Chapter 0

Introduction - Sets and Functions

0.1 Sets

Definition 0.1. A ***set*** is a collection of objects called ***elements*** or ***members*** of the set. To denote a set, we make a complete list $\{x_1, x_2, \cdots, x_N\}$ or use the notation

$$\{x : P(x)\} \quad \text{or} \quad \{x \,|\, P(x)\},$$

where the sentence $P(x)$ describes the property that defines the set. A set A is said to be a ***subset*** of S if every member of A is also a member of S. We write $x \in A$ (or A contains x) if x is a member of A, and write $A \subseteq S$ (or S includes A) if A is a subset of S. The empty set, denoted $\varnothing$, is the set with no member.

Definition 0.2. Let S be a given set, and $A \subseteq S$, $B \subseteq S$. The set $A \cup B$, called the ***union*** of A and B, consists of members belonging to set A or set B. Let $A_1, A_2, \cdots$ be sets. The set $\bigcup_{i=1}^{\infty} A_i = \{x \,|\, x \in A_i \text{ for some } i\}$ is the union of $A_1, A_2, \cdots$. The set $A \cap B$, called the ***intersection*** of A and B, consists of members belonging to both set A and set B. Let $A_1, A_2, \cdots$ be sets. The set $\bigcap_{i=1}^{\infty} A_i = \{x \,|\, x \in A_i \text{ for all } i\}$ is the intersection of $A_1, A_2, \cdots$.

Remark 0.3. Let $\mathscr{F}$ be the collection of some subsets in S. Sometimes we also write the union of sets in $\mathscr{F}$ as $\bigcup_{A \in \mathscr{F}} A$; that is,

$$\bigcup_{A \in \mathscr{F}} A = \{x \in S \,|\, \exists\, A \in \mathscr{F} \ni x \in A\}$$

Similarly, $\bigcap_{A \in \mathscr{F}} A = \{x \in S \,|\, \forall A \in \mathscr{F} \ni x \in A\}$ is the intersection of sets in $\mathscr{F}$.

Example 0.4. Let $A = \{1,2,3,4,5\}$, $B = \{1,3,7\}$, $S = \{1,2,3,\cdots\}$, and $\mathscr{F} = \{A,B\}$. Then $\bigcup\limits_{E\in\mathscr{F}} E \equiv A \cup B = \{1,2,3,4,5,7\}$, and $\bigcap\limits_{E\in\mathscr{F}} E \equiv A \cap B = \{1,3\}$.

Definition 0.5. Let S be a given set, and $A \subseteq S$, $B \subseteq S$. The ***complement*** of A relative to B, denoted $B\backslash A$, is the set consisting of members of B that are not members of A. When the universal set S under consideration is fixed, the complement of A relative to S or simply the complement of A, is denoted by $A^{\complement}$, or $S\backslash A$.

Theorem 0.6. (De Morgan's Law)

1. $B\backslash \bigcup\limits_{i=1}^{\infty} A_i = \bigcap\limits_{i=1}^{\infty}(B\backslash A_i)$ *or* $B\backslash \bigcup\limits_{A\in\mathscr{F}} A = \bigcap\limits_{A\in\mathscr{F}}(B\backslash A)$.
2. $B\backslash \bigcap\limits_{i=1}^{\infty} A_i = \bigcup\limits_{i=1}^{\infty}(B\backslash A_i)$ *or* $B\backslash \bigcap\limits_{A\in\mathscr{F}} A = \bigcup\limits_{A\in\mathscr{F}}(B\backslash A)$.

Proof. By definition,

$$x \in B\backslash \bigcup_{i=1}^{\infty} A_i \Leftrightarrow x \in B \text{ but } x \notin \bigcup_{i=1}^{\infty} A_i \Leftrightarrow x \in B \text{ and } x \notin A_i \text{ for all } i$$
$$\Leftrightarrow x \in B\backslash A_i \text{ for all } i \Leftrightarrow x \in \bigcap_{i=1}^{\infty}(B\backslash A_i)$$

The proof of the second identity is similar, and is left as an exercise. □

Definition 0.7. An ***ordered pair*** (a,b) is an object formed from two objects a and b, where a is called the first coordinate and b the second coordinate. Two ordered pairs are equal whenever their corresponding coordinates are the same. An ***ordered*** n***-tuples*** $(a_1,a_2,\cdots,a_n)$ is an object formed from n objects $a_1,a_2,\cdots,a_n$, where for each j, a_j is called the j-th coordinate. Two n-tuples $(a_1,a_2,\cdots,a_n)$, $(c_1,c_2,\cdots,c_n)$ are equal if $a_j = c_j$ for all $j \in \{1,\cdots,n\}$.

