Chapter 1

The Real Line and Euclidean Space

1.1 Ordered Fields and the Number Systems

1.1.1 Fields and partial orders

Definition 1.1. A set $\mathcal{F}$ is said to be a ***field*** (體) if there are two operations $+$ and $\cdot$ such that

1. $x+y \in \mathcal{F}$, $x \cdot y \in \mathcal{F}$ if $x, y \in \mathcal{F}$. (封閉性)

2. $x+y=y+x$ for all $x, y \in \mathcal{F}$. (commutativity, 加法的交換性)

3. $(x+y)+z=x+(y+z)$ for all $x, y, z \in \mathcal{F}$. (associativity, 加法的結合性)

4. There exists $0 \in \mathcal{F}$, called 加法單位元素, such that $x+0=x$ for all $x \in \mathcal{F}$. (the existence of zero)

5. For every $x \in \mathcal{F}$, there exists $y \in \mathcal{F}$ (usually y is denoted by $-x$ and is called x 的加法反元素) such that $x+y=0$. One writes $x-y \equiv x+(-y)$.

6. $x \cdot y=y \cdot x$ for all $x, y \in \mathcal{F}$. (乘法的交換性)

7. $(x \cdot y) \cdot z=x \cdot(y \cdot z)$ for all $x, y, z \in \mathcal{F}$. (乘法的結合性)

8. There exists $1 \in \mathcal{F}$, called 乘法單位元素, such that $x \cdot 1=x$ for all $x \in \mathcal{F}$. (the existence of unity)

9. For every $x \in \mathcal{F}$, $x \neq 0$, there exists $y \in \mathcal{F}$ (usually y is denoted by x^{-1} and is called x 的乘法反元素) such that $x \cdot y=1$. One writes $x \cdot y \equiv x \cdot x^{-1}=1$.

10. $x \cdot (y+z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathcal{F}$. (distributive law, 分配律)

11. $0 \neq 1$.

Remark 1.2. Let x and y be both multiplicative inverse（乘法反元素）of a number a in $(\mathcal{F}, +, \cdot)$. Then

$$x \cdot a = 1 \quad \Rightarrow \quad (x \cdot a) \cdot y = 1 \cdot y = y \quad \Rightarrow \quad x \cdot 1 = x \cdot (a \cdot y) = y\,;$$

thus $x = y$. In other words, the multiplicative inverse of a number is unique.

Remark 1.3. A set $\mathcal{F}$ satisfying properties 1 to 10 with $0 = 1$ consists of only one member: By distributive law, $x \cdot 0 = x \cdot (0+0) = x \cdot 0 + x \cdot 0$; thus $-(x \cdot 0) + (x \cdot 0) = -(x \cdot 0) + (x \cdot 0) + (x \cdot 0)$ which implies that $x \cdot 0 = 0$. Therefore, if $0 = 1$, then $x = x \cdot 1 = x \cdot 0 = 0$ for all $x \in \mathcal{F}$. Hence, the set $\mathcal{F}$ consists only one element 0.

Remark 1.4. If $x \in \mathcal{F}$, then $\big((1 + (-1)\big) \cdot x = 0$ which implies that $x + (-1) \cdot x = 0$. Therefore, $(-1) \cdot x = -x + x + (-1) \cdot x = -x + 0 = -x$.

Example 1.5. Let $\mathbb{Q} = \left\{ \frac{q}{p} \,\middle|\, p \neq 0, p, q \in \mathbb{Z} : \text{integers} \right\}$. Then $\mathbb{Q}$ is a field. (Check all the properties from 1 to 11).

Example 1.6. Let $\mathbb{N} = \big\{ n \in \mathbb{Z} \,\big|\, n > 0 \big\}$. Then $\mathbb{N}$ is not a field because there is no zero.

Example 1.7. Let $\mathcal{F} = \{a, b, c\}$ with the operations $+$ and $\cdot$ defined by

$+$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$\cdot$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b

.

Then $\mathcal{F}$ is a field because of the following: Properties 1, 2, 3, 6, 7 are obvious.
Property 4: $\exists$ "0" $\ni x +$ "0" $= x$ for all $x \in \mathcal{F}$. In fact, "0" $= a$.
Property 5: $\forall\, x \in \mathcal{F}$, $\exists\, y \in \mathcal{F} \ni x + y = 0$, here $b = -c$, $c = -b$.
Property 8: $\exists$ "1" $\ni x \cdot$ "1" $= x$ for all $x \in \mathcal{F}$. In fact, "1" $= b$ (so Property 11 holds since $a \neq b$).
Property 9: $\forall\, x \neq 0, \in \mathcal{F}$, $\exists\, z \in \mathcal{F} \ni x \cdot z = 1$, here $z = x$.
The validity of Property 10 is left as an exercise.

Example 1.8. Let $(\mathcal{F}, +, \cdot)$ be a field. Then $(x-y)(x+y) = x^2 - y^2$ for all $x, y \in \mathcal{F}$. In fact,

$$\begin{aligned}
(x-y)(x+y) &= (x-y)\cdot x + (x-y)\cdot y && \text{(by 分配律)}\\
&= x\cdot(x-y) + y\cdot(x-y) && \text{(by 乘法交換律)}\\
&= x\cdot x + x\cdot(-y) + y\cdot x + y\cdot(-y) && \text{(by 分配律)}\\
&= x^2 - x\cdot y + x\cdot y - y^2 && \text{(by Remark 1.4 and 乘法交換律)}\\
&= x^2 + 0 - y^2 && \text{(by Property 5)}\\
&= x^2 - y^2 && \text{(by Property 4)}.
\end{aligned}$$

Definition 1.9. A ***partial order*** over a set P is a binary relation $\preccurlyeq$ which is reflexive, anti-symmetric and transitive (滿足遞移律), in the sense that

1. $x \preccurlyeq x$ for all $x \in P$ (reflexivity).
2. $x \preccurlyeq y$ and $y \preccurlyeq x \Rightarrow x = y$ (anti-symmetry).
3. $x \preccurlyeq y$ and $y \preccurlyeq z \Rightarrow x \preccurlyeq z$ (transitivity).

A set with a partial order is called a ***partially ordered set***.

Example 1.10. Let S be a given set, and 2^S be the ***power set*** of S; that is,

$$P = 2^S = \{A \mid A \subseteq S\} = \text{the collection of all subsets of } S\,.$$

We define $\preccurlyeq$ as $\supseteq$. Then

1. $A \supseteq A$ (reflexivity).
2. $A \supseteq B$ and $B \supseteq A \Rightarrow A = B$ (anti-symmetry).
3. $A \supseteq B$ and $B \supseteq C \Rightarrow A \supseteq C$ (transitivity).

Hence, $\supseteq$ is a partial order over 2^S (or equivalently, $(2^S, \supseteq)$ is a partially ordered set). Similarly, $\subseteq$ on 2^S is also a partial order.

Definition 1.11. Let $(P, \preccurlyeq)$ be a partially ordered set. Two elements $x, y \in P$ are said to be ***comparable*** if either $x \preccurlyeq y$ or $y \preccurlyeq x$.

Definition 1.12. A partial order under which every pair of elements is comparable is called a ***total order*** or ***linear order***.

Definition 1.13. An ***ordered field*** is a totally ordered field $(\mathcal{F}, +, \cdot, \preccurlyeq)$ satisfying that

1. If $x \preccurlyeq y$, then $x + z \preccurlyeq y + z$ for all $z \in \mathcal{F}$ (compatibility of $\preccurlyeq$ and $+$).
2. If $0 \preccurlyeq x$ and $0 \preccurlyeq y$, then $0 \preccurlyeq x \cdot y$ (compatibility of $\preccurlyeq$ and $\cdot$).

Example 1.14. $(\mathbb{Q}, +, \cdot, \geqslant)$ is a totally ordered field, but is **not** an ordered field (since Property 2 in Definition 1.13 is violated). On the other hand, $(\mathbb{Q}, +, \cdot, \leqslant)$ is an ordered field.

From now on, the total order $\preccurlyeq$ of an ordered field will be denoted by $\leqslant$.

Definition 1.15. In an ordered field $(\mathcal{F}, +, \cdot, \leqslant)$, the binary relations $<$, $\geqslant$ and $>$ are defined by:

1. $x < y$ if $x \leqslant y$ and $x \neq y$.
2. $x \geqslant y$ if $y \leqslant x$.
3. $x > y$ if $y < x$.

Adopting the definition above, it is not immediately clear that $x \nleqslant y \Leftrightarrow x > y$. However, this is indeed the case, and to be more precise we have the following

Proposition 1.16. (Law of Trichotomy, 三一律) *If x and y are elements of an ordered field $(\mathcal{F}, +, \cdot, \leqslant)$, then exactly one of the relations $x < y$, $x = y$ or $y < x$ holds.*

Proof. Since $\mathcal{F}$ is a totally ordered field, x and y are comparable. Therefore, either $x \leqslant y$ or $y \leqslant x$. Assume that $x \leqslant y$.

1. If $x = y$, then $x \nless y$ and $x \ngtr y$.
2. If $x \neq y$, then $x < y$. If it also holds that $x > y$, then $x \geqslant y$; thus by the property of anit-symmetry of an order, we must have $x = y$, a contradiction. Therefore, it can only be that $x < y$.

The proof for the case $y \leqslant x$ is similar, and is left as an exercise. $\square$

Proposition 1.17. *Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field, and $a, b, x, y, z \in \mathcal{F}$.*

1. *If $a + x = a$, then $x = 0$.*
   If $a \cdot x = a$ and $a \neq 0$, then $x = 1$.

2. *If* $a + x = 0$, *then* $x = -a$.
   If $a \cdot x = 1$ *and* $a \neq 0$, *then* $x = a^{-1}$.

3. *If* $x \cdot y = 0$, *then* $x = 0$ *or* $y = 0$.

4. *If* $x \leqslant y < z$ *or* $x < y \leqslant z$, *then* $x < z$ (*the transitivity of* $<$).

5. *If* $a < b$, *then* $a + x < b + x$ (*the compatibility of* $<$ *and* $+$).
   If $0 < a$ *and* $0 < b$, *then* $0 < a \cdot b$ (*the compatibility of* $<$ *and* $\cdot$).

6. *If* $a + x = b + x$, *then* $a = b$.
   If $a + x \leqslant (<)\, b + x$, *then* $a \leqslant (<)\, b$.
   If $a \cdot x = b \cdot x$ *and* $x \neq 0$, *then* $a = b$.
   If $a \cdot x \leqslant (<)\, b \cdot x$ *and* $x > 0$, *then* $a \leqslant (<)\, b$.

7. $0 \cdot x = 0$.

8. $-(-x) = x$.

9. $-x = (-1) \cdot x$.

10. *If* $x \neq 0$, *then* $x^{-1} \neq 0$ *and* $(x^{-1})^{-1} = x$.

11. *If* $x \neq 0$ *and* $y \neq 0$, *then* $x \cdot y \neq 0$ *and* $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$.

12. *If* $x \leqslant (<)\, y$ *and* $0 \leqslant (<)\, z$, *then* $x \cdot z \leqslant (<)\, y \cdot z$.
    If $x \leqslant (<)\, y$ *and* $0 \geqslant (>)\, z$, *then* $x \cdot z \geqslant (>)\, y \cdot z$.

13. *If* $x \leqslant (<)\, 0$ *and* $y \leqslant (<)\, 0$, *then* $x \cdot y \geqslant (>)\, 0$.
    If $x \leqslant (<)\, 0$ *and* $y \geqslant (>)\, 0$, *then* $x \cdot y \leqslant (<)\, 0$.

14. $0 < 1$ *and* $-1 < 0$.

15. $x \cdot x \equiv x^2 \geqslant 0$.

16. *If* $x > 0$, *then* $x^{-1} > 0$. *If* $x < 0$, *then* $x^{-1} < 0$.

Proof. 1. $(-a) + a + x = (-a) + a = 0 \Rightarrow x = 0$.
$(a^{-1}) \cdot a \cdot x = (a^{-1}) \cdot a = 1 \Rightarrow x = 1$.

2. $(-a) + a + x = (-a) + 0 = -a \Rightarrow x = -a$.
   $(a^{-1}) \cdot a \cdot x = (a^{-1}) \cdot 1 = a^{-1} \Rightarrow x = a^{-1}$.

3. Assume that $x \neq 0$, then $x^{-1} \cdot x \cdot y = x^{-1} \cdot 0 = 0 \Rightarrow y = 0$.
   Assume that $y \neq 0$, then $x \cdot y \cdot y^{-1} = 0 \cdot y^{-1} = 0 \Rightarrow x = 0$.

4 and 5 are Left as an exercise.

6. $a + 0 = a + x + (-x) = b + x + (-x) = b + 0 \Rightarrow a = b$.
   $a + 0 = a + x + (-x) \leqslant b + x + (-x) = b + 0 \Rightarrow a \leqslant b$ (compatibility of $\leqslant$ and $+$).
   $a \cdot x \cdot x^{-1} = b \cdot x \cdot x^{-1} \Rightarrow a = b$.
   Suppose the contrary that $b < a$. Then $0 = b + (-b) \leqslant a + (-b)$. Since $x > 0$, $x \geqslant 0$; thus
   $$0 \leqslant \big(a + (-b)\big) \cdot x = a \cdot x + (-b) \cdot x\,.$$
   As a consequence, $b \cdot x = 0 + b \cdot x \leqslant a \cdot x + (-b) \cdot x + b \cdot x = a \cdot x$. By assumption, we must have $a \cdot x = b \cdot x$ or $(a - b) \cdot x = 0$. Using 3, $x = 0$ (since $a \neq b$), a contradiction.

7. See Remark 1.3.

8. $(-x) + (-(-x)) = 0 = (-x) + x \Rightarrow x = -(-x)$.

9. See Remark 1.4.

10. Assume $x^{-1} = 0$, $1 = x \cdot x^{-1} = x \cdot 0 = 0$, a contradiction. Therefore, $x^{-1} \neq 0$; thus $(x^{-1})^{-1} \cdot x^{-1} = 1 = x \cdot x^{-1} \Rightarrow (x^{-1})^{-1} = x$ (by 4).

11. That $x \cdot y = 0$ cannot be true since it is against Property 3, so $x \cdot y \neq 0$. Moreover,
    $$(x \cdot y)^{-1}(x \cdot y) = 1 = 1 \cdot 1 = (x \cdot x^{-1}) \cdot (y \cdot y^{-1}) = (x^{-1} \cdot y^{-1}) \cdot (x \cdot y)\,;$$
    thus $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$ (by 4).

12. If $x \leqslant (<)\, y$, then $0 = x + (-x) \leqslant (<)\, y + (-x)$. Since $0 \leqslant (<)\, z$, by the compatibility of $\leqslant (<)$ and $\cdot$ we must have $0 \leqslant (<)\, (y + (-x)) \cdot z = y \cdot z + (-x) \cdot z$. Therefore, by the compatibility of $\leqslant (<)$ and $+$, $x \cdot z = 0 + x \cdot z \leqslant (<)\, y \cdot z + (-x) \cdot z + x \cdot z = y \cdot z$. The second statement can be proved in a similar fashion.

13. Left as an exercise.

14. If $1 \leqslant 0$, then compatibility of $\leqslant$ and $+$ implies that $0 \leqslant -1$. By the compatibility of $\leqslant$ and $\cdot$, using 6 and 7 we find that $0 \leqslant (-1)\cdot(-1) = -(-1) = 1$; thus we conclude that $1 = 0$, a contradiction. As a consequence, $0 < 1$; thus the compatibility of $<$ and $+$ implies that $-1 < 0$.

15. Left as an exercise.

16. If $x > 0$ but $x^{-1} \leqslant 0$, then $1 = x \cdot x^{-1} \leqslant x \cdot 0 = 0$, a contradiction. ▫

Proposition 1.18. *Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field, and $x, y \in \mathcal{F}$.*

1. *If $0 \leqslant x < y$, then $x^2 < y^2$.*

2. *If $0 \leqslant x, y$ and $x^2 < y^2$, then $x < y$.*

Proof. 1. By definition of "$<$", $0 \leqslant x \leqslant y$ and $x \neq y$. Using 12 of Proposition 1.17,

$$x^2 \leqslant y \cdot x < y \cdot y = y^2 .$$

By the transitivity of $<$, we conclude that $x^2 < y^2$.

2. Note that $x \neq y$, for if not, then $x^2 - y^2 = 0$ which contradicts to the assumption $x^2 < y^2$. Assume that $y < x$, then 1 implies that $y^2 < x^2$, a contradiction. ▫

Remark 1.19. Proposition 1.18 can be summarized as follows: if $x, y \geqslant 0$, then

$$x < y \Leftrightarrow x^2 < y^2 .$$

Moreover, Example 1.8, Proposition 1.17 and Proposition 1.18 together suggest that if $x, y \geqslant 0$, then $x \leqslant y$ if and only if $x^2 \leqslant y^2$.

Definition 1.20. The ***magnitude*** or the ***absolute value*** of x, denoted $|x|$, is defined as

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 , \\ -x & \text{if } x < 0 . \end{cases}$$

Proposition 1.21. *Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field. Then*

1. *$|x| \geqslant 0$ for all $x \in \mathcal{F}$.*

2. *$|x| = 0$ if and only if $x = 0$.*

3. $-|x| \leqslant x \leqslant |x|$ *for all* $x \in \mathcal{F}$.

4. $|x \cdot y| = |x| \cdot |y|$ *for all* $x, y \in \mathcal{F}$.

5. $|x + y| \leqslant |x| + |y|$ *for all* $x, y \in \mathcal{F}$ (***triangle inequality***, 三角不等式).

6. $\big||x| - |y|\big| \leqslant |x - y|$ *for all* $x, y \in \mathcal{F}$.

Proof. Left as an exercise. □

Proposition 1.22. *Define* $d(x, y) = |x - y|$. *Then*

1. $d(x, y) \geqslant 0$ *for all* $x, y \in \mathcal{F}$.

2. $d(x, y) = 0$ *if and only if* $x = y$.

3. $d(x, y) = d(y, x)$ *for all* $x, y \in \mathcal{F}$.

4. $d(x, y) \leqslant d(x, z) + d(z, y)$ *for all* $x, y, z \in \mathcal{F}$ (***triangle inequality***, 三角不等式).

Proof. Left as an exercise. □

Remark 1.23. $d(x, y)$ is the "distance" of two elements $x, y \in \mathcal{F}$.

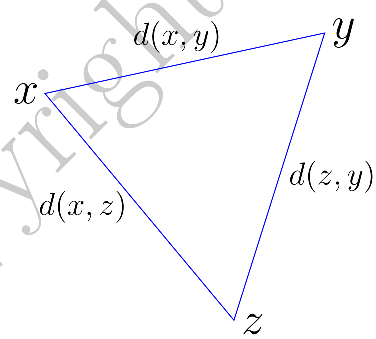


Figure 1.1: An illustration of why 4 of Proposition 1.22 is called the triangle inequality.

