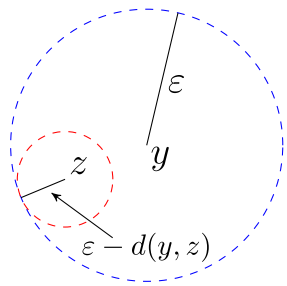


Figure 2.3: $D(z, \varepsilon - d(y,z)) \subseteq D(y,\varepsilon)$ if $z \in D(y,\varepsilon)$

Therefore, $z \in A^{\complement}$ which implies that $D(y,\varepsilon) \subseteq A^{\complement}$; thus A is closed. □

Next we talk about the completeness of a metric space. Recall that the completeness of an order field is defined by the monotone sequence property (or the least upper bound property) which relies on the concept of order, so we cannot define the completeness of a metric space via these two properties. On the other hand, Theorem 1.103 suggests that when the concept of order is out of scope, the convergence of all Cauchy sequences seems a good replacement for completeness. This is in fact how we define the completeness of general metric spaces. To be more precise, we start with the following

Definition 2.74. Let (M,d) be a metric space. A sequence $\{x_k\}_{k=1}^{\infty} \subseteq M$ is said to be ***Cauchy*** if

$$\forall\, \varepsilon > 0,\ \exists\, N > 0 \ni d(x_n, x_m) < \varepsilon \text{ whenever } n, m \geqslant N.$$

Definition 2.75. A metric space (M,d) is said to be ***complete*** if every Cauchy sequence in M converges to a limit in M.

Definition 2.76. A ***Banach space*** is a complete normed vector space.

Definition 2.77. A sequence $\{x_k\}_{k=1}^{\infty}$ in a normed space $(\mathcal{V}, \|\cdot\|)$ is said to be ***bounded*** if

$$\exists\, B > 0 \ni \|x_k\| \leqslant B \ \forall\, k \in \mathbb{N}.$$

Definition 2.78. A sequence $\{x_k\}_{k=1}^{\infty}$ in a metric space (M,d) is said to be ***bounded*** if

$$\exists\, x_0 \in M \text{ and } B > 0 \ni d(x_k, x_0) \leqslant B \ \forall\, k \in \mathbb{N}.$$

Remark 2.79. Adopting the definition of boundedness in a metric space, a sequence $\{x_k\}_{k=1}^{\infty}$ in a nomed space $(\mathcal{V}, \|\cdot\|)$ is bounded if

$$\exists\, x_0 \in \mathcal{V} \text{ and } B > 0 \ni \|x_k - x_0\| \leqslant B \ \forall\, k \in \mathbb{N};$$

thus $\|x_k\| \leqslant \|x_0\| + B \equiv \widetilde{B}$. Therefore, Definition 2.78 implies Definition 2.77.

Proposition 2.80. *A convergent sequence in (M,d) is bounded.*

Proof. Let $\{x_k\}_{k=1}^\infty$ be a convergent sequence in M with limit x_0. Then

$$\forall\,\varepsilon>0,\ \exists\,N>0 \ni d(x_k,x_0)<\varepsilon \text{ whenever } k\geqslant N.$$

Let $C=\max\big\{d(x_1,x_0),d(x_2,x_0),\cdots,d(x_{N-1},x_0),\varepsilon\big\}+1$. Then $d(x_k,x_0)\leqslant C\ \forall\,k\in\mathbb{N}$. ▫

Proposition 2.81.

1. *Every convergent sequence in (M,d) is Cauchy.*
2. *If a subsequence of Cauchy sequence converges, then this Cauchy sequence also converges.*

Proof. See the proof of Proposition 1.96 and Lemma 1.101 by changing $|\cdot|$ to $d(\cdot,\cdot)$. ▫

Theorem 2.82. *A sequence in $\mathbb{R}^n$ converges if and only if the sequence is Cauchy* $\Big($*because of that* $\max\limits_{1\leqslant i\leqslant n}|v_k^{(i)}-u_i|\leqslant\|v_k-u\|_2\leqslant\sqrt{n}\max\limits_{1\leqslant i\leqslant n}|v_k^{(i)}-u_i|\Big)$.

Theorem 2.83. *Let (M,d) be a complete metric space, and $N\subseteq M$ be a closed subset. Then (N,d) is complete* (完備空間中之閉集合是完備的).