Definition 0.8. Given sets A and B, the ***Cartesian product*** $A \times B$ of A and B is the set of all ordered pairs (a,b) with $a \in A$ and $b \in B$, $A \times B = \{(a,b) \,|\, a \in A \text{ and } b \in B\}$. The Cartesian of three or more sets are defined similarly.

Example 0.9. Let $A = \{1,3,5\}$ and $B = \{\star, \diamond\}$. Then

$$A \times B = \{(1,\star),(3,\star),(5,\star),(1,\diamond),(3,\diamond),(5,\diamond)\}.$$

Example 0.10. Let $A = [2,7]$ and $B = [1,4]$. The Cartesian product of A and B is the square plotted below:

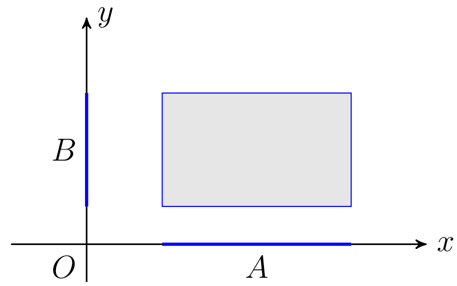
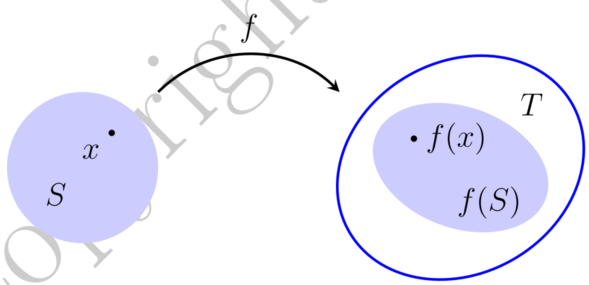


Figure 1: The Cartesian product $[2,7] \times [1,4]$

0.2 Functions

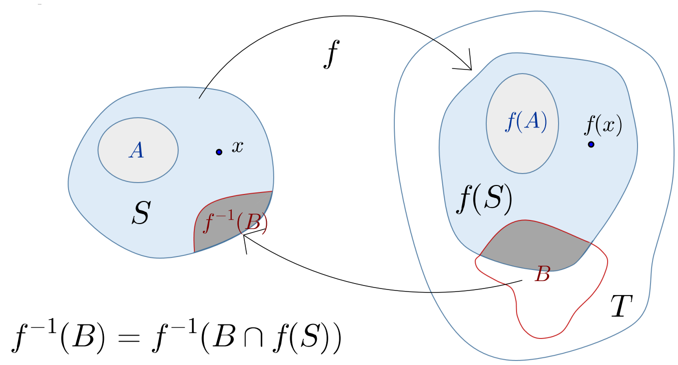
Definition 0.11. Let S and T be given sets. A ***function*** $f : S \to T$ consists of two sets S and T together with a "rule" that assigns to each $x \in S$ a special element of T denoted by $f(x)$. One writes $x \mapsto f(x)$ to denote that x is mapped to the element $f(x)$. S is called the ***domain***（定義域）of f, and T is called the ***target*** or ***co-domain*** of f. The ***range***（值域）of f or the ***image*** of f, is the subset of T defined by $f(S) = \{f(x) \,|\, x \in S\}$.



Definition 0.12. A function $f : S \to T$ is called ***one-to-one***（一對一）, ***injective*** or an ***injection*** if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ (which is equivalent to that $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$). A function $f : S \to T$ is called ***onto***（映成）, ***surjective*** or an ***surjection*** if $\forall\, y \in T$, $\exists\, x \in S$, $\ni f(x) = y$ (that is, $f(S) = T$). A function $f : S \to T$ is called an ***bijection*** if it is one-to-one and onto.

Remark 0.13 (映成函數的反敘述)**.** If $f : S \to T$ is **not** onto, then $\exists\, y \in T$, $\ni \forall\, x \in S$, $f(x) \neq y$. 一般來說，若有一個的數學的敘述 $\forall$statement A, $\exists$statement B $\ni$ statement C 成立，那麼它的相反敘述的寫法為: $\exists$statement A, $\ni \forall$statement B, statement C 不成立。簡單的記法：1. $\forall \leftrightarrow \exists$ 2. $\exists\, P \ni Q \leftrightarrow \ni \forall\, P \sim Q$.

Definition 0.14. For $f : S \to T$, $A \subseteq S$, we call $f(A) = \{f(x) \,|\, x \in A\}$ the ***image*** of A under f. For $B \subseteq T$, we call $f^{-1}(B) = \{x \in S \,|\, f(x) \in B\}$ the ***pre-image*** of B under f.



Example 0.15. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, $B = [-1, 4] \subseteq T$, $f^{-1}(B) = [-2, 2]$.