1.1.2 The natural numbers, the integers, and the rational numbers

Definition 1.24. Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field. The ***natural number system***, denoted by $\mathbb{N}$, is the collection of all the numbers $1, 1+1, 1+1+1, 1+1+\cdots+1$ and etc. in $\mathcal{F}$. We write $2 \equiv 1+1$, $3 \equiv 2+1$, and $n \equiv \underbrace{1 + 1 + \cdots + 1}_{\text{(n times)}}$. In other words, $\mathbb{N} = \{1, 2, 3, \cdots\}$.

The ***integer number system***, denoted by $\mathbb{Z}$, is the set $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$.

Principle of mathematical induction (Peano axiom, 皮亞諾公設):
If S is a subset of $\mathbb{N} \cup \{0\}$ (or $\mathbb{N}$) such that $0 \in S$ (or $1 \in S$) and $k+1 \in S$ if $k \in S$, then $S = \mathbb{N} \cup \{0\}$ (or $S = \mathbb{N}$).

Example 1.25. Prove $\sum\limits_{k=1}^{n} k = \dfrac{n(n+1)}{2}$. $(\star)$

Proof. Let $S = \left\{ n \in \mathbb{N} \,\middle|\, \sum\limits_{k=1}^{n} k = \dfrac{n(n+1)}{2} \right\}$（把所有滿足 $(\star)$ 的 n 收集起來）. Then

1. If $n = 1$, $\sum\limits_{k=1}^{1} k = \dfrac{1 \times 2}{2} = 1$.
2. Assume that $m \in S$, then
$$\sum_{k=1}^{m+1} k = \sum_{k=1}^{m} k + (m+1) = \frac{m(m+1)}{2} + (m+1) = \frac{(m+1)(m+2)}{2}$$
which implies that $m+1 \in S$.

By mathematical induction, we have $S = \mathbb{N}$. □

Example 1.26. Prove that $\dfrac{1}{2^n} < \dfrac{1}{n}$ for all $n \in \mathbb{N}$.

Proof. Let $S = \left\{ n \in \mathbb{N} \,\middle|\, \dfrac{1}{2^n} < \dfrac{1}{n} \right\}$. We show $S = \mathbb{N}$ by mathematical induction as follows:

(i) $1 \in S \Leftrightarrow \dfrac{1}{2} < \dfrac{1}{1}$.

(ii) If $n \in S$, then
$$\frac{1}{2^{n+1}} = \frac{1}{2^n} \cdot \frac{1}{2} < \frac{1}{n} \cdot \frac{1}{2} = \frac{1}{n+n} \leqslant \frac{1}{n+1}\,.$$
which implies that $n+1 \in S$.

By mathematical induction, we have $S = \mathbb{N}$. □

Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field. By the property of being a field, for any non-zero $n \in \mathbb{N}$, there exists a unique multiplicative inverse n^{-1}. This inverse is usually denoted by $\dfrac{1}{n}$. We also use $\dfrac{m}{n}$ to denote $m \cdot n^{-1}$. Giving this notation, we have the following

Definition 1.27. Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an order field. The ***rational number system***, denoted by $\mathbb{Q}$, is the collection of all numbers of the form $\dfrac{q}{p}$ with $p, q \in \mathbb{Z}$ and $p \neq 0$; that is,
$$\mathbb{Q} = \left\{ x \in \mathcal{F} \,\middle|\, x = \frac{q}{p}, p, q \in \mathbb{Z}, p \neq 0 \right\}.$$

Definition 1.28. An order field $(\mathcal{F}, +, \cdot, \leqslant)$ is said to have the ***Archimedean property*** if $\forall\, x \in \mathcal{F}$, $\exists\, n \in \mathbb{Z} \ni x < n$.

Theorem 1.29. *$\mathbb{Q}$ has the Archimedean property.*

Proof. If $x \leqslant 0$, we take $n = 1$. Otherwise if $0 < x = \dfrac{q}{p}$ with $p, q \in \mathbb{N}$, we take $n = q + 1$ and it is obvious that $\dfrac{q}{p} \leqslant q < q + 1 = n$. □

Definition 1.30. A ***well-ordered*** relation on a set S is a total order on S with the property that every non-empty subset of S has a least (smallest) element in this ordering.

Proposition 1.31 (Well-Ordered Property of $\mathbb{N}$)**.** *If $S \subseteq \mathbb{N}$ and $S \neq \varnothing$, then S has a smallest element; that is, $\exists\, s_0 \in S \ni \forall\, x \in S$, $s_0 \leqslant x$.*

Proof. Assume the contrary that there exists a non-empty set $S \subseteq \mathbb{N}$ such that S does not have the smallest element. Define $T = \mathbb{N} \backslash S$, and $T_0 = \big\{ n \in \mathbb{N} \,\big|\, \{1, 2, \cdots, n\} \subseteq T \big\}$. Then we have $T_0 \subseteq T$. Also note that $1 \notin S$ for otherwise 1 is the smallest element in S, so $1 \in T$ (thus $1 \in T_0$).

Assume $k \in T_0$. Since $\{1, 2, \cdots, k\} \subseteq T$, $1, 2, \cdots k \notin S$. If $k + 1 \in S$, then $k + 1$ is the smallest element in S. Since we assume that S does not have the smallest element, $k+1 \notin S$; thus $k + 1 \in T \Rightarrow k + 1 \in T_0$.

Therefore, by mathematical induction we conclude that $T_0 = \mathbb{N}$; thus $T = \mathbb{N}$ (since $T_0 \subseteq T$) which further implies that $S = \varnothing$ (since $T = \mathbb{N} \backslash S$). This contradicts to the assumption $S \neq \varnothing$. □

1.1.3 Countability

Definition 1.32. A set S is called ***denumerable*** or ***countably infinite***（無窮可數的）if S can be put into one-to-one correspondence with $\mathbb{N}$; that is, S is denumerable if and only if $\exists\, f : \mathbb{N} \to S$ which is one-to-one and onto. A set is called ***countable***（可數的）if S is either finite or denumerable.

Remark 1.33. If $f : \mathbb{N} \xrightarrow[\text{onto}]{1-1} S$, then $f^{-1} : S \xrightarrow[\text{onto}]{1-1} \mathbb{N}$. Therefore,

$$S \text{ is denumerable} \Leftrightarrow \exists\, f : \mathbb{N} \xrightarrow[\text{onto}]{1-1} S \Leftrightarrow \exists\, g = f^{-1} : S \xrightarrow[\text{onto}]{1-1} \mathbb{N}.$$

f can be thought as a rule of counting/labeling elements in S since $S = \big\{ f(1), f(2), \cdots \big\}$.

Example 1.34. $\mathbb{N}$ is countable since $f:\mathbb{N}\xrightarrow[onto]{1-1}\mathbb{N}$ with $f(x)=x,\ \forall\, n\in\mathbb{N}$.

Example 1.35. $\mathbb{Z}$ is countable. $f:\mathbb{Z}\to\mathbb{N}$ with $f(x)=\begin{cases}1 & \text{if } x=0\\ 2x & \text{if } x>0\\ -2x+1 & \text{if } x<0\end{cases}$.

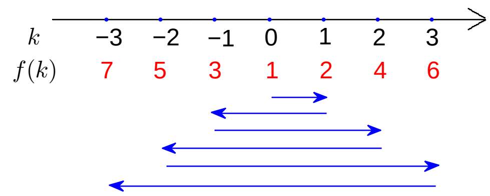


Figure 1.2: An illustration of how elements in $\mathbb{Z}$ are labeled

Example 1.36. The set $\mathbb{N}\times\mathbb{N}=\big\{(a,b)\,\big|\,a,b\in\mathbb{N}\big\}$ is countable. In fact, two ways of mapping are shown in the figures below.

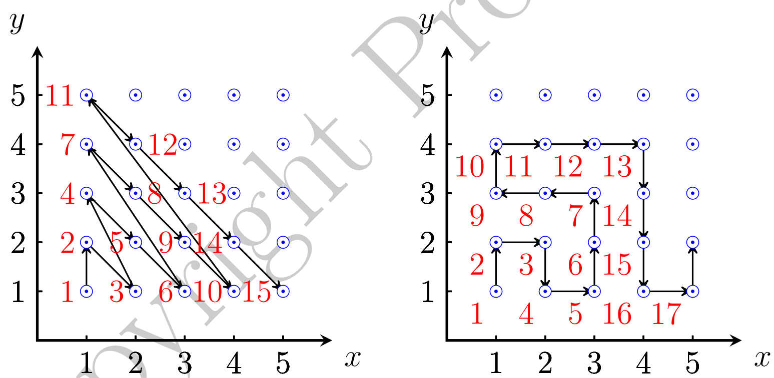


Figure 1.3: The illustration of two ways of labeling elements in $\mathbb{N}\times\mathbb{N}$

Proposition 1.37. *Let S be a non-empty set. The following three statements are equivalent:*

(a) *S is countable;*

(b) *there exists a surjection $f:\mathbb{N}\to S$;*

(c) *there exists an injection $f:S\to\mathbb{N}$.*

Proof. "(a) $\Rightarrow$ (b)" First suppose that $S=\{x_1,\cdots,x_n\}$ is finite. Define $f:\mathbb{N}\to S$ by

$$f(k)=\begin{cases}x_k & \text{if } k<n\,,\\ x_n & \text{if } k\geqslant n\,.\end{cases}$$

Then $f:\mathbb{N}\to S$ is a surjection. Now suppose that S is denumerable. Then by definition of countability, there exists $f:\mathbb{N}\xrightarrow[onto]{1-1}S$.

"(a) $\Leftarrow$ (b)" W.L.O.G. (without loss of generality, 不失一般性) we assume that S is an infinite set. Let $k_1=1$. Since $\#(S)=\infty$, $S_1\equiv S\backslash\{f(k_1)\}\neq\varnothing$; thus $N_1\equiv f^{-1}(S_1)$ is a non-empty subset of $\mathbb{N}$. By the well-ordered property of $\mathbb{N}$ (Proposition 1.31), N_1 has a smallest element denoted by k_2. Since $\#(S)=\infty$, $S_2=S\backslash\{f(k_1),f(k_2)\}\neq\varnothing$; thus $N_2\equiv f^{-1}(S_2)$ is a non-empty subset of $\mathbb{N}$ and possesses a smallest element denoted by k_3. We continue this process and obtain a set $\{k_1,k_2,\cdots\}\subseteq\mathbb{N}$, where $k_1<k_2<\cdots$, and k_j is the smallest element of $N_{j-1}\equiv f^{-1}(S\backslash\{f(k_1),f(k_2),\cdots,f(k_{j-1})\})$.

Claim: $f:\{k_1,k_2,\cdots\}\to S$ is one-to-one and onto.

Proof of claim: The injectivity of f is due to that $f(k_j)\notin\{f(k_1),f(k_2),\cdots,f(k_{j-1})\}$ for all $j\geqslant 2$. For surjectivity, assume that there is $s\in S$ such that $s\notin f(\{k_1,k_2,\cdots\})$. Since $f:\mathbb{N}\to\mathbb{S}$ is onto, $f^{-1}(\{s\})$ is a non-empty subset of $\mathbb{N}$; thus possesses a smallest element k. Since $s\notin f(\{k_1,k_2,\cdots\})$, there exists $\ell\in\mathbb{N}$ such that $k_\ell<k<k_{\ell+1}$. As a consequence, there exists $k\in N_\ell$ such that $k<k_{\ell+1}$ which contradicts to the fact that $k_{\ell+1}$ is the smallest element of N_ℓ.

Define $g:\mathbb{N}\to\{k_1,k_2,\cdots\}$ by $g(j)=k_j$. Then $g:\mathbb{N}\to\{k_1,k_2,\cdots\}$ is one-to-one and onto; thus $h=g\circ f:\mathbb{N}\xrightarrow[onto]{1-1}S$.

"(a) $\Rightarrow$ (c)" If $S=\{x_1,\cdots,x_n\}$ is finite, we simply let $f:S\to\mathbb{N}$ be $f(x_n)=n$. Then f is clearly an injection. If S is denumerable, by definition there exists $g:\mathbb{N}\xrightarrow[onto]{1-1}S$ which suggests that $f=g^{-1}:S\to\mathbb{N}$ is an injection.

"(a) $\Leftarrow$ (c)" Let $f:S\to\mathbb{N}$ be an injection. If f is also surjective, then $f:S\xrightarrow[onto]{1-1}\mathbb{N}$ which implies that S is denumerable. Now suppose that $f(S)\subsetneq\mathbb{N}$. Since S is non-empty, there exists $s\in S$. Let $g:\mathbb{N}\to S$ be defined by

$$g(n)=\begin{cases} f^{-1}(n) & \text{if } n\in f(S)\,,\\ s & \text{if } n\notin f(S)\,.\end{cases}$$

Then clearly $g:\mathbb{N}\to S$ is surjective; thus the equivalence between (a) and (b) implies that S is countable. □

Theorem 1.38. *Any non-empty subset of a countable set is countable.*

Proof. Let S be a countable set, and A be a non-empty subset of S. Since S is countable, by by Proposition 1.37 there exists a surjection $f : \mathbb{N} \to S$. On the other hand, since A is a non-empty subset of S, there exists $a \in A$. Define

$$g(x) = \begin{cases} x & \text{if } x \in A\,, \\ a & \text{if } x \notin A\,. \end{cases}$$

Then $h = g \circ f : \mathbb{N} \to A$ is a surjection, and Proposition 1.37 suggests that A is countable. □

Example 1.39. The set $\mathbb{N} \times \mathbb{N}$ is countable since the map $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f((m,n)) = 2^m 3^n$ is an injection.

Theorem 1.40. *The union of denumerable denumerable sets is denumerable* (無窮可數個無窮可數集的聯集是無窮可數的). *In other words, if $\mathscr{F}$ is a denumerable collection of denumerable sets, then* $\bigcup_{A \in \mathscr{F}} A$ *is denumerable.*

Proof. Let $\mathscr{F} = \{A_i \,|\, i \in \mathbb{N}, A_i \text{ is denumerable}\}$ be an indexed family of denumerable sets, and define $A = \bigcup_{i=1}^{\infty} A_i$. Since A_i is denumerable, $A_i = \{x_{i1}, x_{i2}, x_{i3}, \cdots\}$. Then $A = \{x_{ij} \,|\, i, j \in \mathbb{N}\}$. Let $f : \mathbb{N} \times \mathbb{N} \to A$ be defined by $f((i,j)) = x_{ij}$. Then $f : \mathbb{N} \times \mathbb{N} \to A$ is a surjection. Moreover, Example 1.39 implies that there exists a bijection $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$; thus $h = f \circ g : \mathbb{N} \to A$ is a surjection which, by Proposition 1.37, implies that A is countable. Since $A_1 \subseteq A$, A is infinite; thus A is denumerable. □

Corollary 1.41. *The union of countable countable sets is countable* (可數個可數集的聯集是可數的).

Proof. By adding empty sets into the family or adding $\mathbb{N}$ into a finite set if necessary, we find that the union of countable countable sets is a subset of the union of denumerable denumerable sets. By Theorem 1.38, we find that the union of countable countable sets is countable. □

Example 1.42. $\mathbb{Z} \times \mathbb{Z}$ is countable.

Proof. For $i \in \mathbb{Z}$, let $A_i = \{(i,j) \,|\, j \in \mathbb{Z}\}$. By Example 1.35, A_i is countable for all $i \in \mathbb{Z}$. Since $\mathbb{Z} \times \mathbb{Z} = \bigcup_{i \in \mathbb{Z}} A_i$ which is countable union of countable sets, Theorem 1.40 implies that $\mathbb{Z} \times \mathbb{Z}$ is countable. □

Theorem 1.43. $\mathbb{Q}$ *is countable.*

Proof. Define

$$f(x) = \begin{cases} (p,q), & \text{if } x>0, \quad x=\dfrac{q}{p}, \quad \gcd(p,q)=1,\ p>0. \\ (0,0), & \text{if } x=0. \\ (p,-q), & \text{if } x<0, \quad x=-\dfrac{q}{p}, \quad \gcd(p,q)=1,\ p>0. \end{cases}$$

Then $f : \mathbb{Q} \to \mathbb{Z}\times\mathbb{Z}$ is one-to-one; thus $f : \mathbb{Q} \xrightarrow[onto]{1-1} f(\mathbb{Q})$. Since $\mathbb{Z}\times\mathbb{Z}$ is countable, its non-empty subset $f(\mathbb{Q})$ is also countable. As a consequence, there exists $g : f(\mathbb{Q}) \xrightarrow[onto]{1-1} \mathbb{N}$; thus $h = g\circ f : \mathbb{Q} \xrightarrow[onto]{1-1} \mathbb{N}$. ▫

1.2 Completeness and the Real Number System

1.2.1 Sequences

Definition 1.44. A ***sequence*** in a set S is a function $f : \mathbb{N} \to S$ (not necessary one-to-one or onto). The values of f are called the ***terms*** of the sequence.

Remark 1.45. A sequence in S is a countable list of elements in S arranged in a particular order, and is usually denoted by $\big\{f(n)\big\}_{n=1}^{\infty}$ or $\{x_n\}_{n=1}^{\infty}$ with $x_n = f(n)$.

Definition 1.46. Let $\mathcal{F}$ be an ordered field. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ is said to be ***convergent*** if there exists $x \in \mathcal{F}$ such that for every $\varepsilon > 0$,

$$\#\big\{n\in\mathbb{N}\,\big|\,x_n \notin (x-\varepsilon, x+\varepsilon)\big\} < \infty\,.$$

Such an x is called a ***limit*** of the sequence. In notation,

$$\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{F} \text{ is convergent} \quad \Leftrightarrow \quad \exists\, x\in\mathcal{F} \ni \forall\,\varepsilon>0, \#\{n\in\mathbb{N}\,|\,x_n\notin(x-\varepsilon,x+\varepsilon)\}<\infty.$$

If x is a limit of $\{x_n\}_{n=1}^{\infty}$, we say $\{x_n\}_{n=1}^{\infty}$ converges to x and write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n\to\infty$. If no such x exists we say that $\{x_n\}_{n=1}^{\infty}$ ***diverges*** or $\lim\limits_{n\to\infty} x_n$ does not exist.

Remark 1.47. The number N may depend on ε, and smaller ε usually requires larger N.

In the definition above, it could happen that there are two different limits of a convergent sequence. In fact, this is never the case because of the following

Proposition 1.48. *If $\{x_n\}_{n=1}^{\infty}$ is a sequence in an ordered field $\mathcal{F}$, and $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then $x = y$. (The uniqueness of the limit).*

Proof. Assume the contrary that $x \neq y$. W.L.O.G. we may assume that $x < y$, and let $\varepsilon = \dfrac{y-x}{2} > 0$. Define

$$A_1 = \big\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\big\} \quad \text{and} \quad A_2 = \big\{n \in \mathbb{N} \,\big|\, x_n \notin (y-\varepsilon, y+\varepsilon)\big\}.$$

Then by the definition of the convergence of sequences, $\#A_1 < \infty$ and $\#A_2 < \infty$. Let $N_1 = \max A_1$, $N_2 = \max A_2$ and $N = \max\{N_1, N_2\}$. Since A_1, A_2 are finite, $N < \infty$. On the other hand, $N+1 \notin A_1 \cup A_2$ which implies that $x_{N+1} \in (x-\varepsilon, x+\varepsilon) \cap (y-\varepsilon, y+\varepsilon) = \varnothing$, a contradiction. □

Example 1.49. Let $x_n = \dfrac{(-1)^n}{n+1}$. We show that $\{x_n\}_{n=1}^{\infty}$ converges to 0. By definition, we need to show for every $\varepsilon > 0$ the set $A_\varepsilon = \big\{n \in \mathbb{N} \,\big|\, x_n \notin (-\varepsilon, \varepsilon)\big\}$ is finite. Note that $A_\varepsilon = \big\{n \in \mathbb{N} \,\big|\, |x_n| \geqslant \varepsilon\big\}$; thus

$$A_\varepsilon = \Big\{n \in \mathbb{N} \,\Big|\, \frac{1}{n+1} \geqslant \varepsilon\Big\} = \Big\{n \in \mathbb{N} \,\Big|\, n \leqslant \frac{1}{\varepsilon} - 1\Big\}.$$

Therefore, $\#A_\epsilon = \big[\frac{1}{\varepsilon}\big] - 1 < \infty$ which implies that $\{x_n\}_{n=1}^{\infty}$ converges to 0.