Proof. Let $\{x_k\}_{k=1}^\infty\subseteq N$ be Cauchy sequence. Then

$$\forall\,\varepsilon>0,\exists\,N_0>0\ni d(x_n,x_m)<\varepsilon \text{ if } n,m\geqslant N_0.$$

Therefore, $\{x_k\}_{k=1}^\infty$ is Cauchy in (M,d). By completeness of (M,d), $\exists\,x\in M\ni x_k\to x$ as $k\to\infty$. Note that $x\in N$ since N is closed. ▫

Theorem 2.84. *Let (M,d) be a metric space, and A is dense subset of M; that is, $A\subseteq M\subseteq\bar{A}$. If every Cauchy sequence in A converges in M, then (M,d) is complete.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a Cauchy sequence in M. Since A is dense in M, for each $n\in\mathbb{N}$ there exists $\big\{x_k^{(n)}\big\}_{k=1}^\infty$ such that $x_k^{(n)}\to x_n$ as $k\to\infty$, and for each $j\in\mathbb{N}$, there exists $N(j)>0$ such that

$$d\big(x_k^{(n)},x_n\big)<\frac{1}{j}\qquad\forall\,k\geqslant N(j)\,.$$

Let $y_k=x_{N(k)}^{(k)}$. Then

$$d(y_k,y_\ell)\leqslant d\big(x_{N(k)}^{(k)},x_k\big)+d(x_k,x_\ell)+d\big(x_\ell,x_{N(\ell)}^{(\ell)}\big)<\frac{1}{k}+\frac{1}{\ell}+d(x_k,x_\ell)$$

which implies that $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence (for $\lim\limits_{k,\ell\to\infty} d(y_k, y_\ell) = 0$). Since $\{y_n\}_{n=1}^{\infty} \subseteq A$, by assumption it converges to some point $x \in M$; thus for a given $\varepsilon > 0$, there exists $K > 0$ such that

$$d\big(x^{(n)}_{N(n)}, x\big) = d(y_n, x) < \frac{\varepsilon}{2} \qquad \forall\, n \geqslant K\,.$$

Choose $J > 0$ such that $\dfrac{1}{J} < \dfrac{\varepsilon}{2}$. Then if $n \geqslant \max\big\{K, N(J)\big\}$,

$$d(x_n, x) \leqslant d\big(x_n, x^{(n)}_{N(n)}\big) + d\big(x^{(n)}_{N(n)}, x\big) < \frac{1}{J} + \frac{\varepsilon}{2} < \varepsilon\,. \qquad \square$$

2.4 Series of real numbers and vectors

Definition 2.85. Let $(\mathcal{V}, \|\cdot\|)$ be a normed space. A series $\sum\limits_{k=1}^{\infty} x_k$, where $\{x_k\}_{k=1}^{\infty} \subseteq \mathcal{V}$, is said to ***converge*** to $S \in \mathcal{V}$ if the partial sum $S_n = \sum\limits_{k=1}^{n} x_k$ converges to S, and one writes $S = \sum\limits_{k=1}^{\infty} x_k$ if this is the case.

Theorem 2.86. *Let $(\mathcal{V}, \|\cdot\|)$ be a complete normed space (called **Banach space**). A series $\sum\limits_{k=1}^{\infty} x_k$ converges if and only if*

$$\forall\, \varepsilon > 0, \exists\, N > 0 \ni \|x_k + x_{k+1} + \cdots + x_{k+p}\| < \varepsilon \quad \textit{if } k \geqslant N, p \geqslant 0.$$

Proof. Let $S_n = \sum\limits_{k=1}^{n} x_k$ be partial sum of $\sum\limits_{k=1}^{\infty} x_k$. Then

$$\begin{aligned}
\{S_n\}_{n=1}^{\infty} \text{ converges in } \mathcal{V} &\Leftrightarrow \{S_n\}_{n=1}^{\infty} \text{ is Cauchy}\\
&\Leftrightarrow \forall\, \varepsilon > 0, \exists\, N > 0 \ni \|S_n - S_m\| < \varepsilon \text{ if } n, m \geqslant N\\
&\Leftrightarrow \forall\, \varepsilon > 0, \exists\, N > 0 \ni \|x_{n+1} + x_{n+2} + \cdots + x_m\| < \varepsilon \text{ if } m > n \geqslant N\\
&\Leftrightarrow \forall\, \varepsilon > 0, \exists\, N > 0 \ni \|x_k + x_{k+1} + \cdots + x_{k+p}\| < \varepsilon \text{ if } k \geqslant N+1, p \geqslant 0. \qquad \square
\end{aligned}$$

