# Vector Analysis MA2014-* Midterm Exam 1

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| 1(1) | 1(2) | 1(3) | 2(1) | 2(2) | 3 | 4 | 5 | 6 | Total |
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**Problem 1.** Let $\psi(x,y,z) = (x\cos y\sin z, x\sin y\sin z, x\cos z)$.

1. (9%) Compute the Jacobian matrix of $\psi$.

2. (9%) Find the adjoint matrix of the Jacobian matrix of $\psi$.

3. (12%) Verify the Piola identity for the map $\psi$.

*Solution*: Since

$$[D\psi](x,y,z) = \begin{bmatrix} \cos y\sin z & -x\sin y\sin z & x\cos y\cos z \\ \sin y\sin z & x\cos y\sin z & x\sin y\cos z \\ \cos z & 0 & -x\sin z \end{bmatrix},$$

we have

$$\mathrm{Adj}([D\psi])(x,y,z) = \begin{bmatrix} -x^2\cos y\sin^2 z & -x^2\sin y\sin^2 z & -x^2\sin z\cos z \\ x\sin y & -x\cos y & 0 \\ -x\cos y\sin z\cos z & -x\sin y\sin z\cos z & x\sin^2 z \end{bmatrix}.$$

The Piola identity says that each column of the adjoint matrix of $[\nabla\psi]$ is divergence-free. To check this, we note that

$$\begin{aligned}
&\frac{\partial}{\partial x}(-x^2\cos y\sin^2 z) + \frac{\partial}{\partial y}(x\sin y) + \frac{\partial}{\partial z}(-x\cos y\sin z\cos z) \\
&\quad = -2x\cos y\sin^2 z + x\cos y - x\cos y(\cos^2 z - \sin^2 z) \\
&\quad = -x\cos y\sin^2 z + x\cos y - x\cos y\cos^2 z \\
&\quad = -x\cos y(\sin^2 z + \cos^2 z) + x\cos y = 0\,, \\
&\frac{\partial}{\partial x}(-x^2\sin y\sin^2 z) + \frac{\partial}{\partial y}(-x\cos y) + \frac{\partial}{\partial z}(-x\sin y\sin z\cos z) \\
&\quad = -2x\cos y\sin^2 z + x\sin y - x\sin y(\cos^2 z - \sin^2 z) \\
&\quad = -x\cos y\sin^2 z + x\sin y - x\sin y\cos^2 z \\
&\quad = -x\cos y(\sin^2 z + \cos^2 z) + x\sin y = 0\,, \\
&\frac{\partial}{\partial x}(-x^2\sin z\cos z) + \frac{\partial}{\partial y}0 + \frac{\partial}{\partial z}(x\sin^2 z) = -2x\sin z\cos z + 2x\sin z\cos z = 0\,.
\end{aligned}$$

**Problem 2.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

1. (10%) Justify whether $f$ is continuous at $(0,0)$ or not.
2. (10%) Let $\boldsymbol{v} = (\cos\theta, \sin\theta)$ be a unit vector. Compute the the directional derivative of $f$ at $(0,0)$ in the direction $\boldsymbol{v}$.

*Solution*:

1. Using the polar coordinate:

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta}{r^2} = \lim_{r\to 0} r\cos^3\theta = 0\,;$$

thus $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$ which shows that $f$ is continuous at $(0,0)$.

2. The directional derivative of $f$ at $(0,0)$ in the direction $v = (\cos\theta, \sin\theta)$ is

$$\lim_{t\to 0} \frac{f(0+t\cos\theta, 0+t\sin\theta) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^3\cos^3\theta}{t^2} - 0}{t} = \lim_{t\to 0} \cos^3\theta = \cos^3\theta\,.$$