Example 1.50. The sequence $\{y_n\}_{n=1}^{\infty}$ given by $y_n = \dfrac{3+(-1)^n}{2}$ diverges. To see this, we have to show that any real number x cannot be the limit of $\{y_n\}_{n=1}^{\infty}$.

Let y be given and $\varepsilon = \dfrac{1}{2}$. Then $(y-\varepsilon, y+\varepsilon)$ at most contains one integer. Since y_n only takes value 1 or 2 and $\#\big\{n \in \mathbb{N} \,|\, y_n = 1\big\} = \#\big\{n \in \mathbb{N} \,|\, y_n = 2\big\} = \infty$, we find that

$$\#\big\{n \in \mathbb{N} \,\big|\, y_n \notin (y-\varepsilon, y+\varepsilon)\big\} = \infty$$

which implies $\{y_n\}_{n=1}^{\infty}$ cannot converges to y.

Example 1.51. A ***permutation*** of a non-empty set A is a one-to-one function from A onto A. Let $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation of $\mathbb{N}$, and $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence in an ordered field $\mathcal{F}$. Then $\big\{x_{\pi(n)}\big\}_{n=1}^{\infty}$ is also convergent since if x is the limit of $\{x_n\}_{n=1}^{\infty}$ and $\varepsilon > 0$,

$$\#\big\{n \in \mathbb{N} \,\big|\, x_{\pi(n)} \notin (x-\varepsilon, x+\varepsilon)\big\} = \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\big\} < \infty.$$

Proposition 1.52. *Let $\mathcal{F}$ be an ordered field, $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ be a sequence, and $x \in \mathcal{F}$. Then $\lim\limits_{n\to\infty} x_n = x$ if and only if for every $\varepsilon > 0$, there exists $N > 0$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$. In notation,*

$$\lim_{n\to\infty} x_n = x \quad \Leftrightarrow \quad \forall\, \varepsilon > 0, \exists\, N > 0 \ni n \geqslant N \Rightarrow |x_n - x| < \varepsilon\,.$$

Proof. "$\Rightarrow$" Let $\varepsilon > 0$ be given, and $A_\epsilon = \big\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\big\}$. Since $\{x_n\}_{n=1}^{\infty}$ converges to x, $k \equiv \# A_\epsilon < \infty$. Suppose that $n_1 < n_2 < \cdots < n_k$ belongs to A_ϵ. Let $N = n_k + 1$. Then if $n \geqslant N$, $n \notin A_\epsilon$ which implies that if $n \geqslant N$, $x_n \in (x-\varepsilon, x+\varepsilon)$ or equivalently,

$$|x_n - x| < \varepsilon \quad \text{whenever} \quad n \geqslant N\,.$$

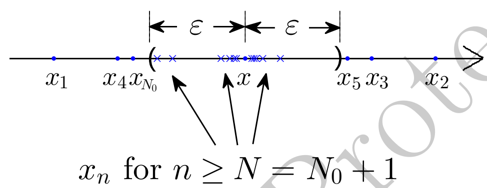


Figure 1.4: Let N_0 be the largest index of those x_n's outside $(x-\varepsilon, x+\varepsilon)$. Then $x_n \in (x-\varepsilon, x+\varepsilon)$ whenever $n \geqslant N = N_0 + 1$.

"$\Leftarrow$" Let $\varepsilon > 0$ be given. Then for some $N > 0$, if $n \geqslant N$, we have $|x_n - x| < \varepsilon$ or equivalently, if $n \geqslant N$, $x_n \in (x-\varepsilon, x+\varepsilon)$. This implies that

$$\#\big\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\big\} < N < \infty\,. \qquad \square$$

Remark 1.53. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ diverges if (and only if)

$$\forall\, x \in \mathcal{F}, \exists\, \varepsilon > 0 \ni \#\{n \in \mathbb{N} \,|\, x_n \notin (x-\varepsilon, x+\varepsilon)\} = \infty$$

which is equivalent to that

$$\forall\, x \in \mathcal{F}, \exists\, \varepsilon > 0 \ni \{n \in \mathbb{N} \,|\, x_n \notin (x-\varepsilon, x+\varepsilon)\} = \{n_1 < n_2 < \cdots < n_j < \cdots\}\,.$$

Therefore, $\{x_n\}_{n=1}^{\infty}$ diverges if (and only if)

$$\forall\, x \in \mathcal{F}, \exists\, \varepsilon > 0 \ni \forall\, N > 0, \exists\, n \geqslant N \text{ such that } |x_n - x| \geqslant \varepsilon\,.$$

Example 1.54. Now we use the ε-N argument as the definition of the convergence of sequences to re-establish the convergence of sequences in Example 1.49, 1.50 and 1.51.

Example 1.49 - revisit: Let $\varepsilon > 0$ be given, and $x_n = \dfrac{(-1)^n}{n+1}$. Let $N = \left[\frac{1}{\varepsilon}\right] + 1$. Since $\left[\frac{1}{\varepsilon}\right] > \dfrac{1}{\varepsilon} - 1$, if $n \geqslant N$ we must have $n > \dfrac{1}{\varepsilon} - 1$; thus if $n \geqslant N$, $\dfrac{1}{n+1} < \varepsilon$. Therefore,

$$|x_n - 0| < \varepsilon \quad \text{whenever} \quad n \geqslant N$$

which implies that $\{x_n\}_{n=1}^{\infty}$ converges to 0.

Example 1.50 - revisit: Let y be given, $\varepsilon = \dfrac{1}{2}$, and $N \in \mathbb{N}$. Define

$$n = \begin{cases} N+1 & \text{if } |y_N - y| < \varepsilon\,, \\ N+2 & \text{if } |y_N - y| \geqslant \varepsilon\,. \end{cases}$$

Then $n \geqslant N$. Moreover, if $|y_N - y| < \varepsilon$, then $|y_n - y| \geqslant |y_n - y_N| - |y_N - y| > 1 - \varepsilon = \varepsilon$, while if $|y_N - y| \geqslant \varepsilon$ then clearly $|y_n - y| \geqslant \varepsilon$. Therefore,

$$\forall\, y \in \mathcal{F}, \exists\, \varepsilon > 0 \ni \forall\, N > 0, \exists\, n > N \ni |y_n - y| \geqslant \varepsilon\,.$$

Example 1.51 - revisit: Now suppose that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence with limit x, and $\varepsilon > 0$ be given. Then there exists $N_1 > 0$ such that if $n \geqslant N_1$, we have $|x_n - x| < \varepsilon$. Let $N = \max\left\{\pi^{-1}(1), \pi^{-1}(2), \cdots, \pi^{-1}(N_1)\right\} + 1$. Then if $n \geqslant N$, $\pi(n) \geqslant N_1$ which implies that

$$\left|x_{\pi(n)} - x\right| < \varepsilon \quad \text{whenever} \quad n \geqslant N\,.$$

Therefore, $\lim\limits_{n\to\infty} x_{\pi(n)} = x$.

From the example above, we notice that proving the convergence using the ε-N argument seems more complicated; however, it is a necessary evil so we encourage the readers to major it.

Lemma 1.55 (Sandwich)**.** *If* $\lim\limits_{n\to\infty} x_n = L$, $\lim\limits_{n\to\infty} y_n = L$, $\{z_n\}_{n=1}^{\infty}$ *is a sequence such that* $x_n \leqslant z_n \leqslant y_n$, *then* $\lim\limits_{n\to\infty} z_n = L$.

Proof. Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} x_n = L$ and $\lim\limits_{n\to\infty} y_n = L$, by definition

$$\exists\, N_1 > 0 \ni L - \varepsilon < x_n < L + \varepsilon \ \text{ whenever } \ n \geqslant N_1$$

and

$$\exists\, N_2 > 0 \ni L - \varepsilon < y_n < L + \varepsilon \ \text{ whenever } \ n \geqslant N_2\,.$$

Let $N = \max\{N_1, N_2\}$. Then for $n \geqslant N$, $L - \varepsilon < x_n \leqslant z_n \leqslant y_n < L + \varepsilon$; thus $\lim\limits_{n\to\infty} z_n = L$. ▫

Proposition 1.56. *If* $a \leqslant x_n \leqslant b$ *and* $\lim\limits_{n\to\infty} x_n = x$, *then* $a \leqslant x \leqslant b$.

Proof. Assume the contrary that $x \notin [a,b]$. If $x < a$, let $\varepsilon = a - x > 0$. Since $\lim\limits_{n\to\infty} x_n = x$, $\exists\, N > 0 \ni x_n \in (x-\varepsilon, x+\varepsilon)$ whenever $n \geqslant N$. Therefore, $x_n < a$ for all $n \geqslant N$, a contradiction. So $a \leqslant x$.

We can prove $x \leqslant b$ in a similar way, and the proof is left as an exercise. □

Corollary 1.57. *If* $a < x_n < b$ *and* $\lim\limits_{n\to\infty} x_n = x$, *then* $a \leqslant x \leqslant b$.

Definition 1.58. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in an order field $\mathcal{F}$.

1. $\{x_n\}_{n=1}^{\infty}$ is said to be ***bounded***（有界的）if there exists $M > 0$ such that $|x_n| \leqslant M$ for all $n \in \mathbb{N}$.

2. $\{x_n\}_{n=1}^{\infty}$ is said to be ***bounded from above***（有上界）if there exists $B \in \mathcal{F}$, called an ***upper bound*** of the sequence, such that $x_n \leqslant B$ for all $n \in \mathbb{N}$.

3. $\{x_n\}_{n=1}^{\infty}$ is said to be ***bounded from below***（有下界）if there exists $A \in \mathcal{F}$, called a ***lower bound*** of the sequence, such that $A \leqslant x_n$ for all $n \in \mathbb{N}$.

Proposition 1.59. *A convergent sequence is bounded*（數列收斂必有界）.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence with limit x. Then there exists $N > 0$ such that

$$x_n \in (x-1, x+1) \qquad \forall\, n \geqslant N.$$

Let $M = \max\big\{|x_1|, |x_2|, \cdots, |x_{N-1}|, |x|+1\big\}$. Then $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. □

Theorem 1.60. *Suppose that* $x_n \to x$ *and* $y_n \to y$ *as* $n \to \infty$, λ *is a constant. Then*

1. $x_n \pm y_n \to x \pm y$ *as* $n \to \infty$.

2. $\lambda \cdot x_n \to \lambda \cdot x$ *as* $n \to \infty$.

3. $x_n \cdot y_n \to x \cdot y$ *as* $n \to \infty$.

4. *If* $y_n, y \neq 0$, *then* $\dfrac{x_n}{y_n} \to \dfrac{x}{y}$ *as* $n \to \infty$.

Proof. The proof of 1 and 2 are left as an exercise.

3. Since $x_n \to x$ and $y_n \to y$ as $n \to \infty$, by Proposition 1.59 $\exists\, M > 0 \ni |x_n| \leqslant M$ and $|y_n| \leqslant M$. Let $\varepsilon > 0$ be given. Moreover,

$$\exists\, N_1 > 0 \ni |x_n - x| < \frac{\varepsilon}{2M} \;\forall\, n \geqslant N_1$$

and

$$\exists\, N_2 > 0 \ni |y_n - y| < \frac{\varepsilon}{2M} \;\forall\, n \geqslant N_2\,.$$

Define $N = \max\{N_1, N_2\}$. Then for all $n \geqslant N$,

$$\begin{aligned}|x_n \cdot y_n - x \cdot y| &= |x_n \cdot y_n - x_n \cdot y + x_n \cdot y - x \cdot y| \leqslant |x_n \cdot (y_n - y)| + |y \cdot (x_n - x)| \\ &\leqslant M \cdot |y_n - y| + M \cdot |x_n - x| < M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon.\end{aligned}$$

4. It suffices to show that $\lim\limits_{n\to\infty} \dfrac{1}{y_n} = \dfrac{1}{y}$ if $y_n, y \neq 0$ (because of 3). Since $\lim\limits_{n\to\infty} y_n = y$, $\exists\, N_1 > 0 \ni |y_n - y| < \dfrac{|y|}{2}$ for all $n \geqslant N_1$. Therefore, $|y| - |y_n| < \dfrac{|y|}{2}$ for all $n \geqslant N_1$ which further implies that $|y_n| > \dfrac{|y|}{2}$ for all $n \geqslant N_1$.

   Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} y_n = y$, $\exists\, N_2 > 0 \ni |y_n - y| < \dfrac{|y|^2}{2}\varepsilon$ for all $n \geqslant N_2$. Define $N = \max\{N_1, N_2\}$. Then for all $n \geqslant N$,

$$\Big|\frac{1}{y_n} - \frac{1}{y}\Big| = \frac{|y_n - y|}{|y_n||y|} < \frac{|y|^2}{2}\varepsilon \cdot \frac{1}{|y|}\frac{2}{|y|} = \varepsilon\,. \qquad \square$$

1.2.2 Monotone sequence property and completeness

Definition 1.61. A sequence $\{x_n\}_{n=1}^{\infty}$ is said to be ***increasing/non-decreasing***, ***decreasing/non-increasing***, ***strictly increasing*** and ***strictly decreasing*** if $x_n \leqslant x_{n+1}$, $x_n \geqslant x_{n+1}$, $x_n < x_{n+1}$ and $x_n > x_{n+1}$ $\forall\, n \in \mathbb{N}$, respectively. A sequence is called (strictly) ***monotone*** if it is either (strictly) increasing or (strictly) decreasing.

Definition 1.62. An ordered field $\mathcal{F}$ is said to satisfy the ***(strictly) monotone sequence property*** if every bounded (strictly) monotone sequence converges to a limit in $\mathcal{F}$.

Remark 1.63. An equivalent definition of the monotone sequence property is that every monotone *increasing* sequence *bounded above* converges; that is, if each sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ satisfying

(i) $x_n \leqslant x_{n+1}$ for all $n \in \mathbb{N}$,

(ii) $\exists\, M \in \mathcal{F} \ni \forall\, n \in \mathbb{N} : x_n \leqslant M$,

is convergent, then we say $\mathcal{F}$ satisfies the monotone sequence property.

Example 1.64. $(\mathbb{Q}, +, \cdot, \leqslant)$ is an ordered field.
Question: Is there any bounded monotone sequence in $\mathbb{Q}$ that does not converge to a limit in $\mathbb{Q}$?
Answer: Yes! Consider the sequence

$$x_1 = \frac{1}{2}, \quad x_2 = \frac{1}{2 + \frac{1}{2}}, \quad x_3 = \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, \quad \cdots, \quad x_{n+1} = \frac{1}{2 + x_n}.$$

Then $\{x_n\}_{n=1}^\infty$ is a monotone decreasing sequence in $\mathbb{Q}$. If $\lim\limits_{n\to\infty} x_n = x$, then Theorem 1.60 implies that $x = \dfrac{1}{2+x}$ from which we conclude that $x = -1 + \sqrt{2}$. Since $x \notin \mathbb{Q}$, $\{x_n\}_{n=1}^\infty$ does not converge (to a limit) in $\mathbb{Q}$. In other words, $\mathbb{Q}$ does not have the monotone sequence property.

Proposition 1.65. *An ordered field satisfying the monotone sequence property has the Archimedean property; that is, if $\mathcal{F}$ is an ordered field satisfying the monotone sequence property, then $\forall\, x \in \mathcal{F}$, $\exists\, n \in \mathbb{N} \ni x < n$.*

Proof. Assume the contrary that there exists $x \in \mathcal{F}$ such that $n \leqslant x$ for all $n \in \mathbb{N}$. Let $x_n = n$. Then $\{x_n\}_{n=1}^\infty$ is increasing and bounded above. By the monotone sequence property of $\mathcal{F}$, there exists $\widehat{x} \in \mathcal{F}$ such that $x_n \to \widehat{x}$ as $n \to \infty$; thus $\exists\, N > 0$ such that

$$|x_n - \widehat{x}| < \frac{1}{4} \qquad \forall\, n \geqslant N.$$

In particular, $|N - \widehat{x}| < \frac{1}{4}$, $|N + 1 - \widehat{x}| < \frac{1}{4}$; thus

$$1 = |N + 1 - N| \leqslant |N + 1 - \widehat{x}| + |\widehat{x} - N| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

a contradiction. □

Example 1.66. Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the monotone sequence property, and $y \in \mathcal{F}$ be a given positive number (that is, $y > 0$). Define $x_n = \dfrac{N_n}{2^n}$, where N_n is the largest integer such that $x_n^2 \leqslant y$; that is, $\big(\frac{N_n}{2^n}\big)^2 \leqslant y$ but $\big(\frac{N_n + 1}{2^n}\big)^2 > y$ (for example, if $y = 2$, then $x_1 = \frac{2}{2^1}$, $x_2 = \frac{5}{2^2}$, $x_3 = \frac{11}{2^3}$, $\cdots$). Then

1. x_n is bounded above: since $x_n^2 \leqslant y \leqslant 2y + y^2 + 1 = (y+1)^2$, by the non-negativity of x_n and y and Remark 1.19 we must have $0 \leqslant x_n \leqslant y + 1$.

2. x_n is increasing: by the definition of N_n,

$$N_n^2 \leqslant 2^{2n} \cdot y \Rightarrow 4 \cdot N_n^2 \leqslant 2^{2n+2} \cdot y = 2^{2(n+1)} \cdot y \Rightarrow \left(\frac{2N_n}{2^{n+1}}\right)^2 \leqslant y \Rightarrow 2N_n \leqslant N_{n+1}\,.$$

Therefore, $x_n = \dfrac{N_n}{2^n} = \dfrac{2N_n}{2^{n+1}} \leqslant \dfrac{N_{n+1}}{2^{n+1}} = x_{n+1}$. Since $\mathcal{F}$ satisfies the monotone sequence property, $\exists\, x \in \mathcal{F} \ni x_n \to x$ as $n \to \infty$. By Theorem 1.60, $x_n^2 \to x^2$, and by Proposition 1.56, $x^2 \leqslant y$.