Corollary 2.87. *If $\sum\limits_{k=1}^{\infty} x_k$ converges, then $\|x_k\| \to 0$ as $k \to \infty$, and if $\|x_k\| \nrightarrow 0$ as $k \to \infty$, then $\sum\limits_{k=1}^{\infty} x_k$ diverges.*

Proof. Take $p = 0$ in Theorem 2.86. $\square$

Definition 2.88. A series $\sum\limits_{k=1}^{\infty} x_k$ is said to ***converge absolutely*** if $\sum\limits_{k=1}^{\infty} \|x_k\|$ converges in $\mathbb{R}$. A series that is convergent but not absolutely convergent is said to be ***conditionally convergent***.

Example 2.89. $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k}$ is conditionally convergent.

Theorem 2.90. *In a complete normed space, if* $\sum\limits_{k=1}^{\infty} x_k$ *converges absolutely, then* $\sum\limits_{k=1}^{\infty} x_k$ *converges.*

Proof. If $\sum\limits_{k=1}^{\infty} x_k$ converges absolutely, then $S_n = \sum\limits_{k=1}^{n} \|x_k\|$ converges in $\mathbb{R}$. Then

$$\forall\, \varepsilon > 0, \exists\, N > 0 \ni \big|\|x_k\| + \|x_{k+1}\| + \cdots + \|x_{k+p}\|\big| < \varepsilon \text{ if } k \geqslant N, p \geqslant 0.$$

Therefore, if $k \geqslant N, p \geqslant 0$,

$$\|x_k + x_{k+1} + \cdots + x_{k+p}\| \leqslant \|x_k\| + \cdots + \|x_{k+p}\| < \varepsilon\,.$$

□

Theorem 2.91. 1. ***Geometric series****:*

(a) *If* $|r| < 1$*, then* $\sum\limits_{k=1}^{\infty} r^k$ *converges absolutely to* $\frac{r}{1-r}$.

(b) *If* $|r| > 1$*, then* $\sum\limits_{k=1}^{\infty} r^k$ *does not converge (diverge).*

2. ***Comparison test****:*

(a) *If* $\sum\limits_{k=1}^{\infty} a_k$ *converges,* $a_k \geqslant 0$*, and* $0 \leqslant b_k \leqslant a_k$*, then* $\sum\limits_{k=1}^{\infty} b_k$ *converges.*

(b) *If* $\sum\limits_{k=1}^{\infty} a_k$ *diverges,* $a_k \geqslant 0$*, and* $a_k \leqslant b_k$*, then* $\sum\limits_{k=1}^{\infty} b_k$ *diverges.*

3. ***p-series****:*
$\sum\limits_{k=1}^{\infty} \frac{1}{k^p}$ *converges if* $p > 1$ *and diverges if* $p \leqslant 1$.

4. ***Root test****:*

(a) *If* $\limsup\limits_{k\to\infty} \sqrt[k]{|x_k|} < 1$*, then* $\sum\limits_{k=1}^{\infty} x_k$ *converges absolutely.*

(b) *If* $\limsup\limits_{k\to\infty} \sqrt[k]{|x_k|} > 1$*, then* $\sum\limits_{k=1}^{\infty} x_k$ *diverges.*

5. ***Ratio and comparison test****:*
Let $\sum\limits_{k=1}^{\infty} a_k$ *and* $\sum\limits_{k=1}^{\infty} b_k$ *be series, and* $b_k > 0$ *for all* $k \in \mathbb{N}$.

(a) $\limsup\limits_{k\to\infty} \frac{|a_k|}{b_k} < \infty$, $\sum\limits_{k=1}^{\infty} b_k$ *is convergent, then* $\sum\limits_{k=1}^{\infty} a_k$ *converges absolutely.*

(b) $\liminf\limits_{k\to\infty} \frac{a_k}{b_k} > 0$, $\sum\limits_{k=1}^{\infty} b_k$ *is divergent, then* $\sum\limits_{k=1}^{\infty} a_k$ *diverges.*

6. ***Integral test****:*

If f is continuous, non-negative, and monotone decreasing on $[1,\infty)$, then $\sum\limits_{k=1}^{\infty} f(k)$ converges if and only if the improper integral $\int_1^{\infty} f(x)dx < \infty$.