**Problem 3.** (15%) Suppose that $\boldsymbol{f}, \boldsymbol{g} : \mathbb{R} \to \mathbb{R}^m$ are differentiable at $a$ and there is a $\delta > 0$ such that $\boldsymbol{g}(x) \neq \mathbf{0}$ for all $0 < |x - a| < \delta$. If $\boldsymbol{f}(a) = \boldsymbol{g}(a) = \mathbf{0}$ and $[D\boldsymbol{g}(a)] \neq \mathbf{0}$, show that

$$\lim_{x \to a} \frac{\|\boldsymbol{f}(x)\|}{\|\boldsymbol{g}(x)\|} = \frac{\|[D\boldsymbol{f}(a)]\|}{\|[D\boldsymbol{g}(a)]\|} .$$

*Proof.* Since $\boldsymbol{f}$ and $\boldsymbol{g}$ are differentiable at $a$, by the definition of the differentiability,

$$\lim_{x \to a} \frac{\big|\boldsymbol{f}(x) - \boldsymbol{f}(a) - [D\boldsymbol{f}(a)](x - a)\big|}{|x - a|} = 0$$

and

$$\lim_{x \to a} \frac{\big|\boldsymbol{g}(x) - \boldsymbol{g}(a) - [D\boldsymbol{g}(a)](x - a)\big|}{|x - a|} = 0$$

which implies that

$$\lim_{x \to a} \frac{\boldsymbol{f}(x) - \boldsymbol{f}(a) - [D\boldsymbol{f}(a)](x - a)}{x - a} = \lim_{x \to a} \frac{\boldsymbol{g}(x) - \boldsymbol{g}(a) - [D\boldsymbol{g}(a)](x - a)}{x - a} = 0$$

Therefore, by the fact that $\boldsymbol{f}(a) = \boldsymbol{g}(a) = \mathbf{0}$,

$$\begin{aligned} \lim_{x \to a} \frac{\|\boldsymbol{f}(x)\|}{\|\boldsymbol{g}(x)\|} &= \lim_{x \to a} \frac{\|\boldsymbol{f}(x) - f(a) - [D\boldsymbol{f}(a)](x - a) + [D\boldsymbol{f}(a)](x - a)\|}{\|\boldsymbol{g}(x) - g(a) - [D\boldsymbol{g}(a)](x - a) + [D\boldsymbol{g}(a)](x - a)\|} \\ &= \lim_{x \to a} \frac{\|\frac{f(x) - f(a) - [D\boldsymbol{f}(a)](x - a)}{x - a} + [D\boldsymbol{f}(a)]\|}{\|\frac{g(x) - g(a) - [D\boldsymbol{g}(a)](x - a)}{x - a} + [D\boldsymbol{g}(a)]\|} = \frac{\|[D\boldsymbol{f}(a)]\|}{\|[D\boldsymbol{g}(a)]\|} . \end{aligned}$$

□

**Problem 4.** (15%) Prove that if $\alpha > \frac{1}{2}$, then

$$f(x,y) = \begin{cases} |xy|^\alpha \log(x^2+y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is differentiable at $(0,0)$.

*Proof.* First we compute $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$. By definition,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0\,,$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0\,.$$

As a consequence,

$$\lim_{(h,k)\to(0,0)} \frac{\left|f(h,k) - f(0,0) - [0,0]\begin{bmatrix} h \\ k \end{bmatrix}\right|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{-|hk|^\alpha \log(h^2+k^2)}{\sqrt{h^2+k^2}}\,;$$

thus letting $(h,k) = (r\cos\theta, r\sin\theta)$, we find that

$$\begin{aligned} \lim_{(h,k)\to(0,0)} \frac{\left|f(h,k) - f(0,0) - [0,0]\begin{bmatrix} h \\ k \end{bmatrix}\right|}{\sqrt{h^2+k^2}} &= \lim_{r\to 0^+} \frac{-2r^{2\alpha}\log r}{r} |\cos\theta|^\alpha |\sin\theta|^\alpha \\ &= -2|\cos\theta\sin\theta|^\alpha \lim_{r\to 0^+} r^{2\alpha-1}\log r = 0\,. \end{aligned}$$

