小波理論與應用 MA5110

Homework Assignment 1

Due Oct. 08. 2024

Problem 1. Let $\{K_\lambda\}_{\lambda>0}$ be summability kernels on $\mathbb{R}$, and $f \in L^p(\mathbb{R})$ for some $p \in [1,\infty)$. In class we show that

$$\lim_{\lambda\to\infty} \|f * K_\lambda - f\|_{L^p(\mathbb{R})} = 0$$

for the case $p = 1$. In this exercise problem, show the validity of the identity above for $1 < p < \infty$.
Hint: You may need the Minkowski inequality:

$$\Big[\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \big|g(x,y)\big|\,dx\Big)^p dy\Big]^{\frac{1}{p}} \leqslant \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \big|g(x,y)\big|^p\,dy\Big)^{\frac{1}{p}} dx$$

as well as the following property: if $f \in L^p(\mathbb{R})$, then

$$\lim_{y\to 0} \int_{\mathbb{R}} \big|f(x) - f(x-y)\big|^p\,dx = 0\,.$$

Problem 2. Similar to the summability kernels that we introduce in class, for 2π-periodic functions we have a similar concept called approximations of the identity. Let $\mathscr{C}(\mathbb{T})$ denote the collection of all continuous 2π-periodic functions. A family of functions $\big\{\varphi_n \in \mathscr{C}(\mathbb{T}) \,\big|\, n \in \mathbb{N}\big\}$ is an approximation of the identity if

(a) $\displaystyle\int_{-\pi}^{\pi} \varphi_n(x)\,dx = 1$ for every $n \in \mathbb{N}$;

(b) $\varphi_n(x) \geqslant 0$ $\Big($or there exists $M > 0$ such that $\displaystyle\int_{-\pi}^{\pi} \big|\varphi_n(x)\big|\,dx \leqslant M$ for all $n \in \mathbb{N}\Big)$;

(c) $\displaystyle\lim_{n\to\infty} \int_{\delta\leqslant|x|\leqslant\pi} \varphi_n(x)\,dx = 0$ for every $\delta > 0$.

Complete the following.

1. Show that if $f \in \mathscr{C}(\mathbb{T})$, then the sequence of functions $\{f \star \varphi_n\}_{n=1}^{\infty}$ converges uniformly to f on $[-\pi,\pi]$, where $f \star \varphi_n$ is the "convolution" of φ_n and f defined by

$$(f \star \varphi_n)(x) = \int_{-\pi}^{\pi} f(y)\varphi_n(x-y)\,dy\,.$$

2. Show that if $f \in L^p(0,2\pi)$ for some $1 \leqslant p < \infty$, then the sequence of functions $\{f \star \varphi_n\}_{n=1}^{\infty}$ converges to f in $L^p(0,2\pi)$; that is,

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \big|(f \star \varphi_n)(x) - f(x)\big|^p\,dx = 0\,.$$

Hint:

1. Make a slight modification of the proof of "$\displaystyle\lim_{\lambda\to 0} f * K_\lambda(c) = f(c)$ whenever $f \in L^\infty(\mathbb{R})$ and f is continuous at c" that we proved in class.

2. Mimic the proof of Problem 1.

Problem 3. In class we state without proving that if $f \in L^2(0, 2\pi)$ and

$$\int_0^{2\pi} f(x) e^{-inx}\, dx = 0 \qquad \forall\, n \in \mathbb{Z}, \tag{$\star$}$$

then $f = 0$. However, it is not clear that the validity of $(\star)$ still implies that $f = 0$ if we only know that $f \in L^1(0, 2\pi)$. Show that

$$\text{if } f \in L^1(0, 2\pi) \text{ and } \int_0^{2\pi} f(x) e^{-inx}\, dx = 0 \text{ for all } n \in \mathbb{Z}, \text{ then } f = 0 \tag{$\star\star$}$$

by complete the following.

1. For a given $f \in L^1(\mathbb{R})$, define

$$\sigma_n(f, x) = \frac{1}{2n+1} \sum_{k=-n}^{n} \sum_{\ell=-k}^{k} \langle f, e_\ell \rangle e^{i\ell x};$$

   that is, $\{\sigma_n(f, \cdot)\}_{n=1}^\infty$ is the Cesàro mean of the Fourier series of f. Show that $\{\sigma_n(f, \cdot)\}_{n=1}^\infty$ converges to f in $L^1(0, 2\pi)$.

2. Show that if $f \in L^1(0, 2\pi)$ satisfies $\langle f, e_n \rangle = 0$ for all $n \in \mathbb{Z}$, then $\sigma_n(f, \cdot) = 0$ for all $n \in \mathbb{N}$. Therefore, Part 1 shows that $(\star\star)$ holds.

Hint: First note that

$$\sum_{\ell=-k}^{k} \langle f, e_\ell \rangle e^{i\ell x} = \sum_{\ell=-k}^{k} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{i(x-y)\ell}\, dy = \int_{-\pi}^{\pi} f(y) D_k(x-y)\, dy,$$

where $D_k(t) = \frac{1}{2\pi} \sum_{\ell=-k}^{k} e^{i\ell t}$. Then

$$\sigma_n(f, x) = \frac{1}{2n+1} \sum_{k=-n}^{n} \int_{-\pi}^{\pi} f(y) D_k(x-y)\, dy = \int_{-\pi}^{\pi} f(y) F_n(x-y)\, dy = (f \star F_n)(x),$$

where $F_n(t) = \frac{1}{2n+1} \sum_{k=-n}^{n} D_k(t)$ and the convolution is defined in Problem 2. Find the precise form of F_n and show that $\{F_n\}_{n=1}^\infty$ is an approximation of the identity introduced in Problem 2, so Part 2 of Problem 2 shows that $\{f \star F_n\}_{n=1}^\infty$ converges to f in $L^1(0, 2\pi)$.

Problem 4. Show that if there exists $\delta > 0$ such that

$$\big|f(x)\big| + \big|\widehat{f}(x)\big| \leqslant C(1 + |x|)^{-1-\delta} \qquad \forall\, x \in \mathbb{R},$$

then

$$\sum_{n=-\infty}^{\infty} \check{\widehat{f}}(x + 2n\pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx} \qquad \forall\, x \in [0, 2\pi].$$

Hint: Note that the condition above implies that $f, \widehat{f} \in L^1(\mathbb{R})$, so $f = \check{\widehat{f}}$ a.e..

1. Show that $\big|\check{\widehat{f}}(x)\big| \leqslant C(1 + |x|)^{-1-\delta}$ for all $x \in \mathbb{R}$ as well.

2. Show that $F(x) \equiv \sum_{n=-\infty}^{\infty} \check{\widehat{f}}(x + 2n\pi)$ and $G(x) \equiv \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx}$ both converges uniformly (by the Weierstrass M-test), so both series are continuous functions.

3. Since F is continuous, $F \in L^2(0, 2\pi)$; thus F equals its Fourier series a.e. Note that G is the Fourier series of F.