7. ***Alternative series****:*

$$\sum_{k=1}^{\infty}(-1)^k a_k \textit{ is convergent if } a_k \geqslant 0,\ a_k \searrow 0 \ (\textit{that is}, a_k \geqslant a_{k+1}, a_k \to 0 \textit{ as } k \to \infty).$$

Remark 2.92. By Problem 1.17,

$$\liminf_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} \leqslant \liminf_{k\to\infty} \sqrt[k]{|x_k|} \leqslant \limsup_{k\to\infty} \sqrt[k]{|x_k|} \leqslant \limsup_{k\to\infty} \frac{|x_{k+1}|}{|x_k|}\,.$$

As a consequence, by the root test we obtain

1. if $\limsup\limits_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} < 1$, the series $\sum\limits_{k=1}^{\infty} x_k$ converges absolutely, and

2. if $\liminf\limits_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} > 1$, the series $\sum\limits_{k=1}^{\infty} x_k$ diverges.

This is called the ***ratio test***.

Example 2.93. Let

$$x_k = \begin{cases} \dfrac{1}{2^{\frac{k+1}{2}}} & \text{if } k \text{ is odd,} \\ \dfrac{1}{3^{\frac{k}{2}}} & \text{if } k \text{ is even,} \end{cases}$$

that is, $\{x_k\}_{k=1}^{\infty} = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{9}, \frac{1}{8}, \frac{1}{27}, \cdots\right\}$, be a sequence in $\mathbb{R}$. Then

1. $\liminf\limits_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} = 0$;

2. $\liminf\limits_{k\to\infty} \sqrt[k]{|x_k|} = \frac{1}{\sqrt{3}}$;

3. $\limsup\limits_{k\to\infty} \sqrt[k]{|x_k|} = \dfrac{1}{\sqrt{2}}$;

4. $\limsup\limits_{k\to\infty} \dfrac{|x_{k+1}|}{|x_k|} = \infty$.

Therefore, $\sum\limits_{k=1}^{\infty} x_k$ converges absolutely.

2.5 Exercises

§2.1 Open Sets and the Interior of Sets

Problem 2.1. Show that every open set in $\mathbb{R}$ is the union of at most countable collection of disjoint open intervals; that is, if $\mathcal{U} \subseteq \mathbb{R}$ is open, then

$$\mathcal{U} = \bigcup_{k\in\mathcal{I}} (a_k, b_k)\,,$$

where $\mathcal{I}$ is countable, and $(a_k, b_k) \cap (a_\ell, b_\ell) = \varnothing$ if $k \neq \ell$.

Problem 2.2. Let (M, d) be a metric space, and $A \subseteq M$. An open cover of A is a collection of open sets whose union contains A; that is, $\{\mathcal{U}_i\}_{i\in\mathcal{I}}$ is called an open cover of A if

1. $\mathcal{U}_i$ is open for all $i \in \mathcal{I}$.

2. $A \subseteq \bigcup\limits_{i\in\mathcal{I}} \mathcal{U}_i$.

Show that

1. if $\big\{(a_k, b_k)\big\}_{k=1}^{\infty}$ is an open cover of $[a, b] \subseteq \mathbb{R}$, then there exists $N > 0$ such that $\bigcup\limits_{k=1}^{N} (a_k, b_k) \supseteq [a, b]$.

2. Using Exercise 2.1 to show that if $\{\mathcal{U}_k\}_{k=1}^{\infty}$ is an open cover of $[a, b]$, then there exists $N > 0$ such that $\bigcup\limits_{k=1}^{N} \mathcal{U}_k \supseteq [a, b]$.

Problem 2.3. Let A and B be subsets of a metric space (M, d). Show that

1. $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$.

2. $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$.

§2.2 Closed Sets, the Closure of Sets, and the Boundary of Sets

Problem 2.4. Let (M, d) be a metric space, and $A \subseteq M$. Show (by definition) that $\bar{A}$ is closed .