Now we show $x^2 = y$. To this end observe that

$$\left(x_n + \frac{1}{2^n}\right)^2 = \left(\frac{N_n}{2^n} + \frac{1}{2^n}\right) = \left(\frac{N_n + 1}{2^n}\right)^2 > y;$$

thus $x_n^2 \leqslant y \leqslant \left(x_n + \frac{1}{2^n}\right)^2$. By the Archimedean propery of $\mathcal{F}$ (Proposition 1.65), $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$; thus Theorem 1.60 implies that $x^2 = \lim\limits_{n\to\infty} x_n^2 = \lim\limits_{n\to\infty} \left(x_n + \dfrac{1}{2^n}\right)^2 = y$. Note that Proposition 1.18 implies that such an x is unique if $x > 0$.

In general, one can define the n-th root of non-negative number y in an ordered field satisfying the monotone sequence property. The construction of the n-th root of $y \in \mathcal{F}$ is left as an exercise.

Definition 1.67. For $n \in \mathbb{N}$, the ***n-th root*** of a non-negative number y in an ordered field satisfying the monotone sequence property is the unique non-negative number x satisfying $x^n = y$. One writes $y^{1/n}$ or $\sqrt[n]{y}$ to denote n-th root of y.

Definition 1.68. An ordered field $\mathcal{F}$ is said to be ***complete*** (完備) (have the completeness property, 具備完備性) if it satisfies the monotone sequence property.

Remark 1.69. In an ordered field, completeness $\Leftrightarrow$ monotone sequence property (在 ordered field 裡，完備性 = 數列單調有界必收斂 = 數列遞增有上界必收斂). Moreover,

1. A complete ordered field is “Archimedean” (Proposition 1.65).

2. For $n \in \mathbb{N}$, the n-th root of a non-negative number in a complete ordered field is well-defined (Definition 1.67).

Proposition 1.70. *Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field. Then $\mathcal{F}$ satisfies the monotone sequence property if and only if $\mathcal{F}$ satisfies the strictly monotone sequence property.*

Proof. The "only if" part is trivial, so we only prove the "if" part. Let $\{x_n\}_{n=1}^\infty$ be a bounded increasing sequence in $\mathcal{F}$. If $\{x_n\}_{n=1}^\infty$ has finite number of values; that is,

$$\#\big\{n \in \mathbb{N} \,\big|\, x_n < x_{n+1}\big\} < \infty\,,$$

then there exists $N \in \mathbb{N}$ such that $x_n = x_N$ for all $n \geqslant N$ which implies that $\{x_n\}_{n=1}^\infty$ converges to x_N. Now suppose that

$$\#\big\{n \in \mathbb{N} \,\big|\, x_n < x_{n+1}\big\} = \infty\,.$$

Then there exists an infinite set $\{n_1, n_2, \cdots\} \subseteq \mathbb{N}$ such that $x_{n_k} \neq x_{n_{k+1}}$ for all $k \in \mathbb{N}$. Let $y_k = x_{n_k}$. Since $\mathcal{F}$ satisfies the strictly monotone sequence property, $y_k \to y$ as $k \to \infty$ for some $x \in \mathcal{F}$. However, it is easy to see that the sequence $\{x_n\}_{n=1}^\infty$ also converges to y since $\{x_n\}_{n=1}^\infty$ is monotone increasing. □

Theorem 1.71. *There is a "unique" complete ordered field, called the real number system* $\mathbb{R}$.

Remark 1.72. Uniqueness means if $\mathcal{F}$ is any other complete ordered field $(\mathcal{F}, \oplus, \odot, \preccurlyeq)$, then there exists an field isomorphism $\phi : \mathbb{R} \to \mathcal{F}$; that is, $\phi : \mathbb{R} \to \mathcal{F}$ is one-to-one and onto, and satisfies that

1. $\phi(x + y) = \phi(x) \oplus \phi(y)$ for all $x, y \in \mathbb{R}$.
2. $\phi(x \cdot y) = \phi(x) \odot \phi(y)$ for all $x, y \in \mathbb{R}$.
3. $x \leqslant y \Rightarrow \phi(x) \preccurlyeq \phi(y)$ for all $x, y \in \mathbb{R}$.

Sketch of proof of Theorem 1.71. Let S be the collection of all bounded increasing sequences in $\mathbb{Q}$ in which all terms in every sequence have the same sign; that is,

$$S = \Big\{\{x_n\}_{n=1}^\infty \,\Big|\, x_n \in \mathbb{Q} \text{ for all } n \in \mathbb{N},\ x_j \cdot x_k \geqslant 0 \text{ for all } k, j \in \mathbb{N}, \text{ and } \{x_n\}_{n=1}^\infty \text{ is increasing and bounded above}\Big\}.$$

Define on S an equivalence relation $\sim$: $\{x_n\}_{n=1}^\infty \sim \{y_n\}_{n=1}^\infty$ if every upper bound of $\{x_n\}_{n=1}^\infty$ is also an upper bound of $\{y_n\}_{n=1}^\infty$, and vice versa. Let $\mathbb{R} = \big\{\big[\{x_n\}_{n=1}^\infty\big] \,\big|\, \{x_n\}_{n=1}^\infty \in S\big\}$ be the set of equivalence class of S (the existence of such a set relies on the *axiom of choice*). We define on $\mathbb{R}$, $+$, $\cdot$, $\leqslant$ as follows: if $r = \big[\{x_n\}_{n=1}^\infty\big]$ and $s = \big[\{y_n\}_{n=1}^\infty\big]$ (where $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \in S$), then

$$1.\ r+s=\big[\{x_n+y_n\}_{n=1}^\infty\big]; \qquad 2.\ r\cdot s=\begin{cases}\big[\{x_n\cdot y_n\}_{n=1}^\infty\big] & \text{if } r,s\geqslant 0\,,\\ -\big((-r)\cdot s\big) & \text{if } r<0 \text{ and } s>0\,,\\ -\big(r\cdot(-s)\big) & \text{if } r>0 \text{ and } s<0\,,\\ (-r)\cdot(-s) & \text{if } r,s<0\,;\end{cases}$$

3. $r \leqslant s$ if every upper bound of $\{y_n\}_{n=1}^\infty$ is also an upper bound for $\{x_n\}_{n=1}^\infty$.

One needs to verify that $\mathbb{R}$ is an ordered field, and this part is left as an exercise (or see Remark 1.73 for some part of the verification).

Claim 1: If $\{x_{n_k}\}_{k=1}^\infty$ is a subsequence of $\{x_n\}_{n=1}^\infty$, then $\big[\{x_{n_k}\}_{k=1}^\infty\big]=\big[\{x_n\}_{n=1}^\infty\big]$.

Claim 2: If $\big[\{x_n\}_{n=1}^\infty\big]<\big[\{y_n\}_{n=1}^\infty\big]$, then for some $N\in\mathbb{N}$, $x_n<y_N$ for all $n\geqslant N$.

The proofs of the claims above are not difficult and are left as an exercise.

Now we show the completeness of $\mathbb{R}$ by showing that $\mathbb{R}$ satisfies the strictly monotone sequence property (Proposition 1.70). Let $\{r_k\}_{k=1}^\infty$ be a bounded, strictly increasing sequence in $\mathbb{R}^+$. Write $r_k=\big[\{x_{k,n}\}_{n=1}^\infty\big]$, where $x_{k,n}\leqslant x_{k,n+1}$ for all $k,n\in\mathbb{N}$. Since $\{r_k\}_{k=1}^\infty$ is bounded in $\mathbb{R}$, there is $M\in\mathbb{Q}$ such that $x_{k,n}\leqslant M$ for all $k,n\in\mathbb{N}$. Moreover, since $r_k<r_{k+1}$ for all $k\in\mathbb{N}$, by claims above we can assume that $x_{k,n}<x_{k+1,1}$ for all $k,n\in\mathbb{N}$; thus

$$x_{k,n}<x_{\ell,m} \qquad \forall\, \ell>k \text{ and } n,m\in\mathbb{N}. \tag{$\star$}$$

Therefore, $\{x_{n,n}\}_{n=1}^\infty$ is bounded and monotone increasing, so $\{x_{n,n}\}_{n=1}^\infty\in S$. Define $r=\big[\{x_{n,n}\}_{n=1}^\infty\big]$. Then $r\in\mathbb{R}$, and

(i) **r is an upper bound of $\{r_k\}_{k=1}^\infty$**: Suppose the contrary that there exists $M\in\mathbb{Q}$ such that $x_{n,n}\leqslant M$ for all $n\in\mathbb{N}$ but $x_{k,\ell}>M$ for some $k,\ell\in\mathbb{N}$.

  (a) If $k\geqslant\ell$, then $x_{k,k}\geqslant x_{k,\ell}>M$ since $\{x_{k,\ell}\}_{\ell=1}^\infty$ is increasing.

  (b) If $k<\ell$, then $x_{\ell,\ell}>x_{k,\ell}>M$ because of ($\star$).

  In either case we conclude that M cannot be an upper bound of r, a contradiction.

(ii) **$r-\varepsilon$ is not an upper bound of $\{r_k\}_{k=1}^\infty$ for all $\varepsilon>0$**: Suppose the contrary that $r-\varepsilon$ is an upper bound of $\{r_k\}_{k=1}^\infty$. Write $\varepsilon=\{\varepsilon_k\}_{k=1}^\infty$, and W.L.O.G. we can assume that there exists $\delta\in\mathbb{Q}$ such that $\varepsilon_k\geqslant 2\delta>0$ for all $k\in\mathbb{N}$. Then for all (fixed) $k\in\mathbb{N}$,

$$\big[\{x_{k,\ell}+\delta\}_{\ell=1}^\infty\big]<\big[\{x_{k,\ell}+2\delta\}_{\ell=1}^\infty\big]\leqslant\big[\{x_{k,\ell}+\varepsilon_k\}_{\ell=1}^\infty\big]\leqslant\big[\{x_{\ell,\ell}\}_{\ell=1}^\infty\big]\,.$$

Let $N_1 = 1$. By claim 2, for each $k \in \mathbb{N}$ there exists $N_{k+1} \in \mathbb{N}$ such that $N_{k+1} \geqslant N_k$ and $x_{N_k,\ell} + \delta < x_{N_{k+1},N_{k+1}}$ for all $\ell \geqslant N_{k+1}$. On the other hand,

$$x_{N_{k+1},N_{k+1}} \geqslant x_{N_k,N_{k+1}} + \delta \geqslant x_{N_k,N_k} + \delta \geqslant \cdots \geqslant x_{1,1} + k\delta$$

which implies that $\{x_{\ell,\ell}\}_{\ell=1}^\infty$ is not bounded, a contradiction.

As a consequence, r is the least upper bound of $\{r_k\}_{k=1}^\infty$. □

From now on $\mathbb{R}$ is the complete ordered field containing $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$.

Remark 1.73 (The existence of additive inverse of real numbers). Suppose that a bounded increasing sequence $\{x_n\}_{n=1}^\infty$ is not equivalent to any rational "number" $\{q\}_{n=1}^\infty$ for any $q \in \mathbb{Q}$, then there exists a decreasing sequence $\{y_n\}_{n=1}^\infty$ such that $x_n - y_n \to 0$ as $n \to \infty$. Such $\{y_n\}_{n=1}^\infty$ can be obtained by choosing y_n to be the smallest upper bound of the form $\dfrac{k}{2^n}$, where $k \in \mathbb{Z}$. By deleting terms if necessary, we can assume that all $y_n's$ have the same sign. Then $\{-y_n\}_{n=1}^\infty$ is a bounded increasing sequence, and $\big[\{-y_n\}_{n=1}^\infty\big]$ is the additive inverse of $\big[\{x_n\}_{n=1}^\infty\big]$.

Example 1.74. In $\mathbb{R}$, define x_n inductively by $x_1 = 0$, $x_2 = \sqrt{2}$, $x_3 = \sqrt{2+\sqrt{2}}$, $\cdots$, $x_{n+1} = \sqrt{2+x_n}$. It is easy to see that $\{x_n\}_{n=1}^\infty$ satisfies $x_n \geqslant 0$ for all $n \in \mathbb{N}$.

1. $x_n \leqslant 2$ for all $n \in \mathbb{N}$ (boundedness): First of all, $x_1 \leqslant 2$. Assume that $x_n \leqslant 2$. Then $x_{n+1} = \sqrt{2+x_n} \leqslant \sqrt{2+2} = 2$. By mathematical induction, $x_n \leqslant 2$ for all $n \in \mathbb{N}$.

2. $x_n \leqslant x_{n+1}$ (monotonicity): Since $x_n - 2 \leqslant 0$ and $x_n + 1 \geqslant 0$, $(x_n - 2)\cdot(x_n + 1) \leqslant 0$. Expanding the product, we obtain that $x_n^2 \leqslant x_n + 2 = x_{n+1}^2$ which implies that $x_n \leqslant x_{n+1}$.

3. $x_n \to 2$ as $n \to \infty$ (convergence): Since $\{x_n\}_{n=1}^\infty$ is a bounded monotone sequence in $\mathbb{R}$, $\lim\limits_{n\to\infty} x_n = x$ for some $x \in \mathbb{R}$. Note that then $x_{n+1} \to x$ as $n \to \infty$. Since $x_{n+1}^2 = x_n + 2$, by Theorem 1.60 we must have $x^2 = x + 2$. Then $(x-2)(x+1) = 0$ which implies $x = 2$ or $x = -1$ (failed). Therefore, $\{x_n\}_{n=1}^\infty$ converges to 2.

Theorem 1.75. *The interval* $(0,1)$ *in* $\mathbb{R}$ *is uncountable* (不可數).

Proof. Assume the contrary that there exists $f : \mathbb{N} \to (0,1)$ which is one-to-one and onto. Write $f(k)$ in decimal expansion (十進位展開); that is,

$$\begin{aligned} f(1) &= 0.d_{11}d_{21}d_{31}\cdots \\ f(2) &= 0.d_{12}d_{22}d_{32}\cdots \\ &\vdots \\ f(k) &= 0.d_{1k}d_{2k}d_{3k}\cdots \\ &\vdots \end{aligned}$$

Here we note that repeated 9's are chosen by preference over terminating decimals; that is, for example, we write $\frac{1}{4} = 0.249999\cdots$ instead of $\frac{1}{4} = 0.250000\cdots$.

Let $x \in (0,1)$ be such that $x = 0.d_1d_2\cdots$, where

$$d_k = \begin{cases} 5 & \text{if } \; d_{kk} \neq 5, \\ 7 & \text{if } \; d_{kk} = 5. \end{cases}$$

(建構一個 x 使其小數點下第 k 位數與 $f(k)$ 的小數點下第 k 位數不相等). Then $x \neq f(k)$ for all $k \in \mathbb{N}$, a contradiction; thus $(0,1)$ is uncountable. □

Corollary 1.76. $\mathbb{R}$ *is uncountable.*

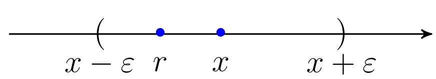
Proposition 1.77. $\mathbb{Q}$ *is dense* (稠密) *in* $\mathbb{R}$*; that is, if* $x, y \in \mathbb{R}$ *and* $x < y$*, then* $\exists\, r \in \mathbb{Q} \ni x < r < y$.

Proof. Since $\frac{1}{n} \to 0$ as $n \to \infty$ (by the Archimedean property of $\mathbb{R}$, Proposition 1.65), there exists $N > 0$ such that $\left|\frac{1}{n} - 0\right| < y - x$ for all $n \geqslant N$.

Claim: $\left\{ \frac{k}{N} \,\middle|\, k \in \mathbb{Z} \right\} \cap (x,y) \neq \varnothing$.

Proof of claim: Suppose the contrary that $\left\{ \frac{k}{N} \,\middle|\, k \in \mathbb{Z} \right\} \cap (x,y) = \varnothing$. Then $\frac{\ell}{N} \leqslant x$ and $\frac{\ell+1}{N} \geqslant y$ for some $\ell \in \mathbb{Z}$, while this fact will imply that $y - x \leqslant \frac{1}{N}$, a contradiction. □

Remark 1.78. The denseness of $\mathbb{Q}$ in $\mathbb{R}$ can be rephrased as follows: if $x \in \mathbb{R}$ and $\varepsilon > 0$, then $\exists\, r \in \mathbb{Q} \ni |x - r| < \varepsilon$.



Corollary 1.79. *The collection of irrational numbers* $\mathbb{Q}^\complement \equiv \mathbb{R}\backslash\mathbb{Q}$ *is dense in* $\mathbb{R}$*; that is, if* $x, y \in \mathbb{R}$ *and* $x < y$*,* $\exists\, c \in \mathbb{Q}^\complement \ni x < c < y$*.*

Proof. Let $x, y \in \mathbb{R}$ with $x < y$. By Proposition 1.77 there exists $r \in \mathbb{Q}$, $r \neq 0$ such that $\dfrac{x}{\sqrt{2}} < r < \dfrac{y}{\sqrt{2}}$. Let $c = \sqrt{2}r$. Then $c \in \mathbb{Q}^\complement$ and $x < c < y$. ▫

Example 1.80. The harmonic sequence

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 + \frac{1}{2} \\ \vdots \quad & \quad \vdots \\ x_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k} \\ \vdots \quad & \qquad\quad \vdots \end{aligned}$$

is (monotone) increasing but not bounded above.

Proof. That the sequence is increasing is trivial. For the unboundedness, we observe that

$$\begin{aligned} x_{2^n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{2^n} \\ &\geqslant 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots + \frac{2^{n-1}}{2^n} \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = 1 + \frac{n}{2} \end{aligned}$$

which is not bounded above (沒有上界). ▫

1.3 Least Upper Bounds and Greatest Lower Bounds

Definition 1.81. Let $\varnothing \neq S \subseteq \mathbb{R}$. A number $M \in \mathbb{R}$ is called an ***upper bound*** (上界) for S if $x \leqslant M$ for all $x \in S$, and a number $m \in \mathbb{R}$ is called a ***lower bound*** (下界) for S if $x \geqslant m$ for all $x \in S$. If there is an upper bound for S, then S is said to be ***bounded from above***, while if there is a lower bound for S, then S is said to be ***bounded from below***. A number $b \in \mathbb{R}$ is called a ***least upper bound*** (最小上界) of S if

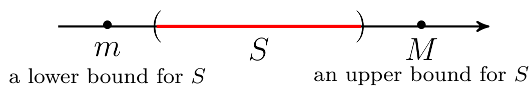
1. b is an upper bound for S, and

2. if M is an upper bound for S, then $M \geqslant b$.

A number a is called a ***greatest lower bound*** (最大下界) of S if

1. a is a lower bound for S, and

2. if m is a lower bound for S, then $m \leqslant a$.



If S is not bounded above, the least upper bound of S is set to be ∞, while if S is not bounded below, the greatest lower bound of S is set to be $-\infty$. The least upper bound of S is also called the ***supremum*** of S and is usually denoted by lub$S$ or $\sup S$, and "the" greatest lower bound of S is also called the ***infimum*** of S, and is usually denoted by glb$S$ or $\inf S$. If $S = \varnothing$, then $\sup S = -\infty$, $\inf S = \infty$.

Example 1.82. Let $S = (0, 1)$. Then $\sup S = 1$, $\inf S = 0$.