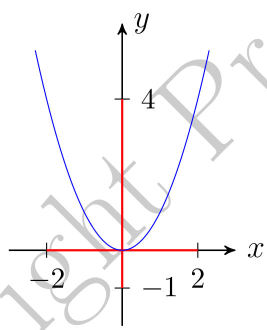


Figure 2: The preimage $f^{-1}([-1, 4])$ is $[-2, 2]$ if $f(x) = x^2$

Proposition 0.16. *Let $f : S \to T$ be a function, C_1 ,$C_2 \subseteq T$ and D_1, $D_2 \subseteq S$.*

(a) $f^{-1}(C_1 \cup C_2) = f^{-1}(C_1) \cup f^{-1}(C_2)$.

(b) $f(D_1 \cup D_2) = f(D_1) \cup f(D_2)$.

(c) $f^{-1}(C_1 \cap C_2) = f^{-1}(C_1) \cap f^{-1}(C_2)$.

(d) $f(D_1 \cap D_2) \subseteq f(D_1) \cap f(D_2)$.

(e) $f^{-1}(f(D_1)) \supseteq D_1$ *("=" if f is one-to-one).*

(f) $f(f^{-1}(C_1)) \subseteq C_1$ *("=" if $C_1 \subseteq f(S)$).*

Proof. We only prove (c) and (d), and the proof of the other statements are left as an exercise.

(c) We first show that $f^{-1}(C_1 \cap C_2) \subseteq f^{-1}(C_1) \cap f^{-1}(C_2)$. Suppose that $x \in f^{-1}(C_1 \cap C_2)$. Then $f(x) \in C_1 \cap C_2$. Therefore, $f(x) \in C_1$ and $f(x) \in C_2$, or equivalently, $x \in f^{-1}(C_1)$ and $x \in f^{-1}(C_2)$; thus $x \in f^{-1}(C_1) \cap f^{-1}(C_2)$.

Next, we show that $f^{-1}(C_1) \cap f^{-1}(C_2) \subseteq f^{-1}(C_1 \cap C_2)$. Suppose that $x \in f^{-1}(C_1) \cap f^{-1}(C_2)$. Then $x \in f^{-1}(C_1)$ and $x \in f^{-1}(C_2)$ which suggests that $f(x) \in C_1$ and $f(x) \in C_2$; thus $f(x) \in C_1 \cap C_2$ or equivalently, $x \in f^{-1}(C_1 \cap C_2)$.

(d) Suppose that $y \in f(D_1 \cap D_2)$. Then $\exists\, x \in D_1 \cap D_2$ such that $y = f(x)$. As a consequence, $y \in f(D_1)$ and $y \in f(D_2)$ which implies that $y \in f(D_1) \cap f(D_2)$. □

Example 0.17. We note it might happen that $f(D_1 \cap D_2) \subsetneq f(D_1) \cap f(D_2)$. Take $D_1 = [-1, 0]$ and $D_2 = [0, 1]$, and define $f : S = \mathbb{R} \to T = \mathbb{R}$ to be $f(x) = x^2$. Then $f(D_1) = f([-1, 0]) = [0, 1]$ and $f(D_2) = f([0, 1]) = [0, 1]$. However,

$$f(D_1 \cap D_2) = f(\{0\}) = \{0\} \subsetneq [0, 1] = f(D_1) \cap f(D_2)\,.$$

0.3 Exercises

§0.1 Sets

Problem 0.1. Let A, B, C be given sets. Show that

1. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
2. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

§0.2 Functions

Problem 0.2. Let S and T be given sets, $A \subseteq S$, $B \subseteq T$, and $f : S \to T$. Show that

1. $f(f^{-1}(B)) \subseteq B$, and $f(f^{-1}(B)) = B$ if $B \subseteq f(S)$.
2. $f^{-1}(f(A)) \supseteq A$, and $f^{-1}(f(A)) = A$ if $f : S \to T$ is one-to-one.

Problem 0.3. Let A and B be two non-empty sets and $f : A \to B$. Show that

1. $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$, $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$, $f^{-1}(D_1 \backslash D_2) = f^{-1}(D_1) \backslash f^{-1}(D_2)$ for all $D_1, D_2 \subseteq B$.

2. $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$, $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$ for all $C_1, C_2 \subseteq A$.

Problem 0.4. Let A and B be two non-empty sets and $f : A \to B$. Show that the following three statements are equivalent; that is, show that each one of the following statements implies the other four.

1. f is one-to-one.

2. For every y in B, the set $f^{-1}(\{y\})$ contains at most one point.

3. $f^{-1}(f(C)) = C$ for all $C \subseteq A$.

4. $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$ for all subsets C_1 and C_2 of A.

5. $f(C_2 \backslash C_1) = f(C_2) - f(C_1)$ for all $C_1 \subseteq C_2 \subseteq A$.