Problem 2.5. Let (M, d) be a metric space, and $A \subseteq M$. Show that $A' = \bar{A} \backslash (A \backslash A')$. In other words, the derived set consists of all limit points that are not isolated points. Also show that $\bar{A} \backslash A' = A \backslash A'$.

Problem 2.6. Let $A \subseteq \mathbb{R}^n$. Define the sequence of sets $A^{(m)}$ as follows: $A^{(0)} = A$ and $A^{(m+1)} =$ the derived set of $A^{(m)}$ for $m \in \mathbb{N}$. Do the following problems.

1. Prove that each $A^{(m)}$ for $m \in \mathbb{N}$ is a closed set; thus $A^{(1)} \supseteq A^{(2)} \supseteq \cdots$.

2. Show that if there exists some $m \in \mathbb{N}$ such that $A^{(m)}$ is a countable set, then A is countable.

3. For any given $m \in \mathbb{N}$, is there a set A such that $A^{(m)} \neq \varnothing$ but $A^{(m+1)} = \varnothing$.

4. Let A be uncountable. Then each $A^{(m)}$ is an uncountable set. Is it possible that $\bigcap_{m=1}^{\infty} A^{(m)} = \varnothing$?

5. Let $A = \left\{ \frac{1}{m} + \frac{1}{k} \,\middle|\, m - 1 > k(k-1), m, k \in \mathbb{N} \right\}$. Find $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$.

Problem 2.7. Let A and B be subsets of a metric space (M, d). Show that

1. $\text{cl}(\text{cl}(A)) = \text{cl}(A)$.

2. $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.

3. $\text{cl}(A \cap B) \subseteq \text{cl}(A) \cap \text{cl}(B)$. Find examples of that $\text{cl}(A \cap B) \subsetneq \text{cl}(A) \cap \text{cl}(B)$.

Problem 2.8. Let (M, d) be a metric space, and $A \subseteq M$ be a subset. Show that

$$\partial A = \big(A \cap \text{cl}(M \backslash A)\big) \cup \big(\text{cl}(A) \backslash A\big).$$

Problem 2.9. Let A and B be subsets of a metric space (M, d). Show that

1. $\partial A = \partial(M \backslash A)$.

2. $\partial(\partial A) \subseteq \partial(A)$. Find examples of that $\partial(\partial A) \subsetneq \partial A$. Also show that $\partial(\partial A) = \partial A$ if A is closed.

3. $\partial(A \cup B) \subseteq \partial A \cup \partial B \subseteq \partial(A \cup B) \cup A \cup B$. Find examples of that equalities do not hold.

4. If $\text{cl}(A) \cap \text{cl}(B) = \varnothing$, then $\partial(A \cup B) = \partial A \cup \partial B$.

5. $\partial(\partial(\partial A)) = \partial(\partial A)$.

Problem 2.10. Let (M, d) be a metric space, and $A \subseteq M$ be a subset. Determine which of the following statements are true.

1. $\text{int}A = A \backslash \partial A$.

2. $\text{cl}(A) = M \backslash \text{int}(M \backslash A)$.

3. $\text{int}(\text{cl}(A)) = \text{int}(A)$.

4. $\text{cl}(\text{int}(A)) = A$.

5. $\partial(\text{cl}(A)) = \partial A$.

6. If A is open, then $\partial A \subseteq M \backslash A$.

7. If A is open, then $A = \text{cl}(A) \backslash \partial A$. How about if A is not open?

Problem 2.11. Let (M, d) be a metric space. A set $A \subseteq M$ is said to be perfect if $A = A'$. The Cantor set is constructed by the following procedure: let $E_0 = [0, 1]$. Remove the segment $\left(\frac{1}{3}, \frac{2}{3}\right)$, and let E_1 be the union of the intervals

$$\left[0, \frac{1}{3}\right], \left[\frac{2}{3}, 1\right].$$

Remove the middle thirds of these intervals, and let E_2 be the union of the intervals

$$\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{3}{9}\right], \left[\frac{6}{9}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right].$$

Continuing in this way, we obtain a sequence of closed set E_k such that

(a) $E_1 \supseteq E_2 \supseteq E_2 \supseteq \cdots$;

(b) E_n is the union of 2^n intervals, each of length 3^{-n}.