□

**Problem 5.** (15%) Let $u : \mathbb{R} \to [0, \infty)$ be differentiable. Prove that for each $(x, y, z) \neq (0, 0, 0)$, the function

$$F(x, y, z) = u(\sqrt{x^2 + y^2 + z^2})$$

satisfies

$$\left(\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right)^{\frac{1}{2}} = \left|u'(\sqrt{x^2 + y^2 + z^2})\right|.$$

*Proof.* Let $F_x, F_y, F_z$ denote $\dfrac{\partial F}{\partial x}$, $\dfrac{\partial F}{\partial y}$ and $\dfrac{\partial F}{\partial z}$, respectively. By the chain rule,

$$F_x(x, y, z) = u'(\sqrt{x^2 + y^2 + z^2})\frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = u'(\sqrt{x^2 + y^2 + z^2})\frac{x}{\sqrt{x^2 + y^2 + z^2}},$$

and similarly,

$$F_y(x, y, z) = u'(\sqrt{x^2 + y^2 + z^2})\frac{y}{\sqrt{x^2 + y^2 + z^2}} \quad \text{and} \quad F_z(x, y, z) = u'(\sqrt{x^2 + y^2 + z^2})\frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

Therefore,

$$(F_x^2 + F_y^2 + F_z^2)(x, y, z) = u'(\sqrt{x^2 + y^2 + z^2})^2\frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} = u'(\sqrt{x^2 + y^2 + z^2})^2$$

which conclude the desired identity. □

**Problem 6.** (15%) Find conditions on a point $(x_0, y_0, u_0, v_0)$ such that there exists real-valued functions $u(x, y)$, $v(x, y)$ which are continuously differentiable near $(x_0, y_0)$ and satisfy the equations

$$\begin{aligned} xu^2 + yv^2 + xy &= 9\,, \\ xv^2 + yu^2 - xy &= 7\,. \end{aligned}$$

Prove that the solution satisfy $u^2 + v^2 = \dfrac{16}{x+y}$.

*Proof.* Let $F(x, y, u, v) = xu^2 + yv^2 + xy - 9$ and $G(x, y, u, v) = xv^2 + yu^2 - xy - 7$. Applying the implicit function theorem: we need

1. $F(x_0, y_0, u_0, v_0) = G(x_0, y_0, u_0, v_0) = 0$, and
2. the matrix $\begin{bmatrix} F_u(x_0, y_0, u_0, v_0) & F_v(x_0, y_0, u_0, v_0) \\ G_u(x_0, y_0, u_0, v_0) & G_v(x_0, y_0, u_0, v_0) \end{bmatrix}$ is invertible.

Computing the matrix $\begin{bmatrix} F_u & F_v \\ G_u & G_v \end{bmatrix}$, we find that the invertibility of this matrix is equivalent to that

$$0 \neq \begin{vmatrix} 2x_0u_0 & 2y_0v_0 \\ 2y_0u_0 & 2x_0v_0 \end{vmatrix} = 4(x_0^2u_0v_0 - y_0^2u_0v_0) = 4u_0v_0(x_0^2 - y_0^2)\,.$$

The implicit function theorem implies that in a neighborhood of $(x_0, y_0)$, there exist $u(x, y)$ and $v(x, y)$ such that $u(x_0, y_0) = u_0$, $v(x_0, y_0) = v_0$, and

$$\begin{aligned} F(x, y, u(x, y), v(x, y)) &= xu(x, y)^2 + yv(x, y)^2 + xy - 9 = 0\,, \\ G(x, y, u(x, y), v(x, y)) &= xv(x, y)^2 + yu(x, y)^2 - xy - 7 = 0\,. \end{aligned}$$

Therefore,

$$0 = F(x, y, u(x, y), v(x, y)) + G(x, y, u(x, y), v(x, y)) = (x + y)\big(u(x, y)^2 + v(x, y)^2\big) - 16$$

which implies that the solutions $u$, $v$ satisfies

$$u^2 + v^2 = \frac{16}{x+y}\,. \qquad \square$$