Example 1.83. Let $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 - x^2 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Define

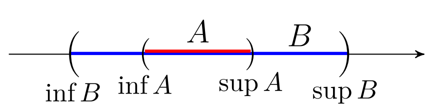
$$S = \{f(x) \mid x \in R\}, \quad T = \big\{x \in \mathbb{R} \mid f(x) > \frac{1}{4}\big\}.$$

We can get $S = (-\infty, 1)$, so $\sup(S) = 1$, $\inf(S) = -\infty$.

Solve $1 - x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{\sqrt{3}}{2}$, then we can get $T = \big(-\frac{\sqrt{3}}{2}, 0\big) \cup \big(0, \frac{\sqrt{3}}{2}\big)$, so $\sup(T) = \frac{\sqrt{3}}{2}$, $\inf(T) = -\frac{\sqrt{3}}{2}$.

Remark 1.84. The least upper bound and the greatest lower bound of S need not be a member of S.

Remark 1.85. The reason for defining $\sup \varnothing = -\infty$ and $\inf \varnothing = \infty$ is as follows: if $\varnothing \neq A \subseteq B$, then $\sup A \leqslant \sup B$ and $\inf A \geqslant \inf B$.



Since $\varnothing$ is a subset of any other sets, we shall have $\sup \varnothing$ is smaller then any real number, and $\inf \varnothing$ is greater than any real number. However, this "definition" would destroy the property that $\inf A \leqslant \sup A$.

The "definition" of $\sup \varnothing$ and $\inf \varnothing$ is purely artificial. One can also define $\sup \varnothing = \infty$ and $\inf \varnothing = -\infty$.

Definition 1.86. An ***open interval*** in $\mathbb{R}$ is of the form (a, b) which consists of all $x \in \mathbb{R} \ni a < x < b$. A ***closed interval*** in $\mathbb{R}$ is of the form $[a, b]$ which consists of all $x \in \mathbb{R} \ni a \leqslant x \leqslant b$.

Proposition 1.87. *Let $S \subseteq \mathbb{R}$ be non-empty. Then*

1. $b = \sup S \in \mathbb{R}$ *if and only if*
   (a) *b is an upper bound of S.*
   (b) $\forall \varepsilon > 0, \exists x \in S \ni x > b - \varepsilon$.
2. $a = \inf S \in \mathbb{R}$ *if and only if*
   (a) *a is a lower bound of S.*
   (b) $\forall \varepsilon > 0, \exists x \in S \ni x < a + \varepsilon$.

Proof. "$\Rightarrow$" (a) is part of the definition of being a least upper bound.

(b) If M is an upper bound of S, then we must have $M \geqslant b$; thus $b - \varepsilon$ is not an upper bound of S. Therefore, $\exists x \in S \ni x > b - \varepsilon$.

"$\Leftarrow$" First, we show that b is an upper bound for S. If not, there exists $x \in S$ such that $b < x$. Let $\varepsilon = x - s > 0$. Then we do not have (i) since $x \in S$ but $x \nless s + \varepsilon$.

Next, we show that if M is an upper bound of S, then $M \geqslant b$. Assume the contrary. Then $\exists M$ such that M is an upper bound of S but $M < b$. Let $\varepsilon = b - M$, then there is no $x \in S \ni x > b - \varepsilon$. $\rightarrow\leftarrow$ □

So far it is not clear that whether the least upper bound or the greatest lower bound for a subset $S \subseteq \mathbb{R}$ exists or not. The following theorem provides the existence of the least upper bound or the greatest lower bound of a set S provided that S has certain properties.

Theorem 1.88. *In $\mathbb{R}$, the following two properties hold:*

1. ***Least upper bound property*** (L.U.B.P.)*:*

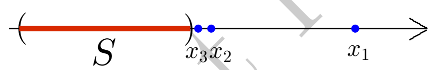
   Let S be a non-empty set in $\mathbb{R}$ that has an upper bound (or is bounded from above), then S has a least upper bound. （非空集合有上界，則有最小上界）

2. ***Greatest lower bound property****:*

   Let S be a non-empty set in $\mathbb{R}$ that has a lower bound (or is bounded from below), then S has a greatest lower bound. (非空集合有下界，則有最大下界)

Proof. We only prove the least upper bound property since the proof of the greatest lower bound property is similar.

Let $\varnothing \neq S \subseteq \mathbb{R}$ be given. Let x_0 be the smallest integer such that x_0 is an upper bound of S. Let $x_1 = x_0 - \dfrac{N_1}{10}$, where N_1 is the largest integer such that x_2 is still an upper bound of S. We continue this process, and define $x_n = x_{n-1} - \dfrac{N_n}{10^n}$, where N_n is the largest integer such that x_n is an upper bound of S.（事實上，x_n 就是十進位下小數點以下只有 n 位的小數裡面，S 的上界中最小的那個數）



Note that in the process of constructing $\{x_n\}_{n=1}^{\infty}$, N_n is always non-negative which implies that $\{x_n\}_{n=1}^{\infty}$ is decreasing. Moreover, any $a \in S$ is a lower bound of $\{x_n\}_{n=1}^{\infty}$. By completeness of $\mathbb{R}$, $\{x_n\}_{n=1}^{\infty}$ converges. Assume that $x_n \to x$ as $n \to \infty$.

Claim: $x = \sup S$ ($\Leftrightarrow$ 1. x is an upper bound of S. 2. $\forall\, \varepsilon > 0, \exists\, s \in S \ni s > x - \varepsilon$).

1. Assume the contrary that x is not an upper bound of S. Then $\exists\, s \in S \ni s > x$. Since $x_n \to x$ as $n \to \infty$, $\exists\, N > 0 \ni |x_n - x| < s - x$ for all $n \geqslant N$; thus

   $$2x - s < x_n < s \qquad n \geqslant N\,.$$

   Therefore, x_n cannot be an upper bound of S for all $n \geqslant N$, a contradiction.

2. Assume the contrary that $\exists\, \varepsilon > 0 \ni \forall\, s \in S$, $s < x - \varepsilon$. Choose $k \in \mathbb{N}$ such that $\varepsilon > \dfrac{1}{10^k}$. Then

   $$x_{k-1} - \frac{N_k + 1}{10^k} = x_k - \frac{1}{10^k} \geqslant x - \varepsilon > s$$

   which suggests that N_k is not the largest integer such that $x_{k-1} - \dfrac{N_k}{10^k}$ is still an upper bound, a contradiction. □

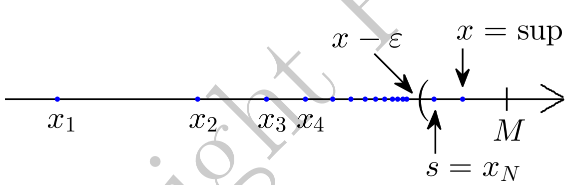
Proposition 1.89. *Suppose that* $\varnothing \neq A \subseteq B \subseteq \mathbb{R}$*. Then* $\inf B \leqslant \inf A \leqslant \sup A \leqslant \sup B$*.*

Proof. We proceed as follows.

1. $\sup A \leqslant \sup B$: Let $b = \sup B$, then $\forall x \in B$, $x \leqslant b$. Since $A \subseteq B$, then $\forall x \in A$, $x \leqslant b$; hence b is also an upper bound for A. Since $\sup A$ is the least upper bound for A and b is an upper bound for A, then $\sup A \leqslant b = \sup B$.
2. It is similar to prove $\inf B \leqslant \inf A$.
3. It is trivially true that $\inf A \leqslant \sup A$. □

Theorem 1.90. *Let* $(\mathcal{F}, +, \cdot, \leqslant)$ *be an ordered field such that* $\mathcal{F}$ *has the least upper bound property, then* $\mathcal{F}$ *is complete.*

Proof. We would like to show that any increasing bounded sequence converges. Let $\{x_n\}_{n=1}^{\infty}$ be increasing and bounded above (by M).



Define $S = \{x_1, x_2, \cdots, x_n, \cdots\}$. Then S is non-empty and has an upper bound; thus by the assumption that $\mathcal{F}$ satisfies the least upper bound property, $\sup S \equiv x$ exists.

1. x is an upper bound of $S \Rightarrow x_n \leqslant x$ for all $n \in \mathbb{N}$.
2. By Proposition 1.87, $\forall\, \varepsilon > 0$, $\exists\, s \in S \ni s > x - \varepsilon$. Note that $s = x_N$ for some $N \in \mathbb{N}$. Since $\{x_n\}_{n=1}^{\infty}$ is increasing, $x_N \leqslant x_n \leqslant x$ for all $n \geqslant N$. Therefore, if $n \geqslant N$,

$$x - \varepsilon < x_N \leqslant x_n \leqslant x < x + \varepsilon$$

which implies that $|x_n - x| < \varepsilon$ if $n \geqslant N$. □

Example 1.91. $\mathbb{Q}$ is not complete. Let $S = \{x_1 = 3, x_2 = 3.1, x_3 = 3.14, \cdots\}$. Then S has 4 as an upper bound, but S has no least upper bound (in $\mathbb{Q}$).

Remark 1.92. The two theorems above suggest that in an ordered field, completeness $\Leftrightarrow$ the least upper bound property.

1.4 Cauchy Sequences

So far the only criteria that we learn (from previous sections) for the convergence of a sequence in an ordered field is that a bounded monotone sequence in $\mathbb{R}$ converges. Are there any other criteria for the convergence of a sequence in an ordered field? By Proposition 1.48, we know that if a sequence $\{x_n\}_{n=1}^{\infty}$ in an ordered field $\mathcal{F}$ converges, then

$$\exists!\, x \in \mathcal{F} \ni \forall\, \varepsilon > 0, \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\big\} < \infty.$$

We would like to investigate if the following much weaker statement

$$\forall\, \varepsilon > 0, \exists\, (\text{a limit candidate})\, y \in \mathcal{F} \ni \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin (y-\varepsilon, y+\varepsilon)\big\} < \infty \qquad (\star)$$

leads to the convergence of a sequence. Note that statement $(\star)$ is equivalent to statement $(\star\star)$ in the following

Definition 1.93. A sequence $\{x_n\}_{n=1}^{\infty}$ in an ordered field is said to be ***Cauchy*** if

$$\forall\, \varepsilon > 0, \exists\, N > 0 \ni |x_n - x_m| < \varepsilon \text{ whenever } n, m \geqslant N\,. \qquad (\star\star)$$

Remark 1.94. $(\star)$ 這個敘述的中心思想是：給定一正值 ε, 我們都能找到一個長度是 2ε 的區間使得落在此區間外的 x_n 只有有限個。因為當對每個長度我們都能找到這樣的區間時，才有機會找到 $\{x_n\}_{n=1}^{\infty}$ 的極限（極限若真的存在的話，那麼這個極限一定落在所有這樣的區間之內）；要是連這樣的區間都找不到，就不可能會收斂了。

Example 1.95. In $\mathbb{Q}$, $x_1 = 3, x_2 = 3.1, x_3 = 3.14, x_4 = 3.141, \cdots$. Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, but is not convergent. Therefore, a Cauchy sequence may not converge.

Proposition 1.96. *Every convergent sequence is Cauchy.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence with limit x. For any $\varepsilon > 0$, $\exists\, N > 0 \ni |x_n - x| < \frac{\varepsilon}{2}$ if $n \geqslant N$. Then by triangle inequality, if $n, m \geqslant N$,

$$|x_n - x_m| \leqslant |x_n - x| + |x - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,;$$

thus $\{x_n\}_{n=1}^{\infty}$ is Cauchy. □

Lemma 1.97. *Every Cauchy sequence is bounded.*

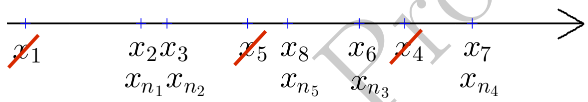
Proof. Let $\{x_n\}_{n=1}^\infty$ be Cauchy. $\exists\, N > 0 \;\ni |x_n - x_m| < 1$ for all $n, m \geqslant N$. In particular, $|x_n - x_N| < 1$ if $n \geqslant N$ or equivalently,

$$x_N - 1 < x_n < x_N + 1 \qquad \forall\, n \geqslant N\,.$$

Let $M = \max\big\{|x_1|, |x_2|, \cdots, |x_{N-1}|, |x_N| + 1\big\}$. Then $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. ▫

Definition 1.98. A sequence $\{y_j\}_{j=1}^\infty$ is called a ***subsequence*** (子數列) of a sequence $\{x_n\}_{n=1}^\infty$ if there exists an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $y_j = x_{f(j)}$. In this case, we often write $f(j) = n_j$ and $y_j = x_{n_j}$.

In other words, a subsequence is a sequence that can be derived from another sequence by deleting some elements without changing the order of remaining elements. Let $f : \mathbb{N} \to \mathbb{R}$ be a sequence an $x_n = f(n)$. A subsequence $\{x_{n_j}\}_{j=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ is the image of an infinite subset $\{n_1, n_2, \cdots\}$ of $\mathbb{N}$ under the map f.

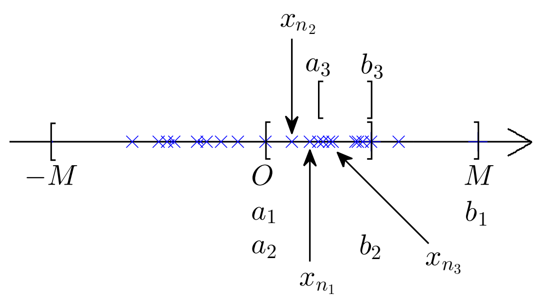


Example 1.99. Let $\{x_n\}_{n=1}^\infty = \big\{1, \frac{1}{2}, \frac{1}{7}, \frac{1}{3}, \frac{2}{3}, \frac{11}{8}, \cdots\big\}$, and $\{y_n\}_{n=1}^\infty = \big\{\frac{1}{2}, \frac{1}{7}, \frac{2}{3}, \frac{11}{8}, \cdots\big\}$. Then $\{y_n\}_{n=1}^\infty$ can be viewed as a subsequence of $\{x_n\}_{n=1}^\infty$ by the relation $y_j = x_{n_j}$; that is, $y_1 = x_2$, $y_2 = x_3$, $y_3 = x_5$, $y_4 = x_6$, and etc. The sequence $\{x_{n_j}\}_{j=1}^\infty$ is obtained by deleting x_1 and x_4 (and maybe more) from the original sequence $\{x_n\}_{n=1}^\infty$. However, if $\{z_n\}_{n=1}^\infty = \{\frac{1}{7}, \frac{11}{8}, 1, \cdots\}$, then $\{z_n\}_{n=1}^\infty$ is not a subsequence of $\{x_n\}_{n=1}^\infty$ (but only a subset) of $\{x_n\}_{n=1}^\infty$ because the order is changed.

Theorem 1.100 (Bolzano-Weierstrass property)**.** *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence; that is, every bounded sequence in $\mathbb{R}$ has a subsequence that converges to a limit in $\mathbb{R}$.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a bounded sequence satisfying $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. Divide $[-M, M]$ into two intervals $[-M, 0], [0, M]$, and denote one of the two intervals containing infinitely many x_n as $[a_1, b_1]$; that is, $\#\big\{n \in \mathbb{N}\,\big|\, x_n \in [a_1, b_1]\big\} = \infty$. Divide $[a_1, b_1]$ into two intervals $\big[a_1, \frac{a_1 + b_1}{2}\big], \big[\frac{a_1 + b_1}{2}, b_1\big]$, and denote one of the two intervals containing infinitely many x_n as $[a_2, b_2]$. We continue this process, and obtain a sequence of intervals $[a_k, b_k]$ such that $\#\{n \in \mathbb{N} \,|\, x_n \in [a_k, b_k]\} = \infty$.

Let x_{n_1} be an element belonging to $[a_1, b_1]$. Since $\#\{n \in \mathbb{N} \,|\, x_n \in [a_1, b_1]\} = \infty$, we can choose $n_2 > n_1$ such that $x_{n_2} \in [a_2, b_2]$, and for the same reason we can choose $n_3 > n_2$ such that $x_{n_3} \in [a_3, b_3]$. We continue this process and obtain $x_{n_k} \in [a_k, b_k]$ with $n_k > n_{k-1}$.



Since $[a_k, b_k] \supseteq [a_{k+1}, b_{k+1}]$ for all $k \in \mathbb{N}$, we find that $\{a_k\}_{k=1}^{\infty}$ is increasing and $\{b_k\}_{k=1}^{\infty}$ is decreasing. Moreover, $a_k \leqslant M$, $b_k \geqslant -M$. As a consequence, by the monotone sequence property, a_k converges to a and b_k converges to b.

On the other hand, we observe that $b_k - a_k = \dfrac{M}{2^{k-1}}$. Then $b - a = \lim\limits_{k\to\infty} \dfrac{M}{2^{k-1}} = 0$; thus $a = b$. Since $a_k \leqslant x_{n_k} \leqslant b_k$, by Sandwich lemma $\lim\limits_{k\to\infty} x_{n_k} = a = b \in \mathbb{R}$. □

Lemma 1.101. *If a subsequence of a Cauchy sequence is convergent, then this Cauchy sequence also converges.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence with a convergent subsequence $\{x_{n_j}\}_{j=1}^{\infty}$. Assume $\lim\limits_{j\to\infty} x_{n_j} = x$. Then $\forall\, \varepsilon > 0$,

$$\exists\, K > 0 \ni |x_{n_j} - x| < \frac{\varepsilon}{2} \qquad \text{if} \quad j \geqslant K, \text{ and}$$
$$\exists\, N > 0 \ni |x_n - x_m| < \frac{\varepsilon}{2} \qquad \text{if} \quad n, m \geqslant N.$$

Choose $j \geqslant \max\{K, N\}$. Then $n_j \geqslant N$; thus if $n \geqslant N$,

$$|x_n - x| \leqslant |x_n - x_{n_j}| + |x_{n_j} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$ □

Theorem 1.102. *Every Cauchy sequence in $\mathbb{R}$ is convergent.*

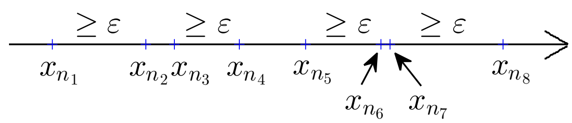
Theorem 1.103. *Suppose that $\mathcal{F}$ is an ordered field with Archimedean property and every Cauchy sequence converges. Then $\mathcal{F}$ is complete.*

Proof. Suppose the contrary that there is a bounded increasing sequence $\{x_n\}_{n=1}^{\infty}$ that does not converge to a limit in $\mathcal{F}$. By assumption, $\{x_n\}_{n=1}^{\infty}$ cannot be Cauchy; thus

$$\exists\, \varepsilon > 0 \ni \forall\, N > 0 \;\exists\, n, m \geqslant N \ni |x_n - x_m| \geqslant \varepsilon\,.$$

Let $N = 1$, $\exists\, n_2 > n_1 \geqslant 1 \ni |x_{n_1} - x_{n_2}| \geqslant \varepsilon$. Let $N = n_2 + 1$, $\exists\, n_4 > n_3 \geqslant n_2 + 1 \ni |x_{n_3} - x_{n_4}| \geqslant \varepsilon$. We continue this process and obtain a sequence $\{x_{n_j}\}_{j=1}^{\infty}$ satisfying

$$\left|x_{n_{2k-1}} - x_{n_{2k}}\right| \geqslant \varepsilon \quad \forall\, k \in \mathbb{N}.$$



Claim: $\left\{x_{n_j}\right\}_{j=1}^{\infty}$ is unbounded (thus a contradiction to the boundedness of $\{x_n\}_{n=1}^{\infty}$).
Proof of claim: Assume the contrary that there exists $M \in \mathcal{F}$ such that $x_{n_j} \leqslant M$ for all $j \in \mathbb{N}$. Since $x_{n_{2k}} \geqslant x_1 + k\varepsilon$ for all $k \in \mathbb{N}$, we must have

$$k \leqslant \frac{M - x_1}{\varepsilon} \qquad \forall\, k \in \mathbb{N}$$

which violates the Archimedean property, a contradiction. ▫

Remark 1.104. In an ordered field with Archimedean property, Completeness ⇔ Cauchy completeness (Every Cauchy sequence converges).