The set $C = \bigcap_{n=1}^{\infty} E_n$ is called the ***Cantor set***.

1. Show that C is a perfect set; that is, $C = C'$.

2. Show that C is uncountable.

3. Find $\text{int}(C)$.

Problem 2.12. In a metric space (M, d), if subsets satisfy $A \subseteq S \subseteq \text{cl}(A)$, then A is said to be dense in S. For example, $\mathbb{Q}$ is dense in $\mathbb{R}$.

1. Show that if A is dense in S and if S is dense in T, then A is dense in T.

2. Show that if A is dense in S and $B \subseteq S$ is open, then $B \subseteq \text{cl}(A \cap B)$.

§2.3 Sequences and Completeness

Problem 2.13. Show that

1. Every convergent sequence in a metric space is a Cauchy sequence.

2. If a subsequence of a Cauchy sequence converges to x, then the sequence converges to x.

3. x is a cluster point of $\{x_k\}_{k=1}^{\infty}$ if and only if $\forall\, \varepsilon > 0$ and $N > 0$, $\exists\, k > N$ with $d(x_k, x) < \varepsilon$.

4. x is a cluster point of $\{x_k\}_{k=1}^{\infty}$ if and only if there is a subsequence converging to x.

5. $x_k \to x$ as $k \to \infty$ if and only if every subsequence of $\{x_k\}_{k=1}^{\infty}$ converges to x.

6. $x_k \to x$ as $k \to \infty$ if and only if every proper subsequence of $\{x_k\}_{k=1}^{\infty}$ has a further subsequence that converges to x.

Problem 2.14. Let (M, d) be a metric space, and $N \subseteq M$. Show that if (N, d) is complete, then N is closed.

Remark: Theorem 2.83 states that if (M, d) is a complete metric space and N is a closed subset of M, then (N, d) is complete. This problem gives a reverse statement.

Problem 2.15. Let a_n be defined by $a_n = \begin{cases} \dfrac{n+1}{2^n} & \text{if } n \text{ is odd}\,, \\ \dfrac{n}{3^n} & \text{if } n \text{ is even}\,. \end{cases}$ Compute the value of $\liminf\limits_{n\to\infty} \sqrt[n]{a_n}$, $\limsup\limits_{n\to\infty} \sqrt[n]{a_n}$, $\liminf\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ and $\limsup\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$, and conclude that whether the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent or not.

Hint: You can use $\lim\limits_{n\to\infty} \sqrt[n]{n} = \lim\limits_{n\to\infty} \sqrt[n]{n+1} = 1$ without proving it.

Problem 2.16. Let $\alpha \in \mathbb{R}$, $\alpha > \dfrac{1}{3}$. Discuss the absolute convergence or the conditional convergence of the series $\sum\limits_{k=2}^{\infty} \dfrac{(-1)^k}{k^\alpha + (-1)^k}$.

Problem 2.17. Determine whether the following series converge or not. Also test for their absolute convergence.

1. $\sum\limits_{n=1}^{\infty} \sin(n^{-\alpha})$, $\alpha > 0$;

2. $\sum\limits_{n=1}^{\infty} \dfrac{\log(n+1) - \log n}{\arctan \frac{2}{n}}$;

3. $\sum\limits_{n=1}^{\infty} \dfrac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{1\cdot 2\cdots n \cdot c(c+1)\cdots(c+n-1)}$;

4. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n+1}\big(1 + \dfrac{1}{3} + \cdots + \dfrac{1}{2n+1}\big)$;

5. $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{\sqrt{n} + (-1)^n}$;

The a, b, c in (3) are not negative integers.

Problem 2.18. Let $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ ba a sequence. A series $\sum\limits_{n=1}^{\infty} b_n$ is said to be a rearrangement of the series $\sum\limits_{n=1}^{\infty} a_n$ if there exists a rearrangement π of $\mathbb{N}$; that is, $\pi : \mathbb{N} \to \mathbb{N}$ is bijective, such that $b_n = a_{\pi(n)}$. Show that if $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely, then any rearrangement of the series $\sum\limits_{n=1}^{\infty} a_n$ converges and has the value $\sum\limits_{n=1}^{\infty} a_n$.

§3.1 Compactness

Problem 2.19. Let (M,d) be a metric space.

1. Show that the union of a finite number of compact subsets of M is compact.

2. Show that the intersection of an arbitrary collection of compact subsets of M is compact.

Problem 2.20. A metric space (M,d) is said to be separable if there is a countable subset A which is dense in M. Show that every compact set is separable.

Problem 2.21. Given $\{a_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ a bounded sequence. Define

$$A = \left\{ x \in \mathbb{R} \,\middle|\, \text{there exists a subsequence } \left\{a_{k_j}\right\}_{j=1}^{\infty} \text{ such that } \lim_{j\to\infty} a_{k_j} = x \right\}.$$

Show that A is a non-empty compact set in $\mathbb{R}$. Furthermore , $\limsup_{k\to\infty} a_k = \sup A$ and $\liminf_{k\to\infty} a_k = \inf A$.

Problem 2.22. Let (M,d) be a compact metric space; that is, M itself is a compact set. If $\{F_k\}_{k=1}^{\infty}$ is a sequence of closed sets such that $\text{int}(F_k) = \varnothing$, then $M \backslash \bigcup_{k=1}^{\infty} F_k \neq \varnothing$.

Problem 2.23. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(x,y) = \begin{cases} |x_1 - y_1| & \text{if } x_2 = y_2\,, \\ |x_1 - y_1| + |x_2 - y_2| + 1 & \text{if } x_2 \neq y_2\,. \end{cases} \quad \text{where } x = (x_1, x_2) \text{ and } y = (y_1, y_2).$$

1. Show that d is a metric on $\mathbb{R}^2$. In other words, $(\mathbb{R}^2, d)$ is a metric space.

2. Find $D(x,r)$ with $r < 1$, $r = 1$ and $r > 1$.

3. Show that the set $\{c\} \times [a,b] \subseteq (\mathbb{R}^2, d)$ is closed and bounded.

4. Examine whether the set $\{c\} \times [a,b] \subseteq (\mathbb{R}^2, d)$ is compact or not.

Problem 2.24. Let (M,d) be a complete metric space, and $A \subseteq M$ be totally bounded. Show that $\text{cl}(A)$ is compact.

Problem 2.25. Let $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence in a metric space, and $x_k \to x$ as $k \to \infty$. Show that the set $A \equiv \{x_1, x_2, \cdots, \} \cup \{x\}$ is compact by

1. showing that A is sequentially compact; and
2. showing that every open cover of A has a finite subcover; and
3. showing that A is totally bounded and complete.

Problem 2.26. Let Y be the collection of all sequences $\{y_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ such that $\sum\limits_{k=1}^{\infty} |y_k|^2 < \infty$. In other words,

$$Y = \Big\{ \{y_k\}_{k=1}^{\infty} \,\Big|\, y_k \in \mathbb{R} \text{ for all } k \in \mathbb{N}, \sum_{k=1}^{\infty} |y_k|^2 < \infty \Big\}.$$

Define $\|\cdot\| : Y \to \mathbb{R}$ by

$$\big\|\{y_k\}_{k=1}^{\infty}\big\| = \Big(\sum_{k=1}^{\infty} |y_k|^2\Big)^{\frac{1}{2}}.$$

1. Show that $\|\cdot\|$ is a norm on Y. The normed space $(Y, \|\cdot\|)$ usually is denoted by ℓ^2.
2. Show that $\|\cdot\|$ is induced by an inner product.
3. Show that $(Y, \|\cdot\|)$ is complete.
4. Let $B = \big\{ y \in Y \,\big|\, \|y\| \leqslant 1 \big\}$. Is E compact or not?

Problem 2.27 (True or False). Determine whether the following statements are true or false. If it is true, prove it. Otherwise, give a counter-example.

1. Every open set in a metric space is a countable union of closed sets.
2. Let $A \subseteq \mathbb{R}$ be bounded from above, then $\sup A \in A'$.
3. An infinite union of distinct closed sets cannot be closed.
4. An interior point of a subset A of a metric space (M, d) is an accumulation point of that set.
5. Let (M, d) be a metric space, and $A \subseteq M$. Then $(A')' = A'$.
6. There exists a metric space in which some unbounded Cauchy sequence exists.
7. Every metric defined in $\mathbb{R}^n$ is induced from some "norm" in $\mathbb{R}^n$.