Example 1.105. $x_n \in \mathbb{R}$, $|x_n - x_{n+1}| < \dfrac{1}{2^{n+1}}\ \forall\, n \in \mathbb{N}$.
Claim: $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Given $\varepsilon > 0$, choose $N > 0 \ni \dfrac{1}{2^N} < \varepsilon$. Then if $N \leqslant n < m$,

$$\begin{aligned}
|x_n - x_m| &\leqslant |x_n - x_{n+1}| + |x_{n+1} - x_m| \\
&\leqslant |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + |x_{n+2} - x_m| \\
&\leqslant \cdots \\
&\leqslant |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\
&\leqslant \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots + \frac{1}{2^m} \\
&\leqslant \frac{1}{2^n} \leqslant \frac{1}{2^N} < \varepsilon\,;
\end{aligned}$$

thus $\{x_n\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$. This implies that the sequence is convergent.

1.5 Cluster Points and Limit Inferior, Limit Superior

Definition 1.106. A point x is called a ***cluster point*** of a sequence $\{x_n\}_{n=1}^{\infty}$ if

$$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n \in (x - \varepsilon, x + \varepsilon)\big\} = \infty\,.$$

Example 1.107. Let $x_n = (-1)^n$. Then 1 and -1 are the only two cluster points of $\{x_n\}_{n=1}^{\infty}$.

Example 1.108. Let $x_n = (-1)^n + \dfrac{1}{n}$.
Claim: 1 and -1 are cluster points of $\{x_n\}_{n=1}^{\infty}$.
Let $\varepsilon > 0$ be given. We observe that

$$\big\{n \in \mathbb{N} \,\big|\, x_n \in (1-\varepsilon, 1+\varepsilon)\big\} \supseteq \Big\{n \in \mathbb{N} \,\Big|\, n \text{ is even}, \frac{1}{n} < \varepsilon\Big\};$$

thus $\#\big\{n \in \mathbb{N} \,\big|\, x_n \in (1-\varepsilon, 1+\varepsilon)\big\} = \infty$. Similarly, -1 is a cluster point.
Claim: $\forall\, a \neq \pm 1$, a is not a cluster point of $\{x_n\}_{n=1}^{\infty}$ (reasoning in the following proposition).

Proposition 1.109. *Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.*

1. *x is a cluster point of $\{x_n\}_{n=1}^{\infty}$ if and only if $\forall\, \varepsilon > 0$, $N > 0$, $\exists\, n \geqslant N \ni |x_n - x| < \varepsilon$.*

2. *x is a cluster point of $\{x_n\}_{n=1}^{\infty}$ if and only if there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converges to x.*

3. *$x_n \to x$ as $n \to \infty$ if and only if every proper subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to x.*

4. *$x_n \to x$ as $n \to \infty$ if and only if $\{x_n\}_{n=1}^{\infty}$ is bounded and x is the only cluster point of $\{x_n\}_{n=1}^{\infty}$.*

5. *$x_n \to x$ as $n \to \infty$ if and only if every proper subsequence of $\{x_n\}_{n=1}^{\infty}$ has a further subsequence that converges to x.*

Proof. We only prove 1-4, and the proof of 5 is left as an exercise.

1. ($\Rightarrow$) Let $\varepsilon > 0$ be given. Since there are infinitely many $n's$ with $|x_n - x| < \varepsilon$, for any fixed $N \in \mathbb{N}$, there are only finite number of the indices that are smaller than N. So there must be some $n \geqslant N$ with $|x_n - x| < \varepsilon$.

   ($\Leftarrow$) Let $\varepsilon > 0$ be given. Pick $n_1 \geqslant 1 \ni |x_{n_1} - x| < \varepsilon$, then pick $n_2 \geqslant n_1 + 1 \ni |x_{n_2} - x| < \varepsilon$. We continue this process and obtain a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ satisfying $|x_{n_j} - x| < \varepsilon$ for all $j \in \mathbb{N}$. Then $\big\{n \in \mathbb{N} \,\big|\, x_n \in (x-\varepsilon, x+\varepsilon)\big\} \supseteq \{n_1, n_2, \cdots\}$.

2. ($\Rightarrow$) By 1, we can pick $n_1 \geqslant 1 \ni |x_{n_1} - x| < 1$ and pick $n_2 \geqslant n_1 + 1 \ni |x_{n_2} - x| < \dfrac{1}{2}$. In general, we can pick $n_k \geqslant n_{k-1} + 1 \ni |x_{n_k} - x| < \dfrac{1}{k}$ for all $k \geqslant 2$. Then

$$x - \frac{1}{k} < x_{n_k} < x + \frac{1}{k} \quad \forall\, k \in \mathbb{N}.$$

By Sandwich lemma, $\lim\limits_{k\to\infty} x_{n_k} = x$.

($\Leftarrow$) $\forall\,\varepsilon > 0,\ \exists\, J > 0\ \ni |x_{n_j} - x| < \varepsilon$ if $j \geqslant J$. Then $\big\{n \in \mathbb{N} \,\big|\, x_n \in (x-\varepsilon, x+\varepsilon)\big\} \supseteq \{n_J, n_{J+1}, \cdots\}$.

3. ($\Rightarrow$) Let $\{x_{n_j}\}_{j=1}^\infty$ be a subsequence of a convergent sequence $\{x_n\}_{n=1}^\infty$ and $\lim\limits_{n\to\infty} x_n = x$. Then $\forall\,\varepsilon > 0$, $\exists\, N > 0 \ni |x_n - x| < \varepsilon$ for all $n \geqslant N$. Since $n_j \to \infty$ as $j \to \infty$, $\exists\, J > 0$ $\ni n_j \geqslant N$; thus $|x_{n_j} - x| < \varepsilon$ whenever $j \geqslant J$.

   ($\Leftarrow$) Assume the contrary that $x_n \nrightarrow x$ as $n \to \infty$. Then
$$\exists\,\varepsilon > 0 \ni \forall\, N > 0, \exists\, n \geqslant N \ni |x_n - x| \geqslant \varepsilon.$$
   Let $n_1 \geqslant 1$ such that $|x_{n_1} - x| \geqslant \varepsilon$, and $n_2 \geqslant n_1 + 1$ such that $|x_{n_2} - x| \geqslant \varepsilon$. In general, we can chose $n_k \geqslant n_{k-1}$ such that $|x_{n_k} - x| \geqslant \varepsilon$ for all $k \geqslant 2$. The subsequence $\{x_{n_j}\}_{j=1}^\infty$ clearly does not converge to x, a contradiction.

4. ($\Rightarrow$) This direction is a direct consequence of Proposition 1.48 and 1.59.

   ($\Leftarrow$) Suppose that $\{x_n\}_{n=1}$ is a bounded sequence in $\mathbb{R}$ and has x as the only cluster point but $\{x_n\}_{n=1}^\infty$ does not converge to x. Then
$$\exists\,\varepsilon > 0 \ni \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\big\} = \infty\,.$$
   Write $\big\{n \in \mathbb{N} \,\big|\, x_n \notin (x-\varepsilon, x+\varepsilon)\big\} = \{n_1, n_2, \cdots, n_k, \cdots\}$. Then we find a subsequence $\big\{x_{n_k}\big\}_{k=1}^\infty$ lying outside $(x-\varepsilon, x+\varepsilon)$. Since $\big\{x_{n_k}\big\}_{k=1}^\infty$ is bounded, the Bolzano-Weierstrass property (Theorem 1.100) suggests that there exists a convergent subsequence $\big\{x_{n_{k_j}}\big\}_{j=1}^\infty$ with limit y. Since $x_{n_{k_j}} \notin (x-\varepsilon, x+\varepsilon)$, $y \notin [x-\varepsilon, x+\varepsilon]$; thus $y \neq x$. On the other hand, 2 suggests that y is a cluster point of $\{x_n\}_{n=1}^\infty$, a contradiction to the assumption that x is the only cluster point of $\{x_n\}_{n=1}^\infty$. □

Definition 1.110. A sequence $\{x_n\}_{n=1}^\infty$ is said to ***diverge to infinity*** if $\forall\, M > 0$, $\exists\, N > 0$ $\ni x_n > M$ whenever $n \geqslant N$. It is said to ***diverge to negative infinity*** if $\{-x_n\}_{n=1}^\infty$ diverge to infinity. We use $\lim\limits_{n\to\infty} x_n = \infty$ or $-\infty$ to denote that $\{x_n\}_{n=1}^\infty$ diverges to infinity or negative infinity, and call ∞ or $-\infty$ the limit of $\{x_n\}_{n=1}^\infty$.

Definition 1.111. The ***extended real number system***, denoted by $\mathbb{R}^*$, is the number system $\mathbb{R} \cup \{\infty, -\infty\}$, where ∞ and $-\infty$ are two symbols satisfying $-\infty < x < \infty$ for all $x \in \mathbb{R}$.

Remark 1.112. 1. $\mathbb{R}^*$ is not a field since ∞ and $-\infty$ do not have multiplicative inverse.

2. The definition of the least upper bound of a set can be simplified as follows: Let $S \subseteq \mathbb{R}^*$ be a set (not necessary non-empty set). A number $b \in \mathbb{R}^*$ is said to be the least upper bound of S if

   (a) b is an upper bound of S (that is, $s \leqslant b$ for all $s \in S$);

   (b) If $M \in \mathbb{R}^*$ is an upper bound of S, then $b \leqslant M$.

   No further discussion (such as $S = \varnothing$ or S is not bounded above) has to be made. The greatest lower bound can be defined in a similar fashion.

3. Any sets in $\mathbb{R}^*$ has a least upper bound and a greatest lower bound in $\mathbb{R}^*$, even the empty set and unbounded set.

4. Proposition 1.87 can be rephrased as follows: Let $S \subseteq \mathbb{R}^*$. Then $b = \sup S \in \mathbb{R}$ if and only if

   (a) b is an upper bound of S;

   (b) $\forall\, \varepsilon > 0, \exists\, s \in S \ni s > b - \varepsilon$.

   Note that $b \in \mathbb{R}$ is crucial since there is no $s \in \mathbb{R}^*$ such that $s > \infty - \varepsilon = \infty$. The greatest lower bound counterpart can be made in a similar fashion.

5. In light of Proposition 1.109 and Definition 1.110, we can redefine cluster points of a real sequence as follows: A number $x \in \mathbb{R}^*$ is said to be a cluster point of a sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ if there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that $\lim\limits_{j\to\infty} x_{n_j} = x$. Note that now we can talk about if ∞ or $-\infty$ is a cluster points of a real sequence.

In the rest of the section, one is allowed to find the least upper bound and the greatest lower bound of a subset in $\mathbb{R}^*$.

Definition 1.113. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$.

1. The ***limit superior*** of $\{x_n\}_{n=1}^{\infty}$, denoted by $\limsup\limits_{n\to\infty} x_n$ or $\overline{\lim\limits_{n\to\infty}}\, x_n$, is the infimum of the sequence $\Big\{ \sup\big\{x_n \,\big|\, n \geqslant k\big\}\Big\}_{k=1}^{\infty}$.

2. The ***limit inferior*** of $\{x_n\}_{n=1}^{\infty}$, denoted by $\liminf\limits_{n\to\infty} x_n$ or $\underline{\lim\limits_{n\to\infty}}\, x_n$, is the supremum of the sequence $\Big\{ \inf\big\{x_n \,\big|\, n \geqslant k\big\}\Big\}_{k=1}^{\infty}$.

Remark 1.114. Let $\sup\limits_{n \geqslant k} x_n$ denote the number $\sup\big\{x_n \,\big|\, n \geqslant k\big\}$ and $\inf\limits_{n \geqslant k} x_n$ denote the number $\inf\big\{x_n \,\big|\, n \geqslant k\big\}$. Then the limit superior and the limit inferior can be written as

$$\limsup_{n\to\infty} x_n = \inf_{k\geqslant 1}\sup_{n\geqslant k} x_n \quad \text{and} \quad \liminf_{n\to\infty} x_n = \sup_{k\geqslant 1}\inf_{n\geqslant k} x_n \,.$$

Remark 1.115. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$, and $y_k = \sup\limits_{n\geqslant k} x_n$ and $z_k = \inf\limits_{n\geqslant k} x_n$. Then $\{y_k\}_{k=1}^{\infty}$ is a decreasing sequence, and $\{z_k\}_{k=1}^{\infty}$ is an increasing sequence. Therefore, the limit of $\{y_k\}_{k=1}^{\infty}$ and the limit of $\{z_k\}_{k=1}^{\infty}$ both "exist" in the sense of Definition 1.46 and 1.110. In fact, the limit of $\{y_k\}_{k=1}^{\infty}$ is the infimum of $\{y_k\}_{k=1}^{\infty}$, and the limit of $\{z_k\}_{k=1}^{\infty}$ is the supremum of $\{z_k\}_{k=1}^{\infty}$. In other words,

$$\lim_{k\to\infty}\sup_{n\geqslant k} x_n = \inf_{k\geqslant 1}\sup_{n\geqslant k} x_n \quad \text{and} \quad \lim_{k\to\infty}\inf_{n\geqslant k} x_n = \sup_{k\geqslant 1}\inf_{n\geqslant k} x_n \,;$$

thus

$$\limsup_{n\to\infty} x_n = \lim_{k\to\infty}\sup_{n\geqslant k} x_n \quad \text{and} \quad \liminf_{n\to\infty} x_n = \lim_{k\to\infty}\inf_{n\geqslant k} x_n \,.$$

Example 1.116. Let $\{x_n\}_{n=1}^{\infty} = \{1, 0, -1, 1, 0, -1, 1, 0, -1, \cdots\}$. Then

$$\begin{aligned} y_k &= \sup_{n\geqslant k} x_n = 1 \;\; \Rightarrow \limsup_{n\to\infty} x_n = 1. \\ z_k &= \inf_{n\geqslant k} x_n = -1 \Rightarrow \liminf_{n\to\infty} x_n = -1. \end{aligned}$$

Example 1.117. Let $x_n = \dfrac{1}{n}$. Then

$$\begin{aligned} y_k &= \sup_{n\geqslant k} x_n = \frac{1}{k} \Rightarrow \limsup_{n\to\infty} x_n = 0. \\ z_k &= \inf_{n\geqslant k} x_n = 0 \;\; \Rightarrow \liminf_{n\to\infty} x_n = 0. \end{aligned}$$

Example 1.118. Let $x_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$; that is, $\{x_n\}_{n=1}^{\infty} = \{1, 0, 3, 0, 5, \cdots\}$. Then

$$\begin{aligned} y_k &= \sup_{n\geqslant k} x_n = \infty \Rightarrow \limsup_{n\to\infty} x_n = \infty. \\ z_k &= \inf_{n\geqslant k} x_n = 0 \;\; \Rightarrow \liminf_{n\to\infty} x_n = 0. \end{aligned}$$

Example 1.119. Let $x_n = \begin{cases} 1 + \dfrac{1}{n} & \text{if } n = 4k+1, \\ -1 - \dfrac{1}{n} & \text{if } n = 4k+2, \\ 1 - \dfrac{1}{n} & \text{if } n = 4k+3, \\ -1 + \dfrac{1}{n} & \text{if } n = 4k. \end{cases}$

$y_k = \sup\limits_{n \geqslant k} x_n = 1 + \dfrac{1}{\bigcirc}$, $z_k = \inf\limits_{n \geqslant k} x_n = -1 - \dfrac{1}{\bigcirc}$. $\limsup\limits_{n\to\infty} x_n = 1$. $\liminf\limits_{n\to\infty} x_n = -1$.

Proposition 1.120. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then*

$$\limsup_{n\to\infty} -x_n = -\liminf_{n\to\infty} x_n \quad \textit{and} \quad \liminf_{n\to\infty} -x_n = -\limsup_{n\to\infty} x_n\,.$$

Proof. By the fact that $\sup\limits_{n\geqslant k} -x_n = -\inf\limits_{n\geqslant k} x_n$,

$$\limsup_{n\to\infty} -x_n = \lim_{k\to\infty} \sup_{n\geqslant k}(-x_n) = \lim_{k\to\infty}\big(-\inf_{n\geqslant k} x_n\big) = -\lim_{k\to\infty}\inf_{n\geqslant k} x_n = -\liminf_{n\to\infty} x_n\,.$$

The second identity holds simply by replacing x_n by $-x_n$ in the first identity. □

Proposition 1.121. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then*

1. $a = \liminf\limits_{n\to\infty} x_n \in \mathbb{R}$ *if and only if*

   (a) $\forall\, \varepsilon > 0$, $\exists\, N > 0$ *such that* $a - \varepsilon < x_n$ *whenever* $n \geqslant N$*; that is,*

   $$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n \leqslant a - \varepsilon\big\} < \infty\,,$$

   and

   (b) $\forall\, \varepsilon > 0$ *and* $N > 0$, $\exists\, n \geqslant N$ *such that* $x_n < a + \varepsilon$*; that is,*

   $$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n < a + \varepsilon\big\} = \infty\,.$$

2. $b = \limsup\limits_{n\to\infty} x_n \in \mathbb{R}$ *if and only if*

   (a) $\forall\, \varepsilon > 0$, $\exists\, N > 0$ *such that* $b + \varepsilon > x_n$ *whenever* $n \geqslant N$*; that is,*

   $$\forall\, \varepsilon > 0,\ \#\big\{n \in \mathbb{N} \,\big|\, x_n \geqslant b + \varepsilon\big\} < \infty\,,$$

   and

(b) $\forall\,\varepsilon > 0$ *and* $N > 0$, $\exists\, n \geqslant N$ *such that* $x_n > b - \varepsilon$*; that is,*

$$\forall\,\varepsilon > 0,\ \#\big\{n \in \mathbb{N}\,\big|\,x_n > b - \varepsilon\big\} = \infty\,.$$

Proof. We only prove 1 since the proof of 2 is similar. Let $z_k = \inf\limits_{n \geqslant k} x_n$, and

$$\sup_{k \geqslant 1} z_k = \lim_{k \to \infty} z_k = a \in \mathbb{R}^*\,.$$

We show that $a \in \mathbb{R}$ if and only if 1-(a) and 1-(b). Nevertheless, by Proposition 1.87 (or Remark 1.112), $a \in \mathbb{R}$ if and only if

(i) a is an upper bound of $\{z_k\}_{k=1}^{\infty}$.

(ii) $\forall\,\varepsilon > 0, \exists\, N \in \mathbb{N} \ni z_N > a - \varepsilon$.

We justify the equivalency between 1-(a) and (ii), as well as the equivalency between 1-(b) and (i) as follows:

(i) a is an upper bound of $\{z_k\}_{k=1}^{\infty}$ $\Leftrightarrow$ $a \geqslant z_k$ for all $k \in \mathbb{N}$ $\Leftrightarrow$ $\forall\,\varepsilon > 0, a + \varepsilon > z_k$ for all $k \in \mathbb{N}$ $\Leftrightarrow$ $\forall\,\varepsilon > 0$ and $k \in \mathbb{N}$, $a + \varepsilon > \inf\limits_{n \geqslant k} x_n$ $\Leftrightarrow$ $\forall\,\varepsilon > 0$ and $k \in \mathbb{N}$, $a + \varepsilon$ is not a lower bound of $\{x_n\}_{n \geqslant k}^{\infty}$ $\Leftrightarrow$ $\forall\,\varepsilon > 0$ and $k \in \mathbb{N}$, $\exists\, n \geqslant k \ni a + \varepsilon > x_n$ $\Leftrightarrow$ 1-(b).

(ii) $\forall\,\varepsilon > 0$, $\exists\, N \in \mathbb{N} \ni z_N > a - \varepsilon$ $\Leftrightarrow$ $\forall\,\varepsilon > 0$, $\exists\, N > 0 \ni \inf\limits_{n \geqslant N} x_n > a - \varepsilon$ $\Leftrightarrow$ $\forall\,\varepsilon > 0$, $\exists\, N > 0$ such that $a - \varepsilon$ is a lower bound of $\{x_N, x_{N+1}, \cdots\}$ $\Leftrightarrow$ $\forall\,\varepsilon > 0$, $\exists\, N > 0$ such that $a - \varepsilon \leqslant x_n$ for all $n \geqslant N$ $\Leftrightarrow$ $\forall\,\varepsilon > 0$, $\exists\, N > 0$ such that $a - \varepsilon < x_n$ for all $n \geqslant N$ $\Leftrightarrow$ 1-(a). □

Remark 1.122. By Proposition 1.121, if $a = \liminf\limits_{n \to \infty} x_n \in \mathbb{R}$, then

$$\forall\,\varepsilon > 0,\ \#\big\{n \in \mathbb{N}\,\big|\,x_n \in (a - \varepsilon, a + \varepsilon)\big\} = \infty$$

which suggests that a is a cluster point of $\{x_n\}_{n=1}^{\infty}$. Moreover, 1-(a) of Proposition 1.121 implies that no other cluster points can be smaller than a. In other words, if $a = \liminf\limits_{n \to \infty} x_n \in \mathbb{R}$, then a is the smallest cluster point of $\{x_n\}_{n=1}^{\infty}$. Similarly, b is the largest cluster point of $\{x_n\}_{n=1}^{\infty}$ if $b = \limsup\limits_{n \to \infty} x_n \in \mathbb{R}$.

Theorem 1.123. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathbb{R}$*. Then*

1. $\liminf\limits_{n \to \infty} x_n \leqslant \limsup\limits_{n \to \infty} x_n$.

2. *If* $\{x_n\}_{n=1}^{\infty}$ *is bounded above by* M*, then* $\limsup\limits_{n\to\infty} x_n \leqslant M$.

3. *If* $\{x_n\}_{n=1}^{\infty}$ *is bounded below by* m*, then* $\liminf\limits_{n\to\infty} x_n \geqslant m$.

4. $\limsup\limits_{n\to\infty} x_n = \infty$ *if and only if* $\{x_n\}_{n=1}^{\infty}$ *is not bounded above.*

5. $\liminf\limits_{n\to\infty} x_n = -\infty$ *if and only if* $\{x_n\}_{n=1}^{\infty}$ *is not bounded below.*

6. *If* x *is a cluster point of* $\{x_n\}_{n=1}^{\infty}$*, then* $\liminf\limits_{n\to\infty} x_n \leqslant x \leqslant \limsup\limits_{n\to\infty} x_n$.

7. *If* $a = \liminf\limits_{n\to\infty} x_n$ *is finite, then* a *is a cluster point.*

8. *If* $b = \limsup\limits_{n\to\infty} x_n$ *is finite, then* b *is a cluster point.*

9. *If* $\{x_n\}_{n=1}^{\infty}$ *converges to* x *in* $\mathbb{R}$ *if and only if* $\liminf\limits_{n\to\infty} x_n = \limsup\limits_{n\to\infty} x_n = x \in \mathbb{R}$.

Proof. Left as an exercise. □

Remark 1.124. Using the definition of cluster points of a sequence in Remark 1.112, Remark 1.122 and Theorem 1.123 together imply that the limit superior/inferior of a sequence is the largest/smallest cluster point of that sequence.

Example 1.125. Let $S = \mathbb{Q} \cap [0,1]$. Then S is countable since it is a subset of a countable set $\mathbb{Q}$. Therefore, $\exists\, f : \mathbb{N} \xrightarrow[onto]{1-1} S$ or equivalently $S = \{q_1, q_2, \cdots, q_n, \cdots\}$. The collection of all cluster points of $\{q_n\}_{n=1}^{\infty}$ is $[0,1]$ since $\mathbb{Q} \cap [0,1]$ is dense in $[0,1]$.

1.6 Euclidean Spaces and Vector Spaces

Definition 1.126. ***Euclidean*** n***-space***, denoted by $\mathbb{R}^n$, consists of all ordered n-tuples of real numbers. Symbolically,

$$\mathbb{R}^n = \left\{x \,\middle|\, x = (x_1, x_2, \cdots, x_n), x_i \in \mathbb{R}\right\}.$$

Elements of $\mathbb{R}^n$ are generally denoted by single letters that stand for n-tuples such as $x = (x_1, x_2, \cdots, x_n)$, and speak of x as a "point" in $\mathbb{R}^n$.

Definition 1.127. A ***real vector space*** $\mathcal{V}$ is a set of elements called vectors, with given operations of vector addition $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and scalar multiplication $\cdot : \mathbb{R} \times \mathcal{V} \to \mathcal{V}$ such that

1. $v + w = w + v$ for all $v, w \in \mathcal{V}$.
2. $(v + w) + u = v + (u + w)$ for all $u, v, w \in \mathcal{V}$.
3. $\exists\, 0$, the zero vector, $\ni v + 0 = v$ for all $v \in \mathcal{V}$.
4. $\forall\, v \in \mathcal{V},\ \exists\, w \in \mathcal{V}\ \ni v + w = 0$.
5. $\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$ for all $\lambda \in \mathbb{R}$ and $v, w \in \mathcal{V}$.
6. $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$ for all $\lambda, \mu \in \mathbb{R}$ and $v \in \mathcal{V}$.
7. $(\lambda \cdot \mu) \cdot v = \lambda \cdot (\mu \cdot v)$ for all $\lambda, \mu \in \mathbb{R}$ and $v \in \mathcal{V}$.
8. $1 \cdot v = v$ for all $v \in \mathcal{V}$.

Example 1.128. Let the vector addition and scalar multiplication on $\mathbb{R}^n$ be defined by

$$x + y = (x_1 + y_1, \cdots, x_n + y_n) \quad \text{if} \quad x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)$$

and

$$\lambda \cdot x = (\lambda x_1, \cdots, \lambda x_n) \quad \text{if} \quad \lambda \in \mathbb{R}, x = (x_1, \cdots, x_n)\,.$$

Then $\mathbb{R}^n$ is a real vector space.

Example 1.129. Let $\mathcal{M} \equiv \big\{n \times m$ matrix with entries in $\mathbb{R}\big\}$. Define

$$A + B \equiv [a_{ij} + b_{ij}], \quad \lambda \cdot A \equiv [\lambda \cdot a_{ij}] \quad \text{if} \quad \lambda \in \mathbb{R}, A = [a_{ij}], B = [b_{ij}] \in \mathcal{M}\,.$$

Then $\mathcal{M}$ is a real vector space.

Definition 1.130. $\mathcal{W}$ is called a ***subspace*** of a real vector space $\mathcal{V}$ if

1. $\mathcal{W}$ is a subset of $\mathcal{V}$.
2. $(\mathcal{W}, +, \cdot)$, with vector addition and scalar multiplication in $\mathcal{V}$, is a real vector space.

Example 1.131. $\mathcal{V} = \mathbb{R}^3$, $W = \mathbb{R}^2 \times \{0\} \equiv \{(x, y, 0) | x, y \in \mathbb{R}\}$. $\mathcal{W}$ is a subspace of $\mathcal{V}$.

Lemma 1.132. *If $\mathcal{W}$ is a subset of a real vector space $\mathcal{V}$, then $\mathcal{W}$ is a subspace if and only if $\lambda \cdot v + \mu \cdot w \in \mathcal{W},\ \forall\, \lambda, \mu \in \mathbb{R},\ v, w \in \mathcal{W}$.*

Remark 1.133. "n" is called the ***dimension*** of $\mathbb{R}^n$.
There are n linearly independent vectors $e_1 = (1, 0, \cdots, 0), e_2 = (0, 1, 0, \cdots, 0), \cdots, e_n = (0, 0, \cdots, 0, 1)$, but if $v_1, v_2, \cdots, v_{n+1}$ are $(n+1)$ vectors in $\mathbb{R}^n$, $\exists\, \lambda_1, \cdots, \lambda_{n+1} \in \mathbb{R}$, $\ni \lambda_1 v_1 + \cdots + \lambda_{n+1} v_{n+1} = 0$, $(\lambda_1, \cdots, \lambda_{n+1}) \neq (0, \cdots, 0)$.

Definition 1.134. A subset $H \subseteq \mathbb{R}^n$ is called a ***hyperplane*** if H is $(n-1)$-dimensional subspace of $\mathbb{R}^n$. An ***affine hyperplane*** is a set $x + H \equiv \{x + y \mid y \in H\}$ for some hyperplane H.

1.7 Normed Vector Spaces, Inner Product Spaces and Metric Spaces

Definition 1.135. A ***nomed vector space*** $(\mathcal{V}, \|\cdot\|)$ is a real vector space $\mathcal{V}$ associated with a function $\|\cdot\| : \mathcal{V} \to \mathbb{R}$ such that

(a) $\|x\| \geqslant 0$ for all $x \in \mathcal{V}$.

(b) $\|x\| = 0$ if and only if $x = 0$.

(c) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and $x \in \mathcal{V}$.

(d) $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in \mathcal{V}$.

A function $\|\cdot\|$ satisfies (a)-(d) is called a ***norm*** on $\mathcal{V}$.

Example 1.136. Let $\mathcal{V} = \mathbb{R}^n$, and $\|x\|_2 \equiv \Big(\sum_{i=1}^n x_i^2\Big)^{\frac{1}{2}}$ if $x = (x_1, x_2, \cdots, x_n)$. Then $\|\cdot\|_2$ is a norm, called 2-norm, on $\mathbb{R}^n$. It suffices to show that (d) in Definition 1.135 holds. Let $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$. Then

$$\begin{aligned}(\|x+y\|_2)^2 &= \sum_{i=1}^n (x_i + y_i)^2 = \sum_{i=1}^n (x_i^2 + 2x_i y_i + y_i^2) = \sum_{i=1}^n x_i^2 + \sum_{i=1}^n y_i^2 + 2\sum_{i=1}^n x_i y_i \\ &\leqslant \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2 \qquad \text{(By Cauchy's inequality)} \\ &= (\|x\|_2 + \|y\|_2)^2\ ;\end{aligned}$$

thus $\|x + y\|_2 \leqslant \|x\|_2 + \|y\|_2$.

Example 1.137. Let $\mathcal{V} = \mathbb{R}^n$, and define

$$\|x\|_p \equiv \begin{cases} \Big(\sum_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty, \\ \max\big\{|x_1|, \cdots, |x_n|\big\} & \text{if } p = \infty, \end{cases} \quad \text{for all } x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n.$$

Then $\|\cdot\|_p$ is a norm, called p-norm, on $\mathbb{R}^n$. Property (d) in Definition 1.135; that is, $\|x+y\|_p \leqslant \|x\|_p + \|y\|_p$, is left as an exercise.

Example 1.138. Let $\mathcal{M}_{n\times m} \equiv \big\{n \times m \text{ matrix with entries in } \mathbb{R}\big\}$, and we remind the readers that if $A \in \mathcal{M}_{n\times m}$, then $A : \begin{cases} \mathbb{R}^m \to \mathbb{R}^n \\ x \mapsto Ax \end{cases}$. Define

$$\|A\|_p = \sup_{\|x\|_p=1} \|Ax\|_p = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p} \quad \forall\, A \in \mathcal{M}_{n\times m}\,;$$

that is, $\|A\|_p$ is the least upper bound of the set $\Big\{\frac{\|Ax\|_p}{\|x\|_p}\,\Big|\, x \neq 0, x \in \mathbb{R}^m\Big\}$. Therefore, $\frac{\|Ax\|_p}{\|x\|_p} \leqslant \|A\|_p \;\forall\, x \neq 0$; thus

$$\|Ax\|_p \leqslant \|A\|_p \|x\|_p \quad \forall\, x \in \mathbb{R}^m\,.$$

Consider the case $p = 1, p = 2$ and $p = \infty$ respectively.

1. $p = 2$: Let $(\cdot,\cdot)_{\mathbb{R}^k}$ denote the inner product in Euclidean space $\mathbb{R}^k$. Then

$$\|Ax\|_2^2 = (Ax, Ax)_{\mathbb{R}^n} = (x, A^{\mathrm{T}}Ax)_{\mathbb{R}^m} = (x, P\Lambda P^{\mathrm{T}}x)_{\mathbb{R}^m} = (P^{\mathrm{T}}x, \Lambda P^{\mathrm{T}}x)_{\mathbb{R}^n}\,,$$

in which we use the fact that $A^{\mathrm{T}}A$ is symmetric; thus diagonalizable by an orthonormal matrix P (that is, $A^{\mathrm{T}}A = P\Lambda P^{\mathrm{T}}$, $P^{\mathrm{T}}P = I$, Λ is a diagonal matrix). Therefore,

$$\begin{aligned} \sup_{\|x\|_2=1} \|Ax\|_2^2 &= \sup_{\|x\|_2=1} (P^{\mathrm{T}}x, \Lambda P^{\mathrm{T}}x) = \sup_{\|y\|_2=1} (y, \Lambda y) \quad (\text{Let } y = P^{\mathrm{T}}x, \text{ then } \|y\|_2 = 1) \\ &= \sup_{\|y\|_2=1} (\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2) \\ &= \max\big\{\lambda_1, \cdots, \lambda_n\big\} = \text{maximum eigenvalue of } A^{\mathrm{T}}A \end{aligned}$$

which implies that $\|A\|_2 = \sqrt{\text{maximum eigenvalue of } A^{\mathrm{T}}A}$.

2. $p = \infty$: $\|A\|_\infty = \sup_{\|x\|_\infty=1} \|Ax\|_\infty = \max\Big\{\sum_{j=1}^m |a_{1j}|, \sum_{j=1}^m |a_{2j}|, \cdots \sum_{j=1}^m |a_{nj}|\Big\}$.

Reason: Let $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ and $A = \big[a_{ij}\big]_{n\times m}$. Then

$$Ax = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1m}x_m \\ a_{21}x_1 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nm}x_m \end{bmatrix}$$

Assume $\max\limits_{1\leqslant i\leqslant n} \sum\limits_{j=1}^{m} |a_{ij}| = \sum\limits_{j=1}^{m} |a_{kj}|$ for some $1 \leqslant k \leqslant n$. Let

$$x = (\operatorname{sgn}(a_{k1}), \operatorname{sgn}(a_{k2}), \cdots, \operatorname{sgn}(a_{kn}))\ .$$

Then $\|x\|_\infty = 1$, and $\|Ax\|_\infty = \sum\limits_{j=1}^{m} |a_{kj}|$.

On the other hand, if $\|x\|_\infty = 1$, then

$$|a_{i1}x_1 + a_{i2}x_2 + \cdots a_{im}x_m| \leqslant \sum_{j=1}^{m} |a_{ij}| \leqslant \max\left\{\sum_{j=1}^{m}|a_{1j}|, \sum_{j=1}^{m}|a_{2j}|, \cdots \sum_{j=1}^{m}|a_{nj}|\right\};$$

thus $\|A\|_\infty = \max\left\{\sum\limits_{j=1}^{m}|a_{1j}|, \sum\limits_{j=1}^{m}|a_{2j}|, \cdots \sum\limits_{j=1}^{m}|a_{nj}|\right\}$. In other words, $\|A\|_\infty$ is the largest sum of the absolute value of row entries.

3. $p = 1$: $\|A\|_1 = \max\left\{\sum\limits_{i=1}^{n}|a_{i1}|, \sum\limits_{i=1}^{n}|a_{i2}|, \cdots, \sum\limits_{i=1}^{n}|a_{im}|\right\}$.

Example 1.139. Let $\mathscr{C}$ be the collection of all continuous real-valued functions on the interval $[0, 1]$; that is,

$$\mathscr{C} = \big\{f : [0,1] \to \mathbb{R} \,\big|\, f \text{ is continuous on } [0,1]\big\}\,.$$

For each $f \in \mathscr{C}$, we define

$$\|f\|_p = \begin{cases} \Big[\displaystyle\int_0^1 |f(x)|^p dx\Big]^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty\,, \\ \max\limits_{x\in[0,1]} |f(x)| & \text{if } p = \infty\,. \end{cases}$$

The function $\|\cdot\|_p : \mathscr{C} \to \mathbb{R}$ is a norm on $\mathscr{C}$ (Minkowski's inequality).

Definition 1.140. An ***inner product space*** $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ is a real vector space $\mathcal{V}$ associated with a function $\langle\cdot,\cdot\rangle : \mathcal{V}\times\mathcal{V} \to \mathbb{R}$ such that

(1) $\langle x, x\rangle \geqslant 0,\ \forall\, x \in \mathcal{V}$.

(2) $\langle x, x\rangle = 0$ if and only if $x = 0$.

(3) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$ for all $x, y, z \in \mathcal{V}$.

(4) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ for all $\lambda \in \mathbb{R}$ and $x, y \in \mathcal{V}$.

(5) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{V}$.

A symmetric bilinear form $\langle\cdot,\cdot\rangle$ satisfies (1)-(5) is called an ***inner product*** on $\mathcal{V}$.

Example 1.141. Let $(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$(x, y) = \sum_{i=1}^{n} x_i y_i \quad \forall\, x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)\,.$$

Then $(\cdot,\cdot)$ is an inner product on $\mathbb{R}^n$.

Example 1.142. Let $\mathscr{C}$ be defined as in Example 1.139. Define

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx\,.$$

Then $\langle\cdot,\cdot\rangle : \mathscr{C} \times \mathscr{C} \to \mathbb{R}$ satisfies all the properties that an inner product has. Note that $\langle f, f\rangle = \|f\|_2^2$.

Proposition 1.143. *If $\langle\cdot,\cdot\rangle$ is an inner product on a real vector space $\mathcal{V}$. Then*

1. $\langle \lambda v + \mu w, u\rangle = \lambda\langle v, u\rangle + \mu\langle w, u\rangle$ *for all* $u, v, w \in \mathcal{V}$.

2. $\langle u, \lambda v + \mu w\rangle = \lambda\langle u, v\rangle + \mu\langle u, w\rangle$ *for all* $u, v, w \in \mathcal{V}$.

3. $\langle v, \lambda w\rangle = \lambda\langle v, w\rangle$ *for all* $v, w \in \mathcal{V}$.

4. $\langle 0, w\rangle = \langle w, 0\rangle = 0$ *for all* $w \in \mathcal{V}$.

Theorem 1.144. *The inner product $\langle\cdot,\cdot\rangle$ on a real vector space induces a norm $\|\cdot\|$ given by $\|x\| = \sqrt{\langle x, x\rangle}$ and satisfies the* ***Cauchy-Schwarz inequality***

$$\big|\langle x, y\rangle\big| \leqslant \|x\| \cdot \|y\| \qquad \forall\, x, y \in \mathcal{V}\,. \tag{1.7.1}$$

Proof. First, we observe that for all $x, y \in \mathcal{V}$ fixed, we must have

$$0 \leqslant \langle \lambda x + y, \lambda x + y \rangle = \|x\|^2 \lambda^2 + 2\langle x, y\rangle \lambda + \|y\|^2$$

for all $\lambda \in \mathbb{R}$. Therefore,

$$\langle x, y\rangle^2 - \|x\|^2 \cdot \|y\|^2 \leqslant 0$$

which implies (1.7.1).

It should be clear that (a)-(c) in Definition 1.135 are satisfied. To show that $\|\cdot\|$ satisfies the triangle inequality, by (1.7.1) we find that

$$\begin{aligned}\big(\|x\| + \|y\|\big)^2 - \|x + y\|^2 &= \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 - \langle x + y, x + y\rangle \\ &= 2\big(\|x\|\|y\| - \langle x, y\rangle\big) \geqslant 0\,;\end{aligned}$$

thus the triangle inequality is also valid. □

Corollary 1.145. *Let $f, g : [0, 1] \to \mathbb{R}$ be continuous. Then*

$$\left|\int_0^1 f(x)g(x)dx\right| \leqslant \left(\int_0^1 |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 |g(x)|^2 dx\right)^{\frac{1}{2}}.$$

Definition 1.146. A ***metric space*** (M, d) is a set M associated with a function $d : M \times M \to \mathbb{R}$ such that

(i) $d(x, y) \geqslant 0$ for all $x, y \in M$.

(ii) $d(x, y) = 0$ if and only if $x = y$.

(iii) $d(x, y) = d(y, x)$ for all $x, y \in M$.

(iv) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in M$.

A function d satisfies (i)-(iv) is called a ***metric*** on M.

Example 1.147 (Discrete metric)**.** Let M be a non-empty set, and $d_0 : M \times M \to \mathbb{R}$ be defined by

$$d_0(x, y) = \begin{cases} 0 & \text{if } x = y\,, \\ 1 & \text{if } x \neq y\,. \end{cases}$$

Then d_0 is a metric on M, and we call d_0 the discrete metric.

Example 1.148 (Bounded metric)**.** Let (M, d) be a metric space. Define $\rho : M \times M \to \mathbb{R}$ by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)} .$$

Then ρ is also a metric on M.

Proposition 1.149. *If $(\mathcal{V}, \|\cdot\|)$ is a normed vector space, then the function $d : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined by $d(x, y) = \|x - y\|$ is a metric on $\mathcal{V}$. In other words, (V, d) is a metric space, and we usually write $(\mathcal{V}, \|\cdot\|)$ as the metric space.*

1.8 Exercises

§1.1 Ordered Fields and the Number Systems

Problem 1.1. Let $(\mathcal{F}, +, \cdot, \leqslant)$ be an ordered field, and $a, b, c, d \in \mathcal{F}$.

1. Show that if $a \leqslant b$ and $c \leqslant d$, then $a + c \leqslant b + d$.

2. Show that if $a \leqslant b$ and $c < d$, then $a + c < b + d$.

Problem 1.2. Let S be a non-empty subset of $\mathbb{N}$ and satisfy that

1. $1,\ 2 \in S$.

2. if m and $m + 1 \in S$, then $m + 2 \in S$.

Show that $S = \mathbb{N}$.

§1.2 Completeness and the Real Number System

Problem 1.3. Let $\mathcal{F}$ be an ordered field with Archimedean property, and $x, y \in \mathcal{F}$. Show that $x \leqslant y$ if and only if $\forall\, \varepsilon > 0$, $x < y + \varepsilon$.

Problem 1.4. Fix $y > 1$. Complete the following.

1. Define $y^{1/n}$ properly. (Hint: see how we define $\sqrt{y}$ in class).

2. Show that $y^n - 1 > n(y - 1)$ for all $n \in \mathbb{N}\backslash\{1\}$; thus $y - 1 > n(y^{1/n} - 1)$.

3. If $t > 1$ and $n > (y - 1)/(t - 1)$, then $y^{1/n} < t$.

4. Show that $\lim_{n\to\infty} y^{1/n} = 1$ as $n \to \infty$.

Problem 1.5. Complete the following.

1. Let $x \geqslant 0$ be a real number such that for any $\varepsilon > 0$, $x \leqslant \varepsilon$. Show that $x = 0$.

2. Let $S = (0,1)$. Show that for each $\varepsilon > 0$ there exists an $x \in S$ such that $x < \varepsilon$.

§1.3 Least Upper Bounds and Greatest Lower Bounds

Problem 1.6. Let A be a non-empty set of $\mathbb{R}$ which is bounded below. Define the set $-A$ by $-A \equiv \big\{ -x \in \mathbb{R} \,\big|\, x \in A \big\}$. Prove that

$$\inf A = -\sup(-A)\,.$$

Problem 1.7. Let A, B be non-empty subset of $\mathbb{R}$. Define $A + B = \{x + y \,|\, x \in A, y \in B\}$. Justify if the following statements are true or false by providing a proof for the true statement and giving a counter-example for the false ones.

1. $\sup(A + B) = \sup A + \sup B$.

2. $\inf(A + B) = \inf A + \inf B$.

3. $\sup(A \cap B) \leqslant \min\{\sup A, \sup B\}$.

4. $\sup(A \cap B) = \min\{\sup A, \sup B\}$.

5. $\sup(A \cup B) \geqslant \max\{\sup A, \sup B\}$.

6. $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Problem 1.8. Let $S \subseteq \mathbb{R}$ be bounded below and non-empty. Show that

$$\inf S = \sup \big\{ x \in \mathbb{R} \,\big|\, x \text{ is a lower bound for } S \big\}\,.$$

Problem 1.9. Let A, B be two sets, and $f : A \times B \to \mathbb{R}$ be a function. Show that

$$\sup_{(x,y)\in A\times B} f(x,y) = \sup_{y\in B}\Big(\sup_{x\in A} f(x,y)\Big) = \sup_{x\in A}\Big(\sup_{y\in B} f(x,y)\Big)\,.$$

Problem 1.10. Fix $b > 1$.

1. Show the law of exponents holds (for rational exponents); that is, show that

(a) if r, s in $\mathbb{Q}$, then $b^{r+s} = b^r \cdot b^s$.

(b) if r, s in $\mathbb{Q}$, then $b^{r \cdot s} = (b^r)^s$.

2. For $x \in \mathbb{R}$, let $B(x) = \{b^t \in \mathbb{R} \,|\, t \in \mathbb{Q}, t \leqslant x\}$. Show that $b^r = \sup B(r)$ if $r \in \mathbb{Q}$. Therefore, it makes sense to define $b^x = \sup B(x)$ for $x \in \mathbb{R}$. Show that the law of exponents (for real exponents) are also valid.

3. Let $y > 0$ be given. Using 4 of Problem 1.4 to show that if $u, v \in \mathbb{R}$ such that $b^u < y$ and $b^v > y$, then $b^{u+1/n} < y$ and $b^{v-1/n} > y$ for sufficiently large n.

4. Let $y > 0$ be given, and A be the set of all w such that $b^w < y$. Show that $x = \sup A$ satisfies $b^x = y$.

5. Prove that if x_1, x_2 are two real numbers satisfying $b^{x_1} = b^{x_2}$, then $x_1 = x_2$.

The number x satisfying $b^x = y$ is called the logarithm of y to the base b, and is denoted by $\log_b y$.

§1.4 Cauchy Sequences

Problem 1.11. Let $a \in \mathbb{R}$. Define a_n through the iterated relation

$$a_n = a_{n-1}^2 - a_{n-1} + 1 \qquad \forall\, n > 1, a_1 = a\,.$$

For what a is the sequence $\{a_n\}_{n=1}^{\infty}$ (1) monotone? (2) bounded? (3) convergent? Compute the limit in the case of convergence.

Problem 1.12. Let $\mathcal{F}$ be an ordered field, and $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{F}$. Show that $\{x_n\}_{n=1}^{\infty}$ is Cauchy if and only if

$$\forall\, \varepsilon > 0, \exists\, y \in \mathcal{F} \ni \#\{n \in \mathbb{N} \,|\, x_n \notin (y - \varepsilon, y + \varepsilon)\} < \infty\,.$$

Problem 1.13. Let $\{a_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be two sequences in $\mathbb{R}$, and define $S_k = \sum\limits_{n=1}^{k} a_n$ (so $\{S_k\}_{k=1}^{\infty}$ is also a sequence). Suppose that $|x_n - x_{n+1}| < a_n$ for all $n \in \mathbb{N}$. Show that $\{x_n\}_{n=1}^{\infty}$ converges if $\{S_k\}_{k=1}^{\infty}$ converges.

Problem 1.14. Let $f : \mathbb{R} \to \mathbb{R}$ be a function so that $|f(x) - f(y)| \leqslant \dfrac{|x-y|}{2}$. Pick an arbitrary $x_1 \in \mathbb{R}$, and define $x_{k+1} = f(x_k)$ for all $k \in \mathbb{N}$. Show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Problem 1.15. Suppose that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are two Cauchy sequence in $\mathbb{R}$. Show that the sequence $\{|x_n - y_n|\}_{n=1}^\infty$ converges.

§1.5 Cluster Points and Limit Inferior, Limit Superior

Problem 1.16. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be sequences in $\mathbb{R}$. Prove the following inequalities:

$$\begin{aligned}\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n &\leqslant \liminf_{n\to\infty}(x_n+y_n) \leqslant \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \\ &\leqslant \limsup_{n\to\infty}(x_n+y_n) \leqslant \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n\,; \\ \big(\liminf_{n\to\infty}|x_n|\big)\big(\liminf_{n\to\infty}|y_n|\big) &\leqslant \liminf_{n\to\infty}|x_ny_n| \leqslant \big(\liminf_{n\to\infty}|x_n|\big)\big(\limsup_{n\to\infty}|y_n|\big) \\ &\leqslant \limsup_{n\to\infty}|x_ny_n| \leqslant \big(\limsup_{n\to\infty}|x_n|\big)\big(\limsup_{n\to\infty}|y_n|\big)\,.\end{aligned}$$

Give examples showing that the equalities are generally not true.

Problem 1.17. Prove that

$$\liminf_{n\to\infty}\frac{|x_{n+1}|}{|x_n|} \leqslant \liminf_{n\to\infty}\sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty}\sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty}\frac{|x_{n+1}|}{|x_n|}\,.$$

Give examples to show that the equalities are not true in general. Is it true that $\lim\limits_{n\to\infty}\sqrt[n]{|x_n|}$ exists implies that $\lim\limits_{n\to\infty}\dfrac{|x_{n+1}|}{|x_n|}$ also exists?

Problem 1.18. Find the following limits.

$$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{n!}\,,\qquad \lim_{n\to\infty}\frac{1}{n}\sqrt[n]{(n+1)(n+2)\cdots(2n)}\,.$$

Problem 1.19. Given the following sets consisting of elements of some sequence of real numbers. Find their sup and inf, and also the limsup and liminf of the sequence.

1. $\{\cos m \,|\, m = 0, 1, 2, \cdots\}$.
2. $\{\sqrt[m]{|\sin m|} \,|\, m = 1, 2, \cdots\}$.
3. $\{(1+\frac{1}{m})\sin\frac{m\pi}{6} \,|\, m = 1, 2, \cdots\}$.

Hint: For 1, first show that for all irrational α, the set

$$S = \{x \in [0,1] \,|\, x = k\alpha \text{ (mod 1) for some } k \in \mathbb{N}\}$$

is dense in $[0,1]$; that is, for all $y \in [0,1]$ and $\varepsilon > 0$, there exists $x \in S \cap (y-\varepsilon, y+\varepsilon)$. Then choose $\alpha = \dfrac{1}{2\pi}$ to conclude that

$$T = \{x \in [0,2\pi] \,|\, x = k \pmod{2\pi} \text{ for some } k \in \mathbb{N}\}$$

is dense in $[0,2\pi]$. To prove that S is dense in $[0,1]$, you might want to consider the following set

$$S_k = \{x \in [0,1] \,|\, x = \ell\alpha \pmod 1 \text{ for some } 1 \leqslant \ell \leqslant k+1\}$$

Note that there must be two points in S_k whose distance is less than $\dfrac{1}{k}$. What happened to (the multiples of) the difference of these two points?

§1.6 Euclidean Spaces and Vector Spaces

Problem 1.20. Show that the p-norm on Euclidean space $\mathbb{R}^n$ given by

$$\|x\|_p \equiv \begin{cases} \Big(\sum\limits_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty\,, \\ \max\{|x_1|, \cdots, |x_n|\} & \text{if } p = \infty\,, \end{cases} \qquad x = (x_1, \cdots, x_n)$$

is indeed a norm.

§1.7 Normed Vector Spaces, Inner Product Spaces and Metric Spaces

Problem 1.21. Let $\mathcal{M}_{n\times m}$ be the collection of all $n \times m$ matrices with real entries as in Example 1.138. Define a function $\|\cdot\| : \mathcal{M} \to \mathbb{R}$ by

$$\|A\| = \sup_{\substack{x\in\mathbb{R}^m \\ x\neq 0}} \frac{\|Ax\|_2}{\|x\|_2}\,,$$

here we recall that $\|\cdot\|_2$ is the 2-norm on Euclidean space given by

$$\|x\|_2 = \Big(\sum_{i=1}^k x_i^2\Big)^{1/2} \quad \text{if} \quad x = (x_1, \cdots, x_k) \in \mathbb{R}^k\,.$$

Show that

1. $\|A\| = \sup\limits_{\substack{x\in\mathbb{R}^m \\ \|x\|_2=1}} \|Ax\|_2 = \inf\{M \in \mathbb{R} \,|\, \|Ax\|_2 \leqslant M\|x\|_2 \ \forall\, x \in \mathbb{R}^m\}$.
2. $\|Ax\|_2 \leqslant \|A\|\|x\|_2$ for all $x \in \mathbb{R}^m$.

3. $\|\cdot\|$ defines a norm on $\mathcal{M}_{n\times m}$.

4. Let $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{M}_{n\times m}$. Show that $\lim\limits_{k\to\infty} \|A_k\| = 0$ if and only if each entry of A_k converges to 0. In other words, by writing $A_k = \big[a_{ij}^{(k)}\big]_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$, show that $\lim\limits_{k\to\infty} \|A_k\| = 0$ if and only if $\lim\limits_{k\to\infty} a_{ij}^{(k)} = 0$ for all $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$. In particular, $A_k \to A$ in the sense that $\|A_k - A\| \to 0$ as $k \to \infty$ if and only if the (i,j)-th entry of A_k converges to (i,j)-th entry of A for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$.

Problem 1.22. Let $(\mathcal{V}, +, \cdot, \langle\cdot,\cdot\rangle)$ be an inner product space, and define $\|v\| = \langle v, v\rangle^{1/2}$ for all $v \in \mathcal{V}$. Show that

1. $2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$ (parallelogram law).

2. $\|x+y\|\|x-y\| \leqslant \|x\|^2 + \|y\|^2$.

3. $4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2$ (polarization identity).

Can the p-norm $\|\cdot\|_p$ on $\mathbb{R}^n$ be induced from any inner product (on $\mathbb{R}^n$) for $p \neq 2$?

Problem 1.23. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be three normed vector spaces such that $X, Y \subseteq Z$ and

$$\|x\|_Z \leqslant C\|x\|_X \quad \forall\, x \in X \qquad \text{and} \qquad \|y\|_Z \leqslant C\|y\|_Y \quad \forall\, y \in Y\,.$$

Define

$$E = \big\{a \in Z \,\big|\, \|a\|_E \equiv \max\{\|a\|_X, \|a\|_Y\} < \infty\big\}$$

and

$$F = \big\{a \in Z \,\big|\, \|a\|_F \equiv \inf_{\substack{a=x+y\\ x\in X, y\in Y}} (\|x\|_X + \|y\|_Y) < \infty\big\}\,.$$

Show that $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are also normed vector spaces, and $E = X \cap Y$. The space F is usually denoted by $X + Y$.

Problem 1.24 (True or False). Determine whether the following statements are true or false. If it is true, prove it. Otherwise, give a counter-example.

1. Given two sets A and B. Then $A\times B$ is countable if and only if A and B are countable.

2. Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ be a sequence and $\limsup\limits_{n\to\infty} x_n = x$. Then $\sup\limits_{n\in\mathbb{N}} x_n = x$.

3. The set $\{(x, y) \in \mathbb{R}^2 \mid x + y \in \mathbb{Q}\}$ is countable.

4. Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ be a sequence such that $|x_n - x_{n+1}| \leqslant \dfrac{1}{n}$. Then $\{x_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$.

5. If a bounded sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ satisfies $x_{n+1} - \epsilon_n \leqslant x_n$ for $n \in \mathbb{N}$, where $\sum_{n=1}^{\infty} \epsilon_n$ is an absolute convergent series; that is, the partial sum $\sum_{n=1}^{k} |\epsilon_n|$ converges as $k \to \infty$, then $\{x_n\}_{n=1}^{\infty}$ converges.

6. Let $\pi : \mathbb{N} \to \mathbb{N}$ be one-to-one and onto (such map π is called a rearrangement), and $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence. Then $\{x_{\pi(n)}\}_{n=1}^{\infty}$ is also convergent.

7. Let $A \subseteq \mathbb{R}$ satisfy
$$\sup\Big\{ \sum_{b \in B} |b| \,\Big|\, B \text{ is a non-empty finite subsets of } A \Big\} < \infty\,.$$
Then $\{x \in A \mid x \neq 0\}$ is countable.

8. Any rearrangement of the series $\sum_{n=1}^{\infty} x_n$ diverges if and only if x_n does not tend to 0 as $n \to \infty$.

9. If $\{x_n\}_{n=1}^{\infty}$ is a sequence of distinct non-zero real numbers such that $\lim_{n \to \infty} x_n = 0$, then the set $\{m x_n